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Degree of Mapping of Manifolds Based on That of Euclidean Open Sets

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In this paper we shall establish a theory of the degree of mapping of manifolds (locally Euclidean spaces) based on the notion of the degree of mapping of Euclidean open sets. In fact, since it is yet an unsolved problem whether a topological manifold is a polyhedron, we can not directly apply the theory of simplicial mappings.

In § 1 we shall state the fundamental properties of the degree of mapping of Euclidean open sets, the definition of manifolds and allied matters. In § 2 α-mappings (mappings with a certain restriction) of open sets of a manifold into another manifold will be treated as a preparation of the following paragraph. In § 3 the definition of the degree of mapping of a general kind will be given. In § 4 will be proved fundamental properties of the degree of mapping defined in § 3.

In this paper we shall use the notation E^m for m-dimensional Euclidean space, K^m (or K) for m-dimensional open disc:

$$K=\{x \mid \sum_{\nu=1}^{m} x_\nu^2 < 1\}.$$

The closure of a set M will be denoted by $\overline{M}$. $\{\}$ means the empty set. Mapping means always continuous mapping.

§ 1. Preliminary Notions

1. 1. First we shall recall fundamental properties of the degree of mapping of the closure of Euclidean open sets. Let D be a bounded open set in E^m and f be a mapping of D into E^m. Let a be a point not on $f(\overline{D}-D)$, then there will be defined an *integer* $\mathrm{A}[a, D, f]$, called *degree of mapping* of D at a by f, with the following properties[1]:

(i) *If f is the identical mapping of D and $a \in D$, then*

1) Cf. Nagumo: A theory of degree of mapping based on infinitesimal analysis, which will appear in Amer. Journ. of Math. and will be denoted by [N].

$$\mathrm{A}[a, D, f]=1.$$

(ii) *If* $a \notin f(\overline{D})$, *then* $\mathrm{A}[a, D, f]=0$.

(iii) *If* $\overline{D}=\bigvee_{i=1}^{k} \overline{D}_i$, $D \supset \bigvee_{i=1}^{k} D_i$ *where* D_i *are open sets and* $a \notin f(\overline{D}_i - D_i)$, *then*

$$\mathrm{A}[a, D, f]=\sum_{i=1}^{k} \mathrm{A}[a, D_i, f].$$

(iv) *If* $f_t(x)$ *and* $a(t)\,(\in E^m)$ *are continuous for* $0 \leqq t \leqq 1, x \in \overline{D}$ *and* $a(t) \notin f_t(\overline{D}-D)$ *for* $0 \leqq t \leqq 1$, *then* $\mathrm{A}[a(t), D, f_t]$ *is constant for* $0 \leqq t \leqq 1$.

(v) *If* $f(D) \subset D'$ [2] *where* D' *is also a bounded open set in* E^m *and* f' *is a mapping of* $\overline{D}'$ *into* E^m *such that* $a \notin f'(\overline{D}'-D') \cup f'f(\overline{D}-D)$, *then*

$$\mathrm{A}[a, D, f'f]=\sum_i \mathrm{A}[a, H_i, f'] \cdot [b_i, D, f],$$

where H_i *are components of* $D'-f(\overline{D}-D)$ *and each* b_i *is any point in* H_i [3].

Theorem 1. 1. *If* D_1 *is an open set such that* $f^{-1}(a) \subset D_1 \subset D$, *then*

$$\mathrm{A}[a, D_1, f]=\mathrm{A}[a, D, f].$$

Proof. Put $D-\overline{D}_1=D_2$ and apply (ii) and (iii).

A mapping f of $\overline{D}(\subset E^m)$ into E^m is said to be *positive* (*negative*) when $\mathrm{A}[p, D, f]>0$ (<0) hold for any point $p \in f(D)$. From (v) we can obtain: *Let* D *and* D' *be open sets in* E^m. *If* f *is a posotive* 1-1 *mapping of* $\overline{D}$ *onto* $\overline{D}'$ *such that* $D'=f(D)$, *then the inverse mapping* f^{-1} *is also positive* [4].

1. 2. Now let us go to the definition of manifold. An *m-dimensional manifold* is a topological space $\mathfrak{M}$ with a *covering system* $\{U_i\}$ as follows:

(i) $\mathfrak{M}$ *is covered by at most a countable number of open sets* U_i.

(ii) *Each* $\overline{U}_i$ *is homeomorphically mapped onto an m-dimensional closed disc* $\overline{K}$ *so that* U_i *corresponds to* K. The homeomorphic mapping φ_i of $\overline{U}_i$ onto $\overline{K}$ such that $K=\varphi_i U_i$ will be called the *local coordinate* of U_i.

(iii) *The covering is locally finite, i. e. any compact set in* $\mathfrak{M}$ *meets only a finite number of* U_i.

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2) In [N] it was $f(\overline{D}) \subset D'$, but an easy artifice will aford us this form.

3) Since $f^{-1}(a)$ is compact and $a \in f(H_i)$ only for a finite number of H_i, then there are at most a finite number of i such that $\mathrm{A}[a, H_i, f'] \neq 0$.

4) Cf. Theorem 1. 2.

(iv) $\mathfrak{M}$ *is connected.*

As manifolds are metrisable we assume that $\mathfrak{M}$ is metric. In this paper we shall use the notation $\mathfrak{M}$ for an m-dimensional manifold.

Let $\{\varphi_i\}$ and $\{\varphi_j'\}$ be two systems of local coordinates of the same $\mathfrak{M}$. φ_i and φ_j' are said to have the *same orientation* (*opposite orientations*) if $\varphi_j'\varphi_i^{-1}$ is positive (negative) on $\varphi_i(U_i \cap U_j')$. $\mathfrak{M}$ is called *orientable* if there exists a covering system $\{U_i\}$ with local coordinates $\{\varphi_i\}$ such that φ_i and φ_j have the same orientation if $U_i \cap U_j \neq \{\}$. If $\mathfrak{M}$ is orientable we take $\{\varphi_i\}$ so that all φ_i have the same orientation. We can prefer a covering system $\{U_i\}$ of $\mathfrak{M}$ and local coordinates $\{\varphi_i\}$ such that any pair of local coordinates φ_i, φ_j have the same or opposite orientations if $U_i \cap U_j \neq \{\}$.

1. 3. Concerning the 1-1 mapping of Euclidean open sets we have:

Theorem 1. 2. *Let D be a bounded open set in E^m and f a 1-1 mapping of $\overline{D}$ into E^m, then $f(D)$ is also an open set in E^m, and for any point $b=f(a), a \in D$ we have*

$$\mathrm{A}[b, D, f]=\mathrm{A}[a, f(D), f^{-1}]=\pm 1.$$

Proof. As f is an 1-1 mapping it holds $b \notin f(\overline{D}-D)$. Let G be a bounded open set containing $f(\overline{D}) \cup \overline{D}$. f^{-1} is continuous on $f(\overline{D})$. Let us extend the mapping f^{-1} to the mapping g of $\overline{G}$ into E^m such that

$$g(x)=f^{-1}(x) \text{ for } x \in f(\overline{D}), \qquad g(x)=x \text{ for } x \in \overline{G}-G.$$

Then $$a \notin (\overline{G}-G) \cup (\overline{D}-D)=g(\overline{G}-G) \cup gf(\overline{D}-D).$$

Thus by (v) in 1. 1.

$$\mathrm{A}[a, D, gf]=\textstyle\sum_i \mathrm{A}[a, H_i, g]\cdot \mathrm{A}[b_i, D, f],$$

where H_i are components of $G-f(\overline{D}-D)$ and each b_i is any point of H_i. But since $gf(x)=x$ for $x \in \overline{D}$ and $a \in D$ we get by (i) $\mathrm{A}[a, D, gf]=1$. Therefore there exists an i such that

$$\mathrm{A}[a, H_i, g]\cdot \mathrm{A}[b_i, D, f] \neq 0.$$

Then $H_i \subset f(D)$ by (ii) in 1. 1 as b_i is any point of H_i and $a \in g(H_i)$. Hence $g(x)=f^{-1}(x)$ for $x \in H_i$ and $a \in f^{-1}(H_i)$.

Thus $$b=f(a) \in H_i \text{ (open set) } \subset f(D).$$

As b is any point of $f(D)$, $f(D)$ is an open set.

Since there is only one H_i which contains b,

$$A[a, H_j, g]\cdot A[b_j, D, f]=0 \quad \text{for } j\neq i,$$

Hence $$1=A[a, H_i, f^{-1}]\cdot A[b, D, f].$$

Thus, since degree of mapping must be integer,

$$A[b, D, f]=A[a, H_i, f^{-1}]=\pm 1.$$

As $f(a)\in H_i\subset f(D)$ we get by Theorem 1.1

$$A[a, H_i, f^{-1}]=A[a, f(D), f^{-1}].$$

Consequently $$A[b, D, f]=A[a, f(D), f^{-1}]=\pm 1.$$

§ 2. α-mappings of Manifords.

2. 1. Throughout this paper we denote by $\mathfrak{M}$ and $\mathfrak{M}'$ m-dimensional manifolds and by $\{U_i\}$ and $\{V_i\}$ covering systems of $\mathfrak{M}$ and $\mathfrak{M}'$ with local coordinates $\{\varphi_i\}$ and $\{\psi_i\}$ respectively. An open set D in $\mathfrak{M}$ is said to be *bounded* if $\bar{D}$ is compact.

f is called an *α-mapping* of D if f is a mapping of $\bar{D}$ such that $f^{-1}(p)\wedge D$ is at most a countable set for any $p\in f(D)$.

Theorem 2. 1. *Let f be a mapping of $\bar{D}$ into $\mathfrak{M}'$ where D is a bounded open set in $\mathfrak{M}$. Then for any given $\varepsilon>0$ there exists an α-mapping f^* of D such that*

$$\operatorname{dist}(f^*(x), f(x))<\varepsilon \text{ for } x\in D, \qquad f^*(x)=f(x) \text{ for } x\in\bar{D}-D. \tag{0}$$

Proof. At first we assume that D is so small that

$$\bar{D}\subset U_k\in\{U_i\}, \qquad f(\bar{D})\subset V_l\in\{V_j\}. \tag{1}$$

Let φ and ψ be the local coordinates of U_k and V_l respect. Put $\psi f\varphi^{-1}=\hat{f}$, then $\hat{f}$ mapps $\varphi(\bar{D})\,(\subset K\subset E^m)$ into K. The open set $\varphi(D)$ in E^m can be regarded as formed from an Euclidean complex C consisting of a countable m-simplexes σ_n and thier sides such that

$$\lim_{n\to\infty}\operatorname{diam}(\sigma_n)=0, \qquad \operatorname{diam}(\hat{f}(\sigma_n))<\delta/2,$$

where δ is a number such that $\operatorname{diam}(A)<\varepsilon$ holds for any set $A\,(\subset V_l)$ with $\operatorname{diam}(\psi(A))<\delta$. Let a_i be the vertices of the complex C, and a point $a_i'(\in K)$ corresponds to a_i so that

$$\operatorname{dist}(a_i', \hat{f}(a_i))<\delta/2, \qquad \lim_{i\to\infty}\operatorname{dist}(a_i', \hat{f}(a_i))=0,$$

and the points $a'_{i(1)}, \dots, a'_{i(m)}$ which correspond to the vertices of any σ_n span a non-degenerated simplex σ_n' in E^m. Let $\hat{f}^*$ be the mapping of

$\varphi(\bar{D})$ ($\subset K$) into K such that $\hat{f}^*(a_i)=a_i'$, $\hat{f}^*(\sigma_n)=\sigma_n'$ (affine in each σ_n). Put $f^*=\psi^{-1}\hat{f}^*\varphi$ then f^* is an α-mapping of D into $\mathfrak{M}'$ such that the relations (0) hold.

Now we remove the assumption (1). Let λ be the Lebesgue's number of the covering of $\bar{D}$ by $\{U_i\}$ and λ' be that of $f(\bar{D})$ by $\{V_i\}$. Then there exists a $\gamma>0$ such that $0<\gamma\leqq\lambda$ and

$$\operatorname{diam}(f(A))<\lambda', \text{ if } A\subset\bar{D} \text{ and } \operatorname{diam}(A)<\gamma.$$

Let $\{W_i\}$ be a countable system of open sets such that $\bigvee_{i=1}^{\infty}W_i=D$, $\operatorname{diam}(W_i)<\gamma$ and $\{W_i\}$ is a locally finite covering of D. Step by step we can find by the first part of the proof, a sequence of mappings $f_i^*(i=1.2,\dots)$ of $\bar{D}$ into $\mathfrak{M}'$ such that $f_0^*=f$, $f_i^*(x)=f_{i-1}^*(x)$ for $x\in\bar{D}-W_i$, $\operatorname{dist}(f_i^*(x), f_{i-1}^*(x))<2^{-i}\varepsilon$ for $x\in W_i$ and f_i^* affords an α-mapping of $\bigvee_{\nu=1}^{i}W_\nu$ into $\mathfrak{M}'$. Thus in the limit $i\to\infty$ we get a desired α-mapping $f^*(x)=\lim_{i\to\infty}f_i^*(x)$.

2. 2. Now let f be an α-mapping of a bounded open set D in $\mathfrak{M}$ into $\mathfrak{M}'$ such that $a\notin f(\bar{D}-D)$ where $a\in\mathfrak{M}'$.

Definition A. *Let $G_\nu(\nu=1,\dots,n)$ be a finite number of disjoint open sets such that*

$$\bar{G}_\nu\subset U_{i(\nu)}\frown D,\quad f(\bar{G}_\nu)\subset V_{j(\nu)},\quad \bigvee_{\nu=1}^{n}G_\nu\supset f^{-1}(a) \tag{1}$$

where $U_{i(\nu)}\in\{U_i\}$ and $V_{j(\nu)}\in\{V_j\}$. Then we define $\mathrm{A}[a, G_\nu, f]$ *by*

$$\mathrm{A}[a, G_\nu, f]=\begin{cases}\mathrm{A}[\psi(a), \varphi(G_\nu), \psi f\varphi^{-1}] & \text{if } a\in V_{j(\nu)}\\ 0 & \text{if } a\notin V_{j(\nu)}\end{cases}$$

where $\psi=\psi_{j(\nu)}$, $\varphi=\varphi_{i(\nu)}$, and $\mathrm{A}[a, D, f]$, *"the degree of mapping of D at a by f"* (α-mapping), *by*

$$\mathrm{A}[a, D, f]=\sum_{\nu=1}^{n}\mathrm{A}[a, G_\nu, f],$$

if $\mathfrak{M}$ is orientable. If $\mathfrak{M}$ is non-orientable we take this by mod 2.

Lemma 2. 1. *Let X be a compact countable set in $\mathfrak{M}$. Then for any given $\varepsilon>0$ there exist a finite number of disjoint open sets G_ν such that* $\operatorname{diam}(G_\nu)<\varepsilon$, $\bigvee_{\nu=1}^{n}G_\nu\supset X$.

Proof. There exists a ρ such that $0<\rho<\varepsilon$, $\rho\neq\operatorname{dist}(x_\mu, x_\nu)$ for any pair $x_\mu, x_\nu\in X$. Let $W_\rho(x_\nu)$ be the ρ-neighborhood of x_ν and put $'G_\mu=W_\rho(x_\mu)-\bigvee_{\nu=1}^{\mu-1}\overline{W}_\rho(x_\nu)$. Then a finite number of $'G_\nu$ will form the desired system $\{G_\nu\}$.

To legitimate Definition A we have the following:

Theorem 2. 2. $A[a, D, f]$ *is independent of the choice of* G_ν, *covering systems of* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *and their local coordinates, provided that they have the same orientation.*

To prove this we use the following:

Lemma 2. 2. *Let* G *and* H *be bounded open sets in* E^m *and* f *be a mapping of* $\overline{G}$ *into* E^m *such that* $f(G)\subset H$. *Let* φ *be a positive* 1-1 *mapping of* $\overline{G}$ *onto* $\overline{G}'(G'=\varphi(G))$ *and* ψ *be a positive* 1-1 *mapping of* $\overline{H}$ *onto* $\overline{H}'(H'=\psi(H))$. *Then, if* $a\notin f(\overline{G}-G)$,

$$A[\psi(a), G', \psi f\varphi^{-1}]=A[a, G, f]. \qquad (0)^{5)}$$

Proof. Put $\psi(a)=a'$ and $\psi f=f'$, then $a'\notin f'(\overline{G}-G)$.
At first let us prove that

$$A[a', G', f'\varphi^{-1}]=A[a', G, f']. \qquad (1)$$

Let G_i be the components of G, then $\varphi(G_i)=G_i'$ are the components of of $G'-\varphi(\overline{G}-G)=G'$. Hence by (v) in §1

$$A[a', G, f']=\sum_i A[a', G_i', f'\varphi^{-1}]\cdot A[a_i, G, \varphi],$$

where a_i is any point of G_i'. As φ is 1-1 and positive and $a_i\in\varphi(G)$, then $A[a_i, G, \varphi]=1$ by Theorem 1. 2.
Hence

$$A[a', G, f']=\sum_i A[a', G_i', f'\varphi^{-1}]. \qquad (2)$$

There are at most a finite number of G_i', $1\leq i\leq l$, such that $a'\in f'\varphi^{-1}(G_i')$. Then by (ii), (iii) in §1 and Theorem 1. 1 we get

$$\sum_i A[a', G_i', f'\varphi^{-1}]=\sum_{i=1}^{l} A[a', G_i', f'\varphi^{-1}]= A[a', G', f'\varphi^{-1}].$$

Hence by (2) we obtain (1).

Now let us prove

$$A[a', G, \psi f]=A[a, G, f]. \qquad (3)$$

Let H_i be the components of $H-f(\overline{G}-G)$ and a_i any point of H_i, then by (v) in §1

$$A[a', G, \psi f]=\sum_i A[a', H_i, \psi]\cdot A[a_i, G, f].$$

Let it be $a\in H_1$. Since ψ is a 1-1 mapping of H and $a'\in\psi(H_1)$, then $A[a', H_i, \psi]=0$ for $i\neq 1$. As ψ is 1-1 and positive we have

5) By Theorem 1. 2. G' and H' are open sets.

$A[a', H_1, \psi]=1$. Hence we get (3). From (1) and (3) follows (0).

Proof of Theorem 2.2. We assume that $\mathfrak{M}$ is orientable, if otherwise the proof goes also similarly. Let $\{U_i'\}$ and $\{V_j'\}$ be other covering systems of $\mathfrak{M}$ and $\mathfrak{M}'$ with local coordinates $\{\varphi_i'\}$ and $\{\psi_j'\}$ respectively. If $\overline{G}_\nu \subset U_i \frown U_{i'}'$ and $f(\overline{G}_\nu) \subset V_j \frown V_{i'}'$, then by Lemma 2.2

$$A[\psi_j(a), \varphi_i(G_\nu), \psi_j f \varphi_i^{-1}]=A[\psi_{i'}'(a), \varphi_{i'}'(G), \psi_{j'}' f \varphi_{i'}'^{-1}],$$

if we take $\psi_{j'}'\psi_j^{-1}$ for ψ, $\varphi_{i'}'\varphi_i^{-1}$ for φ and $\psi_j f \varphi_i^{-1}$ for f, namely $A[a, G_\nu, f]$ is independent of the covering systems of $\mathfrak{M}$ and $\mathfrak{M}'$ or of their local coordinates.

Now let $\{G_\mu\}$ and $\{G_\nu'\}$ be two systems of disjoint open sets satisfying (1) in Definition A and put $G_\mu \frown G_\nu'=G_{\mu\nu}$, then from the definition of $A[a, G, f]$ we get easily

$$\sum_\mu A[a, G_\mu, f]=\sum_\mu\sum_\nu A[a, G_{\mu\nu}, f]=\sum_\nu A[a, G_\nu', f],$$

by applying Therem 1.1. Thus the proof is done.

We can easily prove the following:

Theorem 2.3. (i) *If f is the identical mapping of $D(\subset \mathfrak{M})$ and $a \in D$, then* $A[a, D, f]=1$.

(ii) *If $a \notin f(\overline{D})$, then* $A[a, D, f]=0$.

(iii) *Let $D, D_{i\,(i=1,\dots,k)}$ be bounded open sets in $\mathfrak{M}$ such that*

$$\overline{D}=\bigvee_{i=1}^k \overline{D}_i, \quad D\supset\bigvee_{i=1}^k D_i, \quad D_i\frown D_j=\{\}\ {\scriptstyle (j\neq i)}$$

and f be an α-mapping of D into $\mathfrak{M}'$ such that $a\notin f(\overline{D}_i-D_i)$ $(a\in\mathfrak{M}')$ then

$$A[a, D, f]=\sum_{i=1}^k A[a, D_i, f].$$

Theorem 2.4. *Let D be a bounded open set in $\mathfrak{M}$, f be an α-mapping of D into $\mathfrak{M}'$, and a and a' be two points in a same component of $\mathfrak{M}'-f(\overline{D}-D)$, then*

$$A[a, D, f]=A[a', D, f].$$

Proof. We can prove this easily if a' is sufficiently near to a. Now a and a' can be joined by a curve C on $\mathfrak{M}'$ without touching $f(\overline{D}-D)$. For each point p of C there is a neighborhood $U(p)$ of p where $A[x, D, f]$ $(x\in U(p))$ remains constant. Then by the compactness of C we obtain the desired relation.

§ 3. Degree of General Mappings.

3.1. Symbols $\mathfrak{M}, \mathfrak{M}', \{U_i\}, \{V_j\}, \varphi_i$ and ψ_j have the same meanings as in §2. Let D be a bounded open set in $\mathfrak{M}$.

It will be not difficult to prove the following:

Lemma 3. 1. *For any* $\varepsilon > 0$ *there exists a covering system* $\{U_i\}$ *of* $\mathfrak{M}$ *such that* $\operatorname{diam}(U_i) < \varepsilon$.

Lemma 3. 2. *Let* f_0 *and* f_1 *be two* α*-mappings of* D *into* $\mathfrak{M}'$, *and* Δ *be an open set in* $\mathfrak{M}$ *such that*

$$\bar{\Delta} \subset D \frown U_k, \quad U_k \in \{U_i\}, \quad a \notin f_0(\bar{D}-D), \quad a \in \mathfrak{M}',$$

$$f_0(x) = f_1(x) \quad \textit{for } x \in \bar{D}-\Delta \tag{1}$$

and
$$f_\nu(\bar{\Delta}) \subset V_l \in \{V_j\} \quad {\scriptstyle(\nu=0,1)}.$$

Then
$$\mathrm{A}[a, D, f_0] = \mathrm{A}[a, D, f_1] \tag{*}$$

Proof. At first we assume that $a \notin f_\nu(\bar{\Delta}-\Delta)$ ${\scriptstyle(\nu=0,1)}$.

Then
$$\mathrm{A}[a, D, f_\nu] = \mathrm{A}[a, D-\bar{\Delta}, f_\nu] + \mathrm{A}[a, \Delta, f_\nu]. \tag{2}$$

But by (1)
$$\mathrm{A}[a, D-\bar{\Delta}, f_0] = \mathrm{A}[a, D-\bar{\Delta}, f_1]. \tag{3}$$

And by Definition A

$$\mathrm{A}[a, \Delta, f_\nu] = \mathrm{A}[\psi(a), \varphi(\Delta), \psi f_\nu \varphi^{-1}], \tag{4}$$

where $\varphi(U_k)=K$ and $\psi(V_l)=K$. Put $\psi f_\nu \varphi^{-1} = \hat{f}_\nu$, then $\hat{f}_\nu$ mapps $\varphi(\bar{\Delta})$ $(\subset K)$ into $K \subset E^m$, and $\hat{f}_0(x) = \hat{f}_1(x)$ for $x \in \varphi(\bar{\Delta}-\Delta)$.
If we put $\hat{f}_t(x) = (1-t)\hat{f}_0(x) + t\hat{f}_1(x)$, then

$$\psi(a) \notin \hat{f}_t(\varphi(\bar{\Delta}-\Delta)) = \hat{f}_0(\bar{\Delta}-\Delta) \quad \text{for } 0 \leqq t \leqq 1.$$

Thus by (iv) in §1 $\mathrm{A}[\psi(a), \varphi(\Delta), \hat{f}_t]$ is constant for $0 \leqq t \leqq 1$.

Hence
$$\mathrm{A}[\psi(a), \varphi(\Delta), \psi f_0 \varphi^{-1}] = \mathrm{A}[\psi(a), \varphi(\Delta), \psi f_1 \varphi^{-1}].$$

Thus by (2), (3) and (4) we obtain (*).

Now we shall remove the condition $a \notin f_\nu(\bar{\Delta}-\Delta)$ ${\scriptstyle(\nu=0,1)}$. For this it suffices to prove the existence of an open set Δ' such that

$$\Delta \subset \Delta', \quad \bar{\Delta}' \subset U_k \frown D, \quad a \notin f_\nu(\bar{\Delta}'-\Delta'), \quad f_\nu(\bar{\Delta}') \subset V_l \quad {\scriptstyle(\nu=0,1)}.$$

Put $f_\nu^{-1}(a) = X_\nu$ then X_ν are compact countable sets. For any point $p \in X_\nu \frown (\bar{\Delta}-\Delta)$ there exists a neighborhood $W(p)$ of p such that $W(p) \subset U_k \frown D$, $f_\nu(W(p)) \subset V_l$ and the boundary of $W(p)$ does not meet X_ν. The set $(X_0 \smile X_1) \frown (\Delta-\Delta)$ can be covered by a finite number of such $W(p)$, i.e. by $W(p_r)$ ${\scriptstyle(r=1,\dots,s)}$. Then $\Delta \smile \bigvee_{r=1}^{s} W(p_r) = \Delta'$ has the above mentioned property.

3. 2. Now we proceed to the definition of the degree of mapping of the general kind. Let D be a bounded open set in $\mathfrak{M}$ and f be a mapping of $\bar{D}$ into $\mathfrak{M}'$ such that $a \notin f(\bar{D}-D)$, $(a \in \mathfrak{M}')$.

Definition B. *Let λ be the Lebesgue's number of the finite covering of $f(\bar{D})$ by $\{V_j\}$, where $\{V_j\}$ is a covering system of $\mathfrak{M}'$ such that*

$$\operatorname{diam}(V_j) < \operatorname{dist}(a, f(\bar{D}-D))\,^{6)}.$$

Then we define $A[a, D, f]$, "the degree of mapping of D at a by f," by

$$A[a, D, f] = A[a, D, f^*],$$

where f^* is an α-mapping of D into $\mathfrak{M}'$ such that

$$\operatorname{dist}(f^*(x), f(x)) < \lambda \quad \textit{for } x \in \bar{D}.$$

This definition will be legitimated by the following:

Theorem 3. 1. *Let f, D and λ have the same meanings as in Definition B. Let f_1 and f_2 be two α-mappings of D into $\mathfrak{M}'$ such that*

$$\operatorname{dist}(f_i(x),\ f(x)) < \lambda \quad {\scriptstyle (i=1,2)}.$$

Then

$$A[a, D, f_1] = A[a, D, f_2] \tag{0}$$

Proof. Let p be any point of $\bar{D}$, then there exists a neighborhood $\Delta(p)$ of p such that

$$\operatorname{dist}(f_i(x), f(x')) < \lambda \quad \text{for } x, x' \in \Delta(p) \quad {\scriptstyle (i=1,2)}. \tag{1}$$

Let $\Delta'(p)$ be another neighborhood of p such that $\bar{\Delta}'(p) \subset \Delta(p)$. Then there exists a finite number of points $p_\nu \in \bar{D}$ ${\scriptstyle (\nu=1,\dots,n)}$ such that $\bar{D} \subset \bigvee_{\nu=1}^{n} \Delta'(p_\nu)$. We shall construct α-mappings f_ν^* of D into $\mathfrak{M}'$ such that

$$f_0^* = f_1, \quad f_n^* = f_2, \quad f_\nu^*(x) = f_{\nu-1}^*(x) \text{ for } x \in \bar{D} - \Delta(p_\nu)$$

and $\quad f(\Delta(p_\mu)) \bigvee f_\nu^*(\Delta(p_\mu)) \subset V_{j(\mu)} \in \{V_j\}$ for all ν ${\scriptstyle (\mu, \nu=1,\dots,n)}$.

For this we define f_ν^* step by step as follows:

We put $\quad f\cdot_\nu(x) = f_{\nu-1}^*(x) \quad$ for $x \in \bar{D} - \Delta(\rho_\nu)$,

and $f\cdot_\nu(x) = \psi^{-1}([\rho(x)+\rho'(x)]^{-1}[\rho'(x)\psi f_{\nu-1}^*(x) + \rho(x)\psi f_2(x)])$ for $x \in \Delta(\rho_\nu)$,

where $\rho(x) = \operatorname{dist}(x, \bar{D} - \Delta(\rho_\nu))$, $\rho'(x) = \operatorname{dist}(x, \bar{\Delta}'(\rho_\nu))$ and ψ is the local coordinate of $V_{j(\mu)}$ $(\psi(V_{j(\mu)}) = K)$.

Then $$f\cdot_\nu(\bar{\Delta}(p_\mu)) \bigvee f(\Delta(p_\mu)) \subset V_{j(\mu)}$$

and $$f\cdot_\nu(x) = f_2(x) \quad \text{for } x \in \bar{\Delta}'(p_\nu) \bigvee \{x \mid f_{\nu-1}^*(x) = f_2(x)\}.$$

---

6) Cf. Lemma 3. 1.

Because, from (1) $f_i(\Delta(p_\mu))$ $_{(i=1,2)}$ and $f(\Delta(p_\mu))$ belong to a common V_j, and then by induction we get that $f_\nu^*(\Delta(p_\mu))$ and $f(\Delta(p_\mu))$ belong to the same V_j. We put $\Delta_\nu^*=\{x\,|\,f_{\nu-1}^*(x)\neq f'_\nu(x)\neq f_2(x)\}$. Then Δ_ν^* is an open subset of Δ_ν. By Theorem 2.1 there exists an α-mapping $f_{(\nu)}^*$ of Δ_ν^* into $\mathfrak{M}'$ such that

$$f_{(\nu)}^*(x)=f'_\nu(x) \text{ for } x\in\overline{\Delta}_\nu^*-\Delta_\nu^* \quad\text{and}\quad f_{(\nu)}^*(\overline{\Delta}_\nu^*)\subset V_{j(\nu)}.$$

Now we put

$$f_\nu^*(x)=f_{(\nu)}^*(x) \text{ for } x\in\Delta_\nu^*, \qquad f_\nu^*(x)=f'_\nu(x) \text{ for } \overline{D}-\Delta_\nu^*,$$

Then $f_\nu^*(x)=f_{\nu-1}^*(x)$ for $x\in D-\overline{\Delta}(\rho_\nu)$, $f_\nu^*(x)=f_2(x)$ for $x\in\bigvee_{\mu=1}^{\nu}\Delta'(\rho_\mu)$,

hence f_ν^* are desired mappings.

For any $p\in\overline{D}$ there exists a $\Delta(p_\mu)$ such that $p\in\Delta(p_\mu)$, hence $f_\nu^*(p)$ and $f(p)$ belong to the same $V_{j(\mu)}$. Thus we get $a\notin f_\nu^*(\overline{D}-D)$, since $\operatorname{diam}(V_j)<\operatorname{dist}(a,f(\overline{D}-D))$. Therefore

$$\mathrm{A}[a,D,f_\nu^*]=\mathrm{A}[a,D,f_{\nu-1}^*], \tag{2}$$

if $\overline{\Delta}(p_\nu)\subset D$ by Lemma 3.2. But if not $\overline{\Delta}(p_\nu)\subset D$, then

$$V_{j(\nu)}\frown f(\overline{D}-D)\neq\{\},\qquad \text{hence } a\notin V_{j(\nu)},$$

therefore $\mathrm{A}[a,\Delta(p_\nu),f_\mu^*]=0$, consequently (2) holds also. Since $f_0^*=f_1$ and $f_n^*=f_2$ we obtain (0) from (2).

§ 4. Fundamental Properties of the Degree of Mapping.

4. 1. Let f, D and λ have the same meanings as in Definition B.

Theorem 4. 1. *Theorem* 2. 3 (i), (ii), (iii) *and Theorem* 2. 4 (which will be denoted by (iv)) *remain valid also when f is a general mapping of $\overline{D}$ into $\mathfrak{M}'$.*

Proof. (i) is evident.

To prove (ii) we have to take an α-mapping f^* of D such that

$$\operatorname{dist}(f^*(x),f(x))<\operatorname{Min}\{\lambda,\operatorname{dist}(a,f(\overline{D}))\} \text{ for } x\in\overline{D}$$

and apply Theorem 2. 3 (ii).

To prove (iii) take an α-mapping f^* of D such that

$$\operatorname{dist}(f^*(x),f(x))<\operatorname{Min}\{\operatorname{dist}(a,f(\overline{D}_i-D_i))\,|\,1\leqq i\leqq k\}$$

and apply Theorem 2. 3 (iii).

To prove (iv) we have to choice an α-mapping f^* of D such that

$$\operatorname{dist}(f^*(x), f(x)) < \operatorname{dist}(C, f(\bar{D}-D)) \text{ for } x \in \bar{D},$$

where C is a curve joining a and a' on $\mathfrak{M}'$ not touching $f(\bar{D}-D)$ and apply Theorem 2.4.

Corollary 4.1. *If $\mathfrak{M}$ is closed (compact) and f is a mapping of $\mathfrak{M}$ into $\mathfrak{M}'$, then $\mathrm{A}[p, \mathfrak{M}, f]$ does not depend on $p(\in \mathfrak{M}')$.* (Then we write $\mathrm{A}[p, \mathfrak{M}, f] = \mathrm{A}[\mathfrak{M}', \mathfrak{M}, f]$).

Corollary 4.2. *Let $\mathfrak{M}$ be a closed orientable manifold, $\mathfrak{M}'$ a non-orientable manifold and f be a mapping of $\mathfrak{M}$ into $\mathfrak{M}'$,*

Then
$$\mathrm{A}[\mathfrak{M}', \mathfrak{M}, f] = 0. \tag{0}$$

Proof. On $\mathfrak{M}'$ there exists a simple closed curve C such that; starting from a definite point a of C one can take the local coordinates along C so that every two consecutive local coordinates have the same orientation except that the last has the opposite orientation as the first. Therefore $\mathrm{A}[a, \mathfrak{M}, f] = -\mathrm{A}[a, \mathfrak{M}, f]$, hence we get (0).

Theorem 4.2. *Let f be a mapping of $\bar{D}$ into $\mathfrak{M}'$ and $a \in \mathfrak{M}'$ be a point such that $a \notin f(\bar{D}-D)$. Let λ be the Lebesgue's number of the covering of $f(\bar{D})$ by $\{V_j\}$ where $\{V_j\}$ is a covering system of $\mathfrak{M}'$ such that*

$$\operatorname{diam}(V_j) < \operatorname{dist}(a, f(\bar{D}-D)). \tag{1}$$

If f_1 is a mapping of $\bar{D}$ into $\mathfrak{M}'$ such that

$$\operatorname{dist}(f_1(x), f(x)) < \lambda, \tag{2}$$

then
$$\mathrm{A}[a, D, f_1] = \mathrm{A}[a, D, f].$$

Proof. From (1) and (2) we get $a \notin f_1(\bar{D}-D)$. Then by Lemma 3.1 there exists another covering system $\{V_j'\}$ of $\mathfrak{M}'$ such that

$$\operatorname{diam}(V_j') < \operatorname{dist}(a, f_1(\bar{D}-D)).$$

Let λ' be the Lebesgue's number of the covering of $f_1(\bar{D})$ by $\{V_j'\}$. By Theorem 2.1 there exists an α-mapping f^* of D into $\mathfrak{M}'$ such that

$$\operatorname{dist}(f^*(x), f_1(x)) < \operatorname{Min}[\lambda', \lambda - \operatorname{Max}\{\operatorname{dist}(f_1(x), f(x)) \mid x \in \bar{D}\}].$$

Then $\quad \operatorname{dist}(f^*(x), f(x)) < \lambda, \quad \operatorname{dist}(f^*(x), f_1(x)) < \lambda' \quad$ for $x \in \bar{D}$.

Hence by Definition B

$$\mathrm{A}[a, D, f] = \mathrm{A}[a, D, f^*] = \mathrm{A}[a, D, f_1].$$

Theorem 4.3. *Let f_t be a mapping of $\bar{D}$ into $\mathfrak{M}'$ such that $f_t(x)$ and $a(t)\,(\in \mathfrak{M}')$ are continuous for $0 \leq t \leq 1$, $x \in \bar{D}$ and $a(t) \notin f_t(\bar{D}-D)$ for*

$0 \leqq t \leqq 1$. *Then* $A[a(t), D, f_t]$ *is constant for* $0 \leqq t \leqq 1$.

Proof. Apply Theorem 4.1 (iv) and Theorem 4.2.

4.2. Now let us go to extend (v) in § 1 to the case of manifolds.

Lemma 4.1. *Any open set in* $\mathfrak{M}$ *consists of at most countable open components.*

Proof. For $\mathfrak{M}$ is separable and locally connected.

Lemma 4.2. *Let* D' *be an open set in* $\mathfrak{M}'$ *and* $f(D) \subset D'$, *then* $A[p, D, f]$ $(p \in D' - f(\overline{D} - D))$ *is constant in a component of* $\overline{D}' - f(D - D)$.

Let H be a component of $D' - f(\overline{D} - D)$, then we can write

$$A[p, D, f] = A[H, D, f] \quad \text{if } p \in H.$$

Proof. Cf. Theorem 4.1 (iv).

Theorem 4.4. *Let* $\mathfrak{M}, \mathfrak{M}'$ *and* $\mathfrak{M}''$ *be m-dimensional manifolds,* D *and* D' *be bounded open sets in* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *resp.,* f *be a mapping of* $\overline{D}$ *into* $\mathfrak{M}'$ *such that* $f(D) \subset D'$ *and* f' *that of* $\overline{D}$ *into* $\mathfrak{M}''$ *such that* $a \notin f'f(\overline{D} - D) \cup f'(\overline{D}' - D')$ *where* $a \in \mathfrak{M}''$. *Then*

$$A[a, D, f'f] = \sum_i A[a, H_i, f'] \cdot A[H_i, D, f], \tag{0}$$

where H_i *are the components of* $D' - f(\overline{D} - D)$.

For the proof of this theorem we use the following two lemmas.

Lemma 4.3. *Theorem* 4.4 *holds if* f *and* f' *are* α*-mappings and* D *and* D' *are so small that*

$$\overline{D} \subset U_k \in \{U_i\}, \quad \overline{D}' \subset V_l \in \{V_i\}, \quad f(\overline{D}') \subset W_h \in \{W_n\},$$

where $\{W_n\}$ *is a covering system of* $\mathfrak{M}''$.

Proof. Let φ, ψ and χ be the local coordinates of U_k, V_l and W_h resp. Then by Definition A, putting $\hat{f} = \psi f \varphi^{-1}, \hat{f}' = \chi f' \psi^{-1}$,

$$A[a, D, f'f] = A[\chi(a), \varphi(D), \hat{f}'\hat{f}],$$
$$A[a, H_i, f'] = A[\chi(a), \psi(H_i), \hat{f}'],$$
$$A[H_i, D, f] = A[\psi(H_i), \varphi(D), \hat{f}].$$

But by (v) in § 1 (for mappings in E^m) we get

$$A[\chi(a), \varphi(D), \hat{f}'\hat{f}] = \sum_i A[\chi(a), \psi(H_i), \hat{f}'] \cdot A[\psi(H_i), \varphi(D), \hat{f}].$$

Hence the theorem holds for this case.

Lemma 4.4. *Let* f *be an* α*-mapping of* D *into* $\mathfrak{M}'$ *such that* $a \notin f(\overline{D} - D)$ $(a \in \mathfrak{M}')$, *and* ε *be any positive number. Then there exists a neighborhood* $W(a)$ *of* a *such that* $f^{-1}(W(a))$ *consists of at most countable open components* G_ν *such that* $\operatorname{diam}(G_\nu) < \varepsilon$.

Proof. By Lemma 2.1 there are a finite number of disjoint open sets $'G_i \subset D$ such that $\operatorname{diam}('G_i) < \varepsilon$ and $\bigvee_i 'G_i \supset f^{-1}(a)\,(=X)$. Then

$$\operatorname{dist}(a, f(\bar{D}-\bigvee_i 'G_i))=\delta>0, \qquad \text{since } a \notin f(\bar{D}-\bigvee_i 'G_i).$$

Hence the neighborhood of a with radius δ has the above mentioned property.

Poof of Theorem 4.4. At first we assume that f and f' are α-mappings. $(f'f)^{-1}(a)$ and $f'^{-1}(a)$ are compact countable sets and $f'^{-1}(a) \wedge (f(\bar{D}-D) \vee (\bar{D}'-D'))=\{\}$. We take covering systems $\{V_j\}$ of $\mathfrak{M}'$ and $\{W_n\}$ of $\mathfrak{M}''$ in such a way that

$$\left.\begin{aligned} &\operatorname{diam}(V_j) < \operatorname{dist}[f'^{-1}(a), f(\bar{D}-D) \vee (\bar{D}'-D')], \\ &\operatorname{diam}(W_n) < \operatorname{dist}[a, f'f(\bar{D}-D) \vee f'(\bar{D}'-D')]. \end{aligned}\right\} \quad (1)$$

Let λ be the Lebesgue's number of the covering of $\bar{D}$ by $\{U_i\}$, λ' be that of $\bar{D}'$ by $\{V_j\}$ and λ'' that of $f'(\bar{D}')$ by $\{W_n\}$. Then by Lemma 4.4 there exists a neighborhood $W(a)$ of a such that $\operatorname{diam}(W(a)) < \lambda''$, $\operatorname{diam}(G_\mu') < \lambda'$ for any component G_μ' of $f'^{-1}(W(a))$ and $\operatorname{diam}(G_\nu) < \lambda$ for any component G_ν of $(f'f)^{-1}W(a)$. Then

$$G_\nu \subset U_{i(\nu)} \in \{U_i\}, \quad G_\mu' \subset V_{j(\mu)} \in \{V_j\}, \quad W(a) \subset W_0 \in \{W_n\}. \quad (2)$$

$f(G_\nu)$ (connected) is contained in a G_μ', namely $f(G_\nu) \subset G'_{\mu(\nu)}$. Then $f(\bar{G}_\nu - G_\nu) \subset \bar{G}'_{\mu(\nu)} - G'_{\mu(\nu)}$. (If it was not so, then there would be a $p \in \bar{G}_\nu - G_\nu$ such that $f(p) \in G'_{\mu(\nu)}$, hence $f'f(p) \in W(a)$ and $p \in G_\nu$, which is absurd). Thus $G'_{\mu(\nu)} - f(\bar{G}_\nu - G_\nu)$ has the only one component $G'_{\mu(\nu)}$. Hence by (2) and Lemma 4.3

$$\mathrm{A}[a, G_\nu, f'f] = \mathrm{A}[a, G'_{\mu(\nu)}, f'] \cdot \mathrm{A}[G'_{\mu(\nu)}, G_\nu, f].$$

Then by Definition A

$$\begin{aligned} \mathrm{A}[a, D, f'f] &= \textstyle\sum_\nu \mathrm{A}[a, G_\nu, f'f] \\ &= \textstyle\sum_\mu \sum_{(\nu)\mu} \mathrm{A}[a, G_\mu', f'] \cdot \mathrm{A}[G_\mu', G_\nu, f], \end{aligned} \quad (3)\,[7]$$

where $(\nu)\mu = \{\nu \mid \mu(\nu)=\mu\}$. Let $[\mu]$ be the set of μ such that $\mathrm{A}[a, G_\mu', f'] \neq 0$, then there is a point $b_\mu \in f'^{-1}(a) \wedge G_\mu'$ for $\mu \in [\mu]$.

Hence $$\mathrm{A}[G_\mu', G_\nu, f] = \mathrm{A}[b_\mu, G_\nu, f] \quad \text{for } \mu \in [\mu]. \quad (4)$$

Since $f^{-1}(b_\mu) \subset \bigvee_{(\nu)\mu} G_\nu \subset D$, we get by Theorem 1.1

$$\textstyle\sum_{(\nu)\mu} \mathrm{A}[b_\mu, G_\nu, f] = \mathrm{A}[b_\mu, D, f]. \quad (5)$$

By (2) we have $b_\mu \in f'^{-1}(a) \wedge V_{j(\mu)}$. Thus by (1) $b_\mu \in V_{j(\mu)} \subset H_{i(\mu)}$, where

7) There are only a finite number of μ such that $\mathrm{A}[a, G_\mu, f] \neq 0$ and a finite number of $\nu \in (\nu)\mu$ such that $\mathrm{A}[G_\mu', G_\nu, f] \neq 0$. Cf. also the footnote [3].

$H_{i(\mu)}$ is a component of $D'-f(\bar{D}-D)$. Hence

$$A[b_\mu, D, f]=A[H_{i(\mu)}, D, f] \tag{6}$$

Since $f'^{-1}(a) \wedge H_i \subset \vee_{(\mu)i} G_\mu' \subset H_i$ where $(\mu)i=\{\mu \mid i(\mu)=i\}$, then by Theorem 1.1

$$\sum_{(\mu)i} A[a, G_\mu', f']=A[a, H_i, f'] \tag{7}$$

Consequently by (3), (4), (5), (6) and (7)

$$\begin{aligned} A[a, D, f'f] &= \sum_\mu A[a, G_\mu', f'] \cdot A[H_{i(\mu)}, D, f] \\ &= \sum_i A[a, H_i, f'] \cdot A[H_i, D, f]. \end{aligned}$$

Now we have to consider the general mappings f and f'. By Theorme 2.1 there exists an α-mapping f'^* of D' into $\mathfrak{M}''$ such that

$$\text{dist}\,(f'^*(x), f'(x)) < \lambda' \quad \text{for } x \in \bar{D}',$$

where λ' is the Lebesgue's number of the covering of $f'(\bar{D}')$ by $\{W_n\}$ such that

$$\text{diam}\,(W_n) < \text{dist}\,(a, f'f(\bar{D}-D) \cup f'(\bar{D}'-D')).$$

There exists an α-mapping f^* of D into $\mathfrak{M}'$ such that

$$\text{dist}\,(f'^*f^*(x), f'f(x)) < \lambda' \quad \text{for } x \in \bar{D},$$

$$f^*(x)=f(x) \text{ for } x \in \bar{D}-D, \quad \text{dist}\,(f^*(x), f(x)) < \lambda \text{ for } x \in \bar{D},$$

where λ is the Lebesgue's number of the covering of $f(\bar{D})$ by $\{V_j\}$ such that

$$\text{diam}\,(V_j) < \text{dist}\,(f'^{-1}(a), f(\bar{D}-D)).$$

Hence we have by Definition B

$$A[a, D, f'f]=A[a, D, f'^*f^*], \tag{8}$$

and

$$A[a, H_i, f']=A[a, H_i, f'^*], \tag{9}$$

since $\bar{H}_i-H_i \subset f(\bar{D}-D) \cup (\bar{D}'-D')$. If $A[a, H_i, f'] \neq 0$, then there is a point $b_i \in H_i \wedge f'^{-1}(a)$. Thus by Definition B

$$A[H_i, D, f]=A[b_i, D, f]=A[b_i, D, f^*]=A[H_i, D, f^*], \tag{10}$$

since $f^*(\bar{D}-D)=f(\bar{D}-D)$.

Therefore $\quad A[a, D, f'f]=\sum_i A[a, H_i, f'] \cdot A[H_i, D, f]$

by (8), (9) and (10), because this hols already for f^* and f'^* instead of f and f' respectively.

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