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Random Ergodic Theorem with Finite Possible States

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The purpose of this note is to give a special model of random ergodic theorem.[1]

Let X be the infinite direct product measure space:

$$X = \mathbf{P}_{k=-\infty}^{\infty} \mathrm{H}_k,\ x \in X,\ x = (\dots x_{-1},\ x_0,\ x_1,\ x_2, \dots),\ x_k \in \mathrm{H}_k,$$

$$k = 0,\ \pm 1,\ \pm 2, \dots .$$

We assume that each component space H_k consists of p points, which are described by p figures; 1, 2, ... p, each having the same probability (measure) $1/p$. We denote the k-component x_k of a point x of X by $\eta_k(x)$. The measure on X is denoted by m. Let σ be the shift transformation of X:

$$\eta_k(\sigma x) = \eta_{k+1}(x), \quad k = 0,\ \pm 1,\ \pm 2, \dots .$$

It is well-known that σ is an ergodic transformation of strongly mixing type. Let Ω be another probability field (i. e. measure space). In this note we restrict ourselves to the case in which Ω consists of q points; $\Omega = (\omega_1, \omega_2, \dots, \omega_q)$, each having the same a priori probability $1/q$.

Suppose that it is given a family Φ of permutations $T_1, T_2, \dots, T_p$ of Ω. Starting from any point ω_1 of Ω, we take up at random a point from H_1, if it is x_1, we operate T_{x_1} to ω_1, then ω_1 is transferred to $T_{x_1}\omega_1$, at the second step we take up at random a point from H_2, if it is x_2, we operate T_{x_2} to $T_{x_1}\omega_1$, then we arrive at $T_{x_2}T_{x_1}\omega_1$, and so on.

Continuing this process, the transition probabilty that ω_1 is transferred to ω_2 after the elapse of n units of time is given by

$$m\ \{x \mid \omega_2 = T_{\eta_n(x)} T_{\eta_{n-1}(x)} \cdots T_{\eta_1(x)} \omega_1\}\ .$$

We can represent any permutation T of Ω in a matrix form of degree q; $T = (\tau_{ij})$, $1 \leq i,\ j \leq q$. The i-j element τ_{ij} of T is equal to 1 if $\omega_i = T\omega_j$, $\tau_{ij} = 0$ if $\omega_i \neq T\omega_j$.

1) S. M. ULAM and J. V. NEUMANN: 165. Random ergodic theorems. Bull. Amer. Math. Soc. Vol. 51, No. 9, 1945. p. 660.

Set

$$T_0 = 1/p\ (T_1 + T_2 + \dots + T_p).$$

T_0 is a Markoff matrix. It is easy to verify that the i-j element $\tau_{ij}^{(n)}$ of T_0^n is equal to $m\ \{x \mid T_{\eta_{n}(x)} \cdots T_{\eta_{1}(x)}\ \omega_j = \omega_i\}$, that is the transition probability that ω_j is transferred to ω_i after the elapse of n units of time. It is a well-known fact that if for some integer n, $\tau_{ij}^{(n)} > 0$ for all i, j, then $\lim_{n\to\infty} T_0^n = Q$ exists and all i-j elements of Q are equal to $1/q$. *In this case the family Φ is said to be strongly mixing.*

Let Ξ be the direct product measure space of X and Ω:

$$\Xi = X \times \Omega, \quad \xi \in \Xi, \quad \xi = (x, \omega), \quad x \in X, \quad \omega \in \Omega.$$

Let φ be the measure preserving transformation of Ξ defined by

$$\varphi(x, \omega) = (\sigma x,\ T_{\eta_{1}(x)}\ \omega).$$

THEOREM 1. *φ is strongly mixing if and only if Φ is strongly mixing.*

PROOF: Define the functions $f_i(\omega)$, $i = 1, 2, \dots q$, as follows.

$$f_i(\omega) = \begin{cases} 1 & \text{if} \quad \omega = \omega_i \\ 0 & \text{if} \quad \omega \neq \omega_i. \end{cases}$$

Set

$$F(x, \omega) = f_i(\omega), \ G(x, \omega) = f_j(\omega).$$

Then we have $F(\varphi^n(x, \omega)) = f_i(T_{\eta_{n}(x)} T_{\eta_{n-1}(x)} \cdots T_{\eta_{1}(x)}\ \omega)$.
Assume that φ is strongly mixing, then we have

$$\lim_{n\to\infty} \int F(\varphi^n \xi)\ G(\xi) d\xi = \int F(\xi) d\xi \int G(\xi) d\xi$$
$$= \int f_i(\omega) d\omega \int f_j(\omega) d\omega = 1/q \cdot 1/q = 1/q^2$$

The integral of the left hand side of the above equality is

$$\int F(\varphi^n \xi)\ G(\xi) d\xi = \int \left\{ \int f_i(T_{\eta_{n}(x)} T_{\eta_{n-1}(x)} \cdots T_{\eta_{1}(x)}\ \omega)\ dx \right\} f_j(\omega) d\omega^{2)}$$
$$= 1/q\ \ m\ \{x \mid \omega_i = T_{\eta_{n}(x)} T_{\eta_{n-1}(x)} \cdots T_{\eta_{1}(x)}\ \omega_j\} = 1/q\ \ \tau_{ij}^{(n)}.$$

Therefore we obtain the equality $\lim_{n\to\infty} 1/q\ \tau_{ij}^{(n)} = 1/q^2$, that is, $\lim_{n\to\infty} \tau_{ij}^{(n)} = 1/q$. This shows that Φ is strongly mixing.

Conversely assume that $\lim_{n\to\infty} \tau_{ij}^{(n)} = 1/q$ for all i, j.

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2) In Ω, each point has the positive measure $1/q$, Following the usual custom we should replace the integral notation $\int d\omega$ by the summation notation $\sum$. But, for the sake of simplicity, we use the integral notation.

Set

$$\xi_j(\eta)=\exp\ (2\pi ij\eta/p),\ j=0,\ 1,\ 2,\dots p-1,\ \eta\in \mathrm{H}_k.$$

Obviously $\{\zeta_j(\eta)\}$, $j=0,\ 1,\ 2,\dots p-1$, are the complete orthonormal system of $L^2(\mathrm{H}_k)$ for any k.

Hence $\{\zeta_{k_1}(\eta_{i_1}(x))\,\zeta_{k_2}(\eta_{i_2}(x))\dots\zeta_{k_s}(\eta_{i_s}(x))\}$, $0\leqq s<\infty$, $-\infty<i_1,\dots i_s<\infty$, $0\leqq k_1,\dots k_s\leqq p-1$, are the complete orthonormal system of $L^2(X)$. We denote this system by Ψ.

In order to prove that φ is strongly mixing, it is sufficient to show that

$$(1)\qquad \lim_{n\to\infty}\iint g(\sigma^n x)\ f_i(T_{\eta_n(x)}\cdots T_{\eta_1(x)}\omega)\ h(x)\ f_j(\omega)dxd\omega = 1/q^2\int g(x)dx\int h(x)dx$$

holds for any $g(x)$, $h(x)\in\Psi$.

If $g(x)\equiv 1$, and $h(x)\equiv 1$, then the integral of the left hand side of (1) is equal to $1/q\ \tau_{ij}^{(n)}$, which tends to $1/q^2$ as $n\to\infty$, therefore the equality (1) holds. In general if

$$g(x)=\zeta_{k_1}(\eta_{i_2}(x))\dots\zeta_{k_s}(\eta_{i_s}(x))$$
$$h(x)=\zeta_{l_1}(\eta_{j_1}(x))\dots\zeta_{l_r}(\eta_{j_r}(x)),$$

then the integral of the left hand side of (1) is

$$(2)\qquad 1/q\int\zeta_{k_1}(\eta_{i_1+n}(x))\dots\zeta_{k_s}(\eta_{i_s+n}(x))\ f_i(T_{\eta_n(x)}\cdots T_{\eta_1(x)}\omega_j)\cdot\zeta_{l_1}(\eta_{j_1}(x))\quad\dots\zeta_{l_r}(\eta_{j_r}(x))\ dx\,.$$

Suppose $i_1>i_2>\dots>i_s$ and $j_1>j_2>\dots>j_r$. There is no loss of generality in assuming that $i_s<0$, $j_1>0$, $j_r<0$. We may consider n to be sufficiently large that $i_s+n>j_1>0$.

Set

$$E^{j_1,j_1-1,\ \dots j_r+1,j_r}_{\eta_{j_1},\eta_{j_1-1},\dots\eta_{j_r+1},\eta_{j_r}}=\{x\mid\eta_{j_1}(x)=\eta_{j_1},\ \eta_{j_1-1}(x)=\eta_{j_1-1},\ \dots\eta_{j_r+1}(x)=\eta_{j_r+1},\ \eta_{j_r}(x)=\eta_{j_r}\}.$$

Then the sets $E^{j_1,j_1-1,\ \dots,j_r+1,j_r}_{\eta_{j_1},\eta_{j_1-1},\dots,\eta_{j_r+1},\eta_{j_r}}$ and $E^{i_1+n,i_1-1+n,\dots,i_s+n}_{\eta_{i_1},\eta_{i_1-1},\ \dots,\eta_{i_s}}$ are mutually stochastically independent for any

$$1\leqq\eta_{j_1},\dots,\eta_{j_r},\ \eta_{i_1},\dots,\eta_{i_s}\leqq p\,.$$

Therefore we have

$$m\left(E^{i_1+n,i_1-1+n,\dots,i_s+n}_{\eta_{i_1},\eta_{i_1-1},\ \dots,\eta_{i_s}}\cap E^{j_1,j_1-1,\ \dots,j_r}_{\eta_{j_1},\eta_{j_1-1},\dots,\eta_{j_r}}\right)$$
$$=m\left(E^{i_1+n,i_1-1+n,\dots,i_s+n}_{\eta_{i_1},\eta_{i_1-1},\ \dots,\eta_{i_s}}\right)\ m\left(E^{j_1,j_1-1,\ \dots,j_r}_{\eta_{j_1},\eta_{j_1-1},\dots,\eta_{j_r}}\right)$$

The value of the integral (2) on the set

$$E_{\eta_{i_1},\eta_{i_1-1},\ \ \ \dots,\eta_{i_s}}^{i_1+n,i_1-1+n,\dots,i_s+n} \cap E_{\eta_{j_1},\eta_{j_1-1},\dots,\eta_{j_r}}^{j_1,j_1-1,\ \ \ \dots,j_r}$$

is

$$(3)\qquad 1/q\ m\left(E_{\eta_{i_1},\eta_{i_1-1},\ \ \ \dots,\eta_{i_s}}^{i_1+n,i_1-1+n,\dots,i_s+n}\right)\ m\left(E_{\eta_{j_1},\eta_{j_1-1},\dots,\eta_{j_r}}^{j_1,j_2-1,\ \ \ \dots,j_r}\right)$$
$$\cdot\zeta_{k_1}(\eta_{i_1})\cdots\zeta_{k_s}(\eta_{i_s})\zeta_{l_1}(\eta_{j_1})\cdots\zeta_{l_r}(\eta_{j_r})$$
$$\cdot\int f_i\left(T_{\eta_0}T_{\eta_{-1}}\cdots T_{\eta_{i_s}}T_{\eta_{i_s-1+n}(x)}\cdots T_{\eta_{j_1+1}(x)}T_{\eta_{j_1}}\cdots T_{\eta_1}\omega_j\right)dx$$

The value of the integral in (3) indicates the transition probability that the point $T_{\eta_{j_1}}\dots T_{\eta_{j_r}}\omega_j$ is transferred to the point $T_{\eta_{i_s}}^{-1}\dots T_{\eta_0}^{-1}\omega_i$ after the elapse of $n-1+i_s-j_1$ units of time, which tends to $1/q$ as $n\to\infty$ by our assumption. Therefore the left hand side of (1) exists and is equal to

$$1/q^2\sum_i\sum_j m\left(E_{\eta_{i_1},\ \dots,\eta_{i_s}}^{i_1+n,\dots,i_s+n}\right)\ m\left(E_{\eta_{j_1},\dots,\eta_{j_r}}^{j_1,\ \dots,j_r}\right)$$
$$\cdot\zeta_{k_1}(\eta_{i_1})\cdots\zeta_{k_s}(\eta_{j_s})\ \zeta_{l_1}(\eta_{j_1})\cdots\zeta_{l_r}(\eta_{j_r})$$
$$=1/q^2\int g(\sigma^n x)dx\int h(x)dx=1/q^2\int g(x)dx\int h(x)dx\,.$$

This is the required result.

THEOREM 2. *φ is ergodic if and only if Ω contains no Φ-invariant subset except Ω and the empty set.*

PROOF: If Ω contains a non-trivial Φ-invariant subset A, then $X\times A$ is a non-trivial φ-invariant subset of Ξ, therefore φ is not ergodic.

Conversely assume that $F(x,\omega)$ is a φ-invariant function, which is not a constant:

$$(4)\qquad F(x,\omega)\equiv F(\sigma x,\ T_{\eta_1(x)}\,\omega).$$

In order to conclude the existence of a non-trivial Φ-invariant subset of Ω, it is sufficient to show that $F(x,\omega)$ is a function depending only on the variable ω. If $F(x,\omega)$ depends only on the variable x, then we may conclude from (4) immediately that $F(x,\omega)$ is a constant. Let δ be the least positive value of

$$\int|F(x,\omega_i)-F(x,\omega_j)|^2\ dx,\quad 1\leq i,\ j\leq q\,.$$

Let h be the order of the permutation group $[\Phi]$ of Ω generated by Φ, h is at most $q!$. Let ε be a positive number such that

$$(5)\qquad 6h(1+9ph)\,\varepsilon<\delta\,.$$

By the definition of $L^2(X)$, it is easy to conclude the existence of a function $G(x, \omega)$ and a positive number n such that

(6) $\int |F(x, \omega) - G(x, \omega)|^2 \, dx < \varepsilon$ for all $\omega \in \Omega$,

(7) $G(x, \omega)$ does not depend on the value of $\eta_k(x)$ for $|k| > n$,

(8) $\begin{cases} \int |F(x, \omega) - F(Vx, \omega)|^2 \, dx < \varepsilon & \text{for all } \omega \in \Omega, \\ \int |F(\sigma^{2n+1} x, \omega) - F(\sigma^{2n+1} Vx, \omega)|^2 \, dx < \varepsilon & \text{for all } \omega \in \Omega, \end{cases}$

where V is any measure preserving transformation of X satisfying the equalities $\eta_k(x) = \eta_k(Vx)$ for all $|k| \leq n$.
Let S_1 and S_2 be elements of $[\Phi]$.

Set

$$A(S_1) = \{x \mid T\eta_n(x) T\eta_{n-1}(x) \cdots T\eta_1(x) = S_1\}$$

$$A'(S_2) = \{x \mid T\eta_{2n+1}(x) T\eta_{2n}(x) \cdots T\eta_{n+2}(x) = S_2\} = \sigma^{n+1}(A(S_2))$$

$$A(S_1, S_2) = A(S_1) \wedge A'(S_2), \quad B_\eta = \{x \mid \eta_{n+1}(x) = \eta\}.$$

Let η' and η'' be any mutually different integers between 1 and p. Let V be a measure preserving transformation of X such that

$$V\{x \mid \eta_{n+1}(x) = \eta'\} = \{x \mid \eta_{n+1}(x) = \eta''\},$$

$$V\{x \mid \eta_{n+1}(x) = \eta''\} = \{x \mid \eta_{n+1}(x) = \eta'\},$$

and $\quad \eta_k(x) = \eta_k(Vx)$ for all $k \neq n+1$.

$Vx \in A(S_1, S_2)$ if and only if $x \in A(S_1, S_2)$.

From (4) we obtain

(9) $F(x, \omega) \equiv F(\sigma^{2n+1} x, \ T\eta_{2n+1}(x) \cdots T\eta_{n+2}(x) T\eta_{n+1}(x) T\eta_n(x) \cdots T\eta_1(x) \, \omega)$.

If $x \in A(S_1, S_2)$, then from (9) we have

(10) $F(x, \omega) = F(\sigma^{2n+1} x, \ S_2 T\eta_{n+1}(x) S_1 \, \omega)$

If $x \in A(S_1, S_2) \wedge B_{\eta'}$, then from (10) we have

(11) $\begin{cases} F(x, \omega) = F(\sigma^{2n+1} x, \ S_2 T\eta' S_1 \, \omega) \\ F(Vx, \omega) = F(\sigma^{2n+1} Vx, \ S_2 T_{\eta''} S_1 \, \omega). \end{cases}$

From (11) and (8) we have

(12) $\int_{A(S_1, S_2) \cap B_{\eta'}} |F(\sigma^{2n+1} x, \ S_2 T\eta' S_1 \, \omega) - F(\sigma^{2n+1} x, \ S_2 T_{\eta''} S_1 \omega)|^2 \, dx$

$$< 2 \int_{A(S_1, S_2) \cap B_{\eta'}} |F(\sigma^{2n+1} x, \ S_2 T\eta' S_1 \, \omega) - F(\sigma^{2n+1} Vx, \ S_2 T_{\eta''} S_1 \, \omega)|^2 \, dx$$

$$+ 2 \int_X | F(\sigma^{2n+1} Vx,\ S_2 T_{\eta''} S_1\, \omega) - F(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx$$

$$< 2 \int_{A(S_1, S_2) \cap B_{\eta'}} | F(x, \omega) - F(Vx, \omega) |^2 dx + 2\varepsilon < 4\varepsilon .$$

From (6) and (12) we have

$$(13) \quad \int_{A(S_1, S_2) \cap B_{\eta'}} | G(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - G(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx$$

$$< 3 \int_X | G(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - F(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) |^2\, dx$$

$$+ 3 \int_{A(S_1, S_2) \cap B_{\eta'}} | F(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - F(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx$$

$$+ 3 \int_X | F(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) - G(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx$$

$$< 3(\varepsilon + 4\varepsilon + \varepsilon) = 18\varepsilon .$$

The set $B_{\eta'}$ is stochastically independent of the set $A(S_1, S_2)$ and of the functions appearing in the left hand side of (13), therefore the left hand side of (13) ie equal to

$$(14) \quad m(B_{\eta'}) \int_{A(S_1, S_2)} | G(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - G(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx$$

$$= 1/p \int_{A(S_1) \cap A'(S_2)} | G(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - G(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx .$$

The set $A(S_1)$ is stochastically independent of the set $A'(S_2)$ and of the functions in (14), therefore (14) is equal to

$$(15) \quad 1/p\ m(A(S_1)) \int_{A'(S_2)} | G(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - G(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx .$$

Let S_1 be an element of $[\Phi]$ such that $m(A(S_1)) \geq 1/h$, then we have from the inequality (13) and (15),

$$(16) \quad \int_{A'(S_2)} | G(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - G(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx < 18ph\ \varepsilon .$$

From (6) and (16) we have

$$(17) \quad \int_{A'(S_2)} | F(\sigma^{2n+1} x,\ S_2 T_{\eta'} S_1\, \omega) - F(\sigma^{2n+1} x,\ S_2 T_{\eta''} S_1\, \omega) |^2\, dx$$

$$< 3(\varepsilon + 18ph\ \varepsilon + \varepsilon) = 6(1 + 9ph)\ \varepsilon .$$

Summing up (17) over all $S_2 \in [\Phi]$, we obtain

$$(18) \qquad \int | F(\sigma^{2n+1} x,\ T\eta_{2n+1}(x) \cdots T\eta_{n+2}(x) T\eta' S_1 \omega) - F(\sigma^{2n+1} x,\ T\eta_{2n+1}(x) \cdots T\eta_{n+2}(x) T\eta'' S_1 \omega) |^2 \, dx < 6h(1+9ph)\, \varepsilon .$$

Since $F(\sigma^{2n+1} x,\ T\eta_{2n+2}(x) \cdots T\eta_{n+2}(x)\, \omega)$

$$= F(\sigma^n (\sigma^{n+1} x),\ T\eta_n(\sigma^{n+1}x) \cdots T\eta_1(\sigma^{n+1}x)\, \omega),$$

by replacing the variable $\sigma^{n+1} x$ in the right hand side of (18) by x, and by making use of (5), we obtain

$$(19) \qquad \int | F(\sigma^n x,\ T\eta_n(x) T\eta_{n-1}(x) \cdots T\eta_1(x) T\eta' S_1 \omega) - F(\sigma^n x,\ T\eta_n(x) T\eta_{n-1}(x) \cdots T\eta_1(x) T\eta'' S_1 \omega) |^2 \, dx < \delta .$$

Since $F(x, \omega)$ is a φ-invariant function,

$$F(\sigma^n x,\ T\eta_n(x) T\eta_{n-1}(x) \cdots T\eta_1(x)\, \omega) \equiv F(x, \omega).$$

Therefore

$$\int | F(x, T\eta' S_1 \omega) - F(x, T\eta'' S_1 \omega) |^2 \, dx < \delta .$$

By the definition of δ, we have

$$F(x, T\eta' S_1 \omega) \equiv F(x, T\eta'' S_1 \omega) .$$

Since η' and η'' are arbitrary, we obtain

$$F(x, T_1 \omega) \equiv F(x, T_2 \omega) \equiv \ldots \equiv F(x, T_p \omega) .$$

Therefore

$$F(\sigma x,\ T_1 \omega) \equiv F(\sigma x,\ T_2 \omega) \equiv \ldots \equiv F(\sigma x,\ T_p \omega) \equiv F(x, \omega)$$

Let r be the order of T_1, then we have

$$F(\sigma^r x,\ T^r{}_1 \omega) \equiv F(\sigma^r x, \omega) \equiv F(x, \omega) .$$

From the ergodicity of σ^r, we can conclude that $F(x, \omega)$ depende only on the variable ω. This completes the proof of the theorem.

The extension of our results to the general case in which each component space H_k is the continuum of [0, 1]-interval with the usual Lebesgue measure and Φ is a family of measure preserving transformations of an arbitrary measure space Ω was made by S. KAKUTANI.

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