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On Modified Bent-Functions and Phragmén-Lindelöf's Principle

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§ 1. Phragmén-Lindelöf's principle is usually stated as follows:

Let $f(z)$ be a function, regular in the right half-plane and bounded on the imaginary axis, e.g. we assume

$$\limsup_{z \to iy} |f(z)| \leq 1, \quad z = x + iy, \tag{1.1}$$

and we shall denote by $M(r)$ the least upper bound of the absolute values $|f(z)|$ on the semi-circle $|z| = r$ and $|\theta| < \frac{\pi}{2}$, i.e. $M(r) = \underset{|\theta| < \pi/2}{\text{l.u.b.}} |f(re^{i\theta})|$, and put for the sake of simplicity

$$\alpha = \liminf_{r \to +\infty} \frac{\log M(r)}{r}, \qquad \beta = \limsup_{r \to +\infty} \frac{\log M(r)}{r}.$$

Then, there may happen two cases: Either the absolute value $|f(z)|$ increases to infinity so that α is positive, or the function $f(z)$ is bounded so that we have $|f(z)| \leq 1$ at every point of the half-plane. Especially, if $\alpha = -\infty$, then $f(z)$ is identically zero [2].

Since E. Phragmén and E. Lindelöf established this famous principle [1], many authors have studied on this subject. E. and R. Nevanlinna introduced the bentfunction $m(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |f(re^{i\theta})| \cos \theta d\theta$ [2], [3] and proved the monotonousness of $m(r)/r$. A. Dinghas has obtained this result by using the Poisson representation [6]. L. Ahlfors discussed the same property from a standpoint of a certain differential inequality and proposed a question if we have $\alpha = \beta$ so that the limit of $\frac{\log M(r)}{r}$ for $r \to +\infty$ exists [5]. M. Heins has answered this question, showing that for $0 \leq \alpha < +\infty$ we have $\alpha = \beta$ and the case $-\infty < \alpha < 0$ does not occur [8].

In this Note, we first introduce *Modified Bent-Functions* as follows:

$$\mu(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta \tag{1.2}$$

and we shall prove the montonousness of $\mu(r)/r$, so that we can put

$$(1.3)\qquad \eta=\lim_{r\to+\infty}\frac{\mu(r)}{r},\qquad \eta^{+}=\lim_{r\to+\infty}\frac{m(r)}{r}.$$

Next, in § 4, we shall establish the following *Fundamental Inequality*:

$$(1.4)\qquad \log|f(\rho e^{i\varphi})|\leq\frac{2}{\pi}\eta\cdot\rho\cos\varphi,\qquad |\varphi|<\frac{\pi}{2}$$

from which we can deduce the relations between η and η^{+}, as well as Heins' results containing the Phragmén-Lindelöf principle (Theorem 5):

i) if $\eta=+\infty$, then $\alpha=\beta=\eta=+\infty$.

ii) if $0<\eta<+\infty$, then $\alpha=\beta=\frac{2}{\pi}\eta$.

iii) if $-\infty<\eta\leq 0$, then $\alpha=\beta=0$.

iv) if $\eta=-\infty$, then $\alpha=\beta=\eta=-\infty$.

Simple examples show that the quantity η may be really negative.

§ 2. We use the following inequality as the starting point of our study, which can be derived from the theory of harmonic majoration:

$$(2.1)\qquad \log|f(\rho e^{i\varphi})|\leq\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\log|f(re^{i\theta})|\left\{\frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\varphi-\theta)}\right.$$
$$\left.-\frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}\right\}d\theta.$$[1)]

The Poisson's kernels $\frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\theta-\varphi)}$ and $\frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}$ are expressible as power series of ρ/r which converge uniformly for $\rho\leq r_1<r$. That is,

$$(2.2)\qquad \frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\theta-\varphi)}=1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{r}\right)^n\cos n(\theta-\varphi)$$

and

$$(2.3)\qquad \frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}=1+2\sum_{n=1}^{\infty}\left(\frac{\rho}{r}\right)^n\cos n(\theta+\varphi-\pi).$$

Hence we have

$$\frac{r^2-\rho^2}{r^2+\rho^2-2\rho r\cos(\theta-\varphi)}-\frac{r^2-\rho^2}{r^2+\rho^2+2\rho r\cos(\theta+\varphi)}.$$

1) The detailed proof of this inequality is found in [2], p. 5-18.

$$= 4 \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2}-\varphi\right) \sin n\left(\frac{\pi}{2}-\theta\right),$$

which converges uniformly for $\rho \leq r_1 < r$. Consequently, we can interchange the signs $\sum$ and $\int$ in (2.1):

$$\text{(2.4)} \qquad \log |f(\rho e^{i\varphi})| \leq \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2}-\varphi\right)$$

$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2}-\theta\right) d\theta .$$

By multiplying both sides of the inequality by $\cos\varphi$ and integrating them with respect to φ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we obtain for $\rho < r$,

$$\text{(2.5)} \qquad \frac{1}{\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(\rho e^{i\varphi})| \cos\varphi d\varphi \leq \frac{1}{r} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos\theta d\theta ,$$

Thus we have

THEOREM 1. *$\mu(r)/r$ is a non-decreasing function of r.*

The monotonousness of $m(r)/r$ was given by Nevanlinna, Ahlfors and Dinghas.

From (2.5), we have for $\rho < r$,

$$\text{(2.6)} \qquad \frac{\mu(\rho)}{\rho} \leq \frac{m(\rho)}{\rho} \leq 2 \frac{\log M(r)}{r} .$$

Let r tend to infinity, then $\frac{\mu(\rho)}{\rho} \leq \frac{m(\rho)}{\rho} \leq 2\alpha$.

Now, as $\mu(\rho)/\rho$ increases monotonously, $\lim_{\rho\to+\infty} \mu(\rho)/\rho$ must exist. Hence we have the following theorem.

THEOREM 2. *For $\alpha \neq +\infty$, η exists and we have furthermore*

$$\text{(2.7)} \qquad \eta \leq \eta^+ \leq 2\alpha .$$

§ 3. Now let $\text{l. u. b.}_{|\varphi|<\pi/2} |f(\rho e^{i\varphi})|$ be attained at a point $z = \rho e^{i\varphi_1(\rho)}$, $\rho < r$, $|\varphi_1(\rho)| < \frac{\pi}{2}$, that is

$$M(\rho) = |f(\rho e^{i\varphi_1(\rho)})| = \text{l. u. b.}_{|\varphi|<\pi/2} |f(\rho e^{i\varphi})|, ^{1)}$$

then from (2.4), we have

1) In case $|\varphi_1(\rho)| = \pi/2$, by the hypothesis (1.1), $|f(z)| \leqq 1$ for $|z| = \rho$. And then by the maximal principle, we can proceed our discussion as in § 5. Hence we obtain $-\infty < \eta \leqq 0$ and $\alpha = \beta = 0$.

$$\text{(3.1)} \qquad \log M(\rho) \leq \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\rho}{r}\right)^n \sin n\left(\frac{\pi}{2} - \varphi_1(\rho)\right)$$

$$\times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \, n\left(\frac{\pi}{2} - \theta\right) d\theta .$$

By using the inequality $n \sin \theta \geq |\sin n\theta|$, $0 \leq \theta \leq \pi$, we can estimate $\frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta$ as follows:

$$\left| \frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta \right|$$

$$\leq \frac{1}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \sin n\left(\frac{\pi}{2} - \theta\right) d\theta$$

$$\leq \frac{n}{r^n} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta$$

$$= \frac{n}{r^n} \left\{ 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log^+ |f(re^{i\theta})| \cos \theta d\theta - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |f(re^{i\theta})| \cos \theta d\theta \right\} .$$

If α is finite, we can see from (2.7) that $m(r)/r$ and $\mu(r)/r$ are bounded for $r \to +\infty$. Thus, if we let r tend to $+\infty$, ρ fixed, all the terms but the first in the right-hand side of (3.1) vanish. Consequently, we have, if α is finite,

$$\frac{\log M(\rho)}{\rho} \leq \frac{2}{\pi} \eta \cos \varphi_1(\rho) .$$

If $\eta \geq 0$, then $\log M(\rho)/\rho \leq \frac{2}{\pi} \eta$. Finally, be making ρ tend to infinity, we have

$$\text{(3.2)} \qquad \frac{\pi}{2} \limsup_{\rho \to +\infty} \frac{\log M(\rho)}{\rho} \leq \eta .$$

From the inequalities (2.7) and (3.2), we have

THEOREM 3. *If η is not negative and α is finite, then*

$$\frac{\pi}{2} \beta \leq \eta \leq 2\alpha .$$

COROLLARY. *Under the same conditions as* THEOREM 3,

$$0 \leq \beta \leq \frac{4}{\pi} \alpha .$$

REMARK. If η is $+\infty$, then α and β are also $+\infty$, and conversely, if α is $+\infty$, then η is also $+\infty$.

§ 4. For a fixed ρ, let r tend to infinity in the inequality (2. 4), then

$$\text{(1. 4)} \qquad \log |f(\rho e^{i\varphi})| \leq \frac{2}{\pi} \eta \rho \cos \varphi .$$

If η is not negative, then we have

$$\text{(4. 1)} \qquad \log |f(\rho e^{i\varphi})| \leq \frac{4}{\pi} \alpha \rho \cos \varphi \leq \frac{4}{\pi} \alpha \rho ,$$

by virtue of (2. 6). Consequently, we obtain

THEOREM 4. *If η is not negative, then*

$$\frac{\log M(\rho)}{\rho} \leq \frac{4}{\pi} \alpha ,$$

for any positive ρ. Accordingly, if the equality holds identically in the above inequality, then $f(z)$ must have the form $Ce^{\frac{4}{\pi}\alpha z}$, where C is a constant having absolute valve 1.

From (1. 4), we have the following inequality for any positive ρ and φ

$$\text{(4. 2)} \qquad |f(\rho e^{i\varphi})| \leq e^{\frac{2}{\pi}\eta\rho \cos \varphi} , \qquad \left(|\varphi| < \frac{\pi}{2}\right).$$

Consequently, if η is not positive, then we have for any positive ρ and φ

$$\text{(4. 4)} \qquad |f(\rho e^{i\varphi})| \leq 1 , \qquad \left(|\varphi| < \frac{\pi}{2}\right).$$

These results prove the famous Phragmén-Lindelöf principle.

REMARK. From (4. 1), we can see that, when α is not positive, we have also $|f(z)| \leq 1$ identically.

§ 5. We are in a position to make the Theorem 3 more precise and to deduce Heins' results.

We shall express $f(z)$ as follows, that is

$$\text{(5. 1)} \qquad f(z) \equiv e^{\frac{2}{\pi}\eta z} g(z) .$$

By the relation (1. 4) and (1. 1), $g(z)$ is analytic and of modulus less than 1 in the half-plane $\Re z > 0$, and further $\limsup_{z \to iy} |g(z)| \leq 1$, that is $g(z)$ is a function of Phragmén-Lindelöf's type.

By virtue of (1. 4) and (5. 4), we have

$$|g(z)| \leq 1. \tag{5.2}$$

Now let $E_r(\theta)$ be the set of angles, θ, of the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, for which

$$\log |g(re^{i\theta})| \leq -\varepsilon r, \quad (\rho \text{ fixed}), \tag{5.3}$$

for an arbitrary given positive number ε.

$$\log |f(re^{i\theta})| = \frac{2}{\pi} \eta r \cos\theta + \log |g(re^{i\theta})|.$$

Multiplying the both side by $\cos\theta$ and integrating with θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, it follows that

$$\begin{aligned}
\frac{\mu(r)}{r} &= \eta + \frac{1}{r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log |g(re^{i\theta})| \cos\theta d\theta \\
&= \eta + \frac{1}{r}\int_{E_r(\theta)} \log |g(re^{i\theta})| \cos\theta d\theta \\
&\quad + \frac{1}{r}\int_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]-E_r(\theta)} \log |g(re^{i\theta})| \cos\theta d\theta \\
&\leq \eta + \frac{1}{r}\int_{E_r(\theta)} \log |g(re^{i\theta})| \cos\theta d\theta .
\end{aligned}$$

By (5. 3)

$$\frac{\mu(r)}{r} \leq \eta - \frac{\varepsilon r}{r}\int_{E_r(\theta)} d\sin\theta .$$

Therefore we have $\mu(r)/r + \varepsilon \operatorname{meas}_{\text{sine}} E_r(\theta) \leq \eta$, where we denote by $\operatorname{meas}_{\text{sine}} A$, the measure of A with respect to the mass distribution $d\sin\theta$:

$$\operatorname{meas}_{\text{sine}} A = \int_A d\sin\theta.$$

Let r tend to infinity, then we have

$$\lim_{r\to+\infty} \text{meas}_{\text{sine}}\, E_r(\theta) \leq 0 \text{ i. e. } \lim_{r\to+\infty} \text{meas}_{\text{sine}}\, E_r(\theta) = 0 .$$

Hence, given any positive number ε', there exist a positive number δ and a large R, for $r > R$ we have

$$\text{meas}\left(E_r(\theta) \cap \left[\frac{\pi}{2}-\delta,\ \frac{\pi}{2}\right]\right) < \frac{\varepsilon'}{4} \text{ and}$$

$$\text{meas}\left(E_r(\theta) \cap \left[-\frac{\pi}{2},\ -\frac{\pi}{2}+\delta\right]\right).[1]$$

On the other hand,

$$\text{meas}_{\text{sine}}\left(E_r(\theta) \cap \left[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta\right]\right) = \int_{E_r(\theta)\cap\left[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta\right]} d\sin\theta \geq \cos\left(\frac{\pi}{2}-\delta\right)$$

$$\times \int_{E_r(\theta)\cap\left[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta\right]} d\theta = \cos\left(\frac{\pi}{2}-\delta\right) \text{ meas } \left(E_r(\theta) \cap \left[-\frac{\pi}{2}+\delta,\ \frac{\pi}{2}-\delta\right]\right).$$

Hence we obtain

$$\lim_{r\to\infty} \text{meas } E_r(\theta) = 0 . \tag{5. 4}$$

Case A. $0 < \eta < +\infty$. In this case, for any sufficiently small positive number ε'', there exists an angle $\theta_0(r)$ such that $|\theta_0(r)| < \varepsilon''$ and $\theta_0(r) \overline{\in} E_r(\theta)$.[2] For this angle $\theta_0(r)$,

$$\log M(r) \geq \log |f(re^{i\theta_0(r)})| = \frac{2}{\pi}\, \eta r \cos\theta_0(r) + \log |g(re^{i\theta_0(r)})| .$$

From (5. 3) and the definition of $\theta_0(r)$,

$$\frac{\log M(r)}{r} > \frac{2}{\pi}\, \eta \cos \varepsilon'' - \varepsilon .$$

Let r tend to positive infinity, we have $\alpha \geq \frac{2}{\pi}\, \eta \cos\varepsilon'' - \varepsilon$. As ε and ε'' are arbitrary, we have $\alpha \geq \frac{2}{\pi}\, \eta$ which proves $\alpha = \beta = \frac{2}{\pi}\, \eta$.

Case B. $-\infty < \eta \leq 0$. In this case it is clear that α and β are not positive. For any sufficiently small positive number δ', there would exist an angle $\theta(r)$ such that $\frac{\pi}{2} > \theta(r) > \frac{\pi}{2} - \delta'$ and $\theta(r) \overline{\in} E_r(\theta)$.

For this angle $\theta(r)$,

1) This measure is used in the Lebesgue's sense.

2) $|\theta|$ means the magnitude of the angle θ.

$$\log M(r) \geq \frac{2}{\pi} \eta r \cos \theta(r) + \log |g(re^{i\theta(r)})|$$

$$> \frac{2}{\pi} \eta r \cos \theta(r) - \varepsilon r .$$

Hence

$$\frac{\log M(r)}{r} > \frac{2}{\pi} \eta \cos\left(\frac{\pi}{2} - \delta'\right) - \varepsilon .$$

Let r tend to infinitly, then we have, by the definitions of ε and δ', $\alpha \geq 0$. Consequenity, $\alpha = \beta = 0$.

Case C. $\eta = +\infty$. In this case it is clear by the Remark in § 4 that $\alpha = \beta = \eta = +\infty$.

Case D. $\eta = -\infty$. This case cannot occure except if $f(z) \equiv 0$. Consequently, $\alpha = \beta = \eta = -\infty$.

Hence we have

THEOREM 5. $\lim\limits_{r \to +\infty} \dfrac{\log M(r)}{r}$ *exists and further*

i) *if* $\eta = +\infty$, *then* $\alpha = \beta = \eta = +\infty$.

ii) *if* $0 < \eta < +\infty$, *then* $\alpha = \beta = \dfrac{2}{\pi} \eta$.

iii) *if* $-\infty < \eta \leq 0$, *then* $\alpha = \beta = 0$ *and* $|f(z)| \leq 1$ *for all* $\Re z > 0$.

iv) *if* $\eta = -\infty$, *then* $\alpha = \beta = \eta = -\infty$ *and in this case* $f(z)$ *must be identically zero.*

COROLLARY. *For a sufficiently large* $|z|$, *the functions* $f(z)$, *which are of the Phragmén-Lindelöf's type, are expressible as* $Ce^{\frac{2}{\pi}\eta z}$ *except for a set of almost measure zero, where* C *is a constant of modulus 1.*

§ 6. Let $u(z)$ be a harmonic or a subharmonic function in the half-plane $\Re z > 0$, and further suppose that $u(z)$ is not positive on the imaginary axis. Then we are able to replace $\log |f(z)|$ by $u(z)$ in the inequality (2. 1). Consequently, we obtain the following theorems.

THEOREM 7. $\dfrac{1}{r} \displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} u(re^{i\theta}) \cos \theta d\theta$ *is non-decreasing function of* r *and the limit exists for* $r \to +\infty$.

We denote this limit by K.

THEOREM 8. *Let* $M(r)$ *be maximum of* $u(re^{i\theta})$ *on the semi-circle*

$|z|=r$, $|\theta|<\frac{\pi}{2}$. *Then* $\lim_{r\to+\infty} M(r)/r$ *exists and if* K *is not negative, then*

$$\frac{\pi}{2}\lim_{r\to+\infty}\frac{M(r)}{r}=K.$$

THEOREM 9. *If* K *is finite, then*

$$\lim_{r\to+\infty}\frac{u(re^{i\theta})}{r}=K\cos\theta$$

for almost all θ.

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