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On the Continuous Function Defined on a Sphere

By Hidehiko YAMABE and Zuiman YUJOBO (Tokyo)

1. Prof. S. Kakutani proposed the following problems[1]: Given a bounded convex body in an $(n+1)$-dimensional Euclidean space R^{n+1}, is it always possible to find a circumscribing cube around it ? Or more generally if $f(x)$ is a real-valued continuous function on an n-dimensional sphere S^n with the centor o, then is it possible to find $(n+1)$ points $x_0, x_1, \dots, x_n$ on S^n perpendicular to one another (which means that the vectors $ox_0, ox_1, \dots, ox_n$ are perpendicular to one another), such that

$$f(x_0) = f(x_1) = \dots = f(x_n) \ ? \tag{1}$$

The purpose of this paper is to answer these questions in the affirmative.

2. Let R^{m+1} be an $(m+1)$-dimensional Euclidean space with the origin o and with the rectangular co-ordinate axes $oe_1, oe_2, \dots, oe_{m+1}$. The cartesian co-orninates of a point p will be denoted by $(p^1, p^2, \dots, p^{m+1})$ and the distance from o by $\| p \|$. Let us consider the concentric spheres whose common center coincides with o and whose radius runs over the interval $[1, 2]$. We denote the aggregate of these concentric spheres by C.

Now we shall prove the following

LEMMA A′. *Let S_0 and S_1 be the set $\{p ; \| p \| = 1\}$ and $\{p ; \| p\| = 2\}$ respectively. If L is a closed set in C which intersects any continuous curve joining S_0 and S_1, then L contains $(m+1)$ points $q_0, q_1, \dots, q_m$ such that*

$$\| q_0 \| = \| q_1 \| = \dots = \| q_m \| \tag{2}$$

and $oq_0, oq_1, \dots, oq_m$ are perpendicular to one another.

Proof. If $m = 0$, the lemma is evidently true. Let us assume that this lemma is true when the dimension of the space $< m+1$. We can easily find in an ε-neighbourhood[2] of L an closed m-dimension manifold L_ε[3] which also intersects any continuous curve joining

1) S. Kakutani: "Circumscribing cube around convex body". Annals of Math. vol. 43.

2) ε-neighbourhood is the set of points whose distances from L are $< \varepsilon$.

3) If such a L_ε does not exist we can draw a curve from S_0 to S_1 which does not intersect L.

S_0 and S_1. Clearly there are two points $p(1)$, $p(0)$ in L such that

$$\sup_{p \in L_\varepsilon} \| p \| = \| p(1) \|$$

$$\inf_{p \in L_\varepsilon} \| p \| = \| p(0) \| . \tag{3}$$

Now we join $p(1)$ and $p(0)$ by a curve $p(\tau)$ $(0 \leqq \tau \leqq 1)$ on L_ε. For every $p(\tau)$ we can easily determine a rotation of axes ρ_τ such that

$$\rho_\tau p(\tau) \in \overrightarrow{oe_{m+1}}\,,^{4)}$$

and such that ρ_τ is a continuous function of τ. Let π be the hyperplane $\{p\,;\,p^{m+1}=0\}$, and let $H^{m-1}(p)$ be the aggregate of q such that

$$\| q \| = \| p \| ,$$

and such that $o\,q$ is perpendicular to $o\,p$. Then $\rho_\tau H^{m-1}(p(\tau))$ are all in π. Let C' be the intersection of π and C, and let S_0' and S_1' be $S_0 \frown \pi$ and $S_1 \frown \pi$ respectively. To a point $y \in C'$ corresponds a point $x \in C$ in such a way that

$$x = \rho_\tau^{-1}\{\overrightarrow{o\,y} \frown \rho_\tau H^{m-1}(x(t))\}$$

$$\equiv \varphi(y) ,$$

where $\tau = \| y \| - 1$.

This mapping $\varphi(y)$ is continuous; moreover if $\| y \| = \| z \|$, then $\| \varphi(y) \| = \| \varphi(z) \|$; and if $o\,y$ is perpendicular to $o\,z$, then so is $o\,\varphi(y)$ to $o\,\varphi(z)$.

The closed set $\varphi^{-1}(L_\varepsilon)$ intersects any continuous curve γ joining S_0' and S_1' in C' because $\varphi(S_0')$ and (S_1') are joined by $\varphi(\gamma)$, the former of them being inside or on L_ε, the latter being outside or on L_ε. Therefore by the assumption $\varphi^{-1}(L_\varepsilon)$ contains m points r_1', r_2', ..., r_m' such that

$$\| r_1' \| = \| r_2' \| = \dots = \| r_m' \| = \tau_0 + 1 ,$$

and or_1', or_2', ..., or_m' are perpendicular to one another. Hence

[4] $\overrightarrow{oe_{m+1}}$ and $\overrightarrow{oy}$ denote half lines beginning at o.

$$\| p(\tau_0) \| = \| \varphi(r'_1) \| = \dots = \| \varphi(r'_m) \| ,$$

and $o\,p(\tau_0)$, $o\,\varphi(r'_1)$, ..., $o\,\varphi(r'_m)$ are also perpendicular to one another.

Let us put $p(\tau_0) = q_0[\varepsilon]$, $\varphi(r'_1) = q_1[\varepsilon]$, ..., $\varphi(r'_m) = q_m[\varepsilon]$. It is easy to take a sequence ε_n converging to zero so that $q_0[\varepsilon_n]$, $q_1[\varepsilon_n]$, ..., $q_m[\varepsilon_n]$ may converge to limit points q_0, q_1, ..., q_m respectively. Clearly every $q_i \in L$,

$$\| q_0 \| = \| q_1 \| = \dots = \| q_m \| ,$$

and $o\,q_i$'s are perpendicular to one another. Thus the lemma is proved.

3. A point P of a cylindrical space (I, S^m) which is a topological product of the interval $I = [0, 1]$ and the m-dimensional sphere S^m, is represented by a pair of co-ordinates (t, s) for $t \in I$ and $s \in S^m$, both $t = t(P)$ and $s = s(P)$ being continuous functions of P. Then we have another lemma which will be obtained without difficulty from the Lemma A′.

LEMMA A. *Let S_0^m and S_1^m be the set $\{P ; t(P) = 0\}$ and $S_1\{P ; t(P) = 1\}$ respectively. If L is a closed set on (I, S^m) which intersects any continuous curve that joints S_0 and S_1, then L contains $(m+1)$ points Q_0, Q_1, ..., Q_m such that*

$$t(Q_0) = t(Q_1) = \dots = t(Q_m) \qquad (2)$$

and such that $s(Q_i)$'s $(0 \leqq i \leqq m)$ are perpendicular to one another.

4. For a real-valued function $f(x)$ on S^n, there exist two points $x(1)$ and $x(0)$ with

$$\sup_{t \in S^n} f(x) = f(x(1))$$
$$\inf_{t \in S^n} f(x) = f(x(0)) . \qquad (4)$$

We join $x(0)$ and $x(1)$ by a curve $x(t)$ $(0 \leqq t \leqq 1)$ on S^n. We may consider the S^n as the unit sphere in an $(n+1)$-dimensional space R^{n+1} with the origin o. For a point p in this R^{n+1} the co-ordinates p^i's and $\| p \|$ are similarly defined as in § 2. π denotes the hyperplane $\{p ; p^{n+1} = 0\}$, and e_i $(0 \leqq i \leqq n)$ denotes the point whose i-th co-ordinate is equal to 1 and other co-ordinates are zero.

We take again the rotations of axes ρ_t such that

$$\rho_t\, x(t) = e_{n+1} ,$$

ρ_t being continuous. Then $\rho_t\, H^{n-1}(x(t))$ are all contained in π. Let

S^{n-1} be $S^n \wedge \pi$. Let us consider the topological product (I, S^{n-1}) of I and S^{n-1}, whose point P is represented by $t \in I$ and $u \in S^{n-1}$. We define S_0^{n-1} and S_1^{n-1} in a similar way as in § 4. Put

$$\rho_t^{-1} u(P) = \Psi(P),$$
$$F(P) = f(x(t)) - f(\Psi(P)), \tag{5}$$

and let the set of zero points of $F(P)$ be K. Then any curve which is drawn from S_0^{n-1} to S_1^{n-1} intersects K because $F(P) \leqq 0$ for $P \in S_0^{n-1}$ and $F(P) \geqq 0$ for $P \in S_1^{n-1}$; therefore K contains n points $P_1, P_2, \dots, P_n$ such that

$$t(P_1) = t(P_2) = \dots = t(P_n) = t_0,$$

and such that $u(P_i)$'s are perpendicular to one another. On the other hand

$$f(x(t_0)) = f(\Psi(P_1)) = \dots = f(\Psi(P_n)),$$

and $x(t_0)$, $\Psi(P_1)$, ..., $\Psi(P_n)$ are clearly perpendicular to one another. Thus we have the theorem:

THEOREM. *For a continuous function $f(x)$ on S^n, there exist $(n+1)$ points $x_0, x_1, \dots, x_n$ perpendicular to one another on S^n such that*

$$f(x_0) = f(x_1) = \dots = f(x_n).$$

From the above theorem we can obtain by the same argument as Kakutani[1]) the following

THEOREM. *For a bounded convex body in an $(n+1)$-dimensional Euclidean space there exists a circumscribing cube around it.*

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