Osaka Mathematical Journal,
Vol. 2, No. 1, March, 1950

On an Extension of the Helly's Theorem

By Hidehiko YAMABE

The theorem that we shall prove in this note is an extension of the Helly's theorem[1]), which makes it possible to improve the Weierstrass's approximation theorem. The theorem is as follows.

THEOREM. *Let M be a dense and convex set in a normed linear space E, not necessarily complete, and let $f_1, f_2, \dots, f_n$ be n linear functionals over E, Then for any element a in E there exists $b \in M$ in an ε-neighbourhood*[2]) *of a so that*

$$f_i(a) = f_i(b), \qquad (i = 1, 2, \dots, n).$$

Proof. We may assume that

$$\sum_{i=1}^{n} \kappa_i f_i(x) \neq \text{const}, \tag{1}$$

unless every κ_i is zero.[3])

If for $(n+1)$ points $x_0, x_1, \dots, x_n$ arbitrary taken in E, there should exist $(n+1)$ real numbers $\lambda_0, \lambda_1, \dots, \lambda_n$ with

$$\sum_{j=0}^{n} \lambda_j = 0, \qquad \sum_{j=0}^{n} |\lambda_j| \neq 0 \tag{2}$$

so that

$$\sum_{j=0}^{n} \lambda_j f_i(x_j) = 0, \qquad (i = 1, 2, \dots, n) \tag{3}$$

then

$$\det \begin{vmatrix} f_1(x_0) & \dots\dots & f_1(x_n) \\ \dots & \dots\dots & \dots \\ f_n(x_0) & \dots\dots & f_n(x_n) \\ 1 & \dots\dots & 1 \end{vmatrix} = 0$$

Therefore there would exist n real numbers κ_i not all of which are zero so that

$$\sum_{i=1}^{n} \kappa_i f_i(x_j) = 0.$$

This contradicts the hypothesis (1). So there do not exist $(n+1)$ real numbers λ_j $(j = 0, 1, \dots, n)$ satisfying the condition (2), (3).

---

1) Helly. Monatshefte für Math. und Phys. Bd. 31 (1921).

2) ε-neighbourhood of a means the set $\{x ; \| x - a \| < \varepsilon\}$.

3) In this note small greek letters denote real numbers.

Now let us consider the continuous mapping $F(x)$ from E into the n-dimensional euclidean space R^n in such a way that

$$F(x) = (f_1(x), \ldots\ldots, f_n(x)).$$

Then we may choose $(n+1)$ points $a, x_1, \ldots\ldots, x_n$, such that $F(a)$, $F(x_1)$, $E(x_2)$, $\ldots\ldots$, $F(x_n)$ are linearly independent in R^n because of that fact above shown. Furthermore we may take $b_j \in M$ $(j=1, 2, \ldots\ldots, n)$ sufficiently near to x_j so that $F(a), F(b_1) \ldots\ldots, F(b_n)$ are also linearly independent, and

$$\| b_j - a \| < \varepsilon . \tag{4}$$

Near the point $z = (1+n\delta)a - \delta\sum_{j=1}^{n} b_j$, where the positive number δ is so small that $\| z-a \| < \varepsilon/2$, we can take b_0 so that

$$\| b_0 - a \| < \varepsilon , \tag{5}$$

and for some $\mu_j\,(1 \leq j \leq n)$,

$$F(b_0) = (1 + n\delta + \sum_{j=1}^{n} \mu_j) F(a) - \\ -(\sum_{j=1}^{n} (\delta + \mu_j)) F(b_j). \tag{6}$$

This means

$$f_i(b_0) = (1+n\delta + \sum_{j=1}^{n} \mu_i) f_i(a) - \\ -\sum_{j=1}^{n} (\delta+\mu_j) f_i(b^j). \qquad (i=1, 2, \ldots\ldots, n) \tag{7}$$

From (7)

$$f_i(a) = \frac{1}{1+n\delta+\sum_{j=1}^{n} \mu_k} f_i(b_0) + \\ +\sum_{j=1}^{n} \frac{\delta+\mu_j}{1+n\delta+\sum_{k=1}^{n} \mu_k} f_i(b_0). \tag{8}$$

Put

$$\left. \begin{aligned} \alpha_0 &= \frac{1}{1+n\delta+\sum_{k=1}^{n} \mu_k} \\ \alpha_j &= \frac{\delta+\mu_j}{1+n\delta+\sum_{k=1}^{n} \mu_k} \end{aligned} \right\} \tag{9}$$

and take b_0 so near to z that $| \mu_j | < \delta$, then

$$\alpha_j \geqq 0 . \qquad (j=0, 1, \ldots, n) \tag{10}$$

$$f_i(a) = \sum_{j=0}^{n} \alpha_j f_i(b_j) = f_i(\sum_{j=0}^{n} \alpha_j b_j). \tag{11}$$

Put $\bar{b} = \sum_{j=0}^{n} \alpha_j b_j$ in M. Then

$$f_i(a) = f_i(\bar{b}), \qquad (i=1, 2, \ldots, n)$$

and we have from (5) and (11)

$$\| a - b \| \leq \sum_{j=0}^{n} \alpha_j \| a - b_j \| \leq \sum_{j=0}^{n} \alpha_j \varepsilon = \varepsilon,$$

and thus the theorem is proved.

Now let us take "(C)", the space of continuous functions defined on $[0, 1]$, in place of E with the norm defined as usual, and P, the aggregate of polynomials, in place of M, then we shall have the theorem :

THEOREM. *For a $g(t) \in (C)$ we can take a polynomial $p(t) \in P$ such that for an arbitrarily small positive ε and for n linear functionals Φ_i $(i = 1, 2, \dots, n)$,*

$$\sup_{t \in [0.1]} | g(t) - p(t) | < \varepsilon ,$$

and

$$\Phi_i (g(t)) = \Phi_i (p(t)) . \qquad (i = 1, 2, \dots, n)$$

(Received October 20, 1949)