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On Lattices of Functions on Topological Spaces and of Functions on Uniform Spaces.

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G. ŠILOV, I. GELFAND and A. KOLMOGOROFF have shown that the structure of the ring of continuous functions on a bicompact topological space defines the space up to a homeomorphism [1], [2].

We shall give in this paper an extension of their results to completely regular, not necessarily bicompact, topological spaces and to uniform spaces.

In § 1 we consider completely regular (not necessarily bicompact) spaces. In § 2 we consider chiefly uniformities (uniform topologies) of totally bounded uniform spaces and of metric spaces. In § 3 we discuss the special case of complete metric spaces.

§ 1. Let R be a completely regular topological space. We denote by $L(R)$ the lattice of all functions defined on R, which are bounded, ≥ 0, and which are defined as the infimum of certain (a finite or an infinite number of) continuous functions, the order being defined as usual. Then $\varphi(x) = \inf_{\gamma} \varphi_{\gamma}(x)$ is the infimum of φ_{γ} in $L(R)$, which is denoted by $\bigwedge_{\gamma} \varphi_{\gamma}$. We mean by an *ideal* of a lattice a subset I of the lattice such that $f \in I$, $g \in I$ imply $f \bigvee g \in I$, and that $f \in I$, $f \geq g$ imply $g \in I$. But the lattice itself and the null set ϕ are not regarded as ideals in this paper.

Theorem 1. *In order that two completely regular spaces R_1 and R_2 are homeomorphic, it is necessary and sufficient that the lattices $L(R_1)$ and $L(R_2)$ are isomorphic.*

Proof. Since the necessity of the condition is obvious, we shall prove only the sufficiency.

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1) G. Šilov, Ideals and subrings of the rings of continuous functions, C. R. URSS, 22 (1939.)

2) I. Gelfand and A. Kolmogoroff, On rings of continuous functions on topological spaces, C. R. URSS, 22 (1939).

1. Let R be a completely regular space. We call an ideal I of $L(R)$ an *open ideal,* when $\varphi_\gamma \notin I$ (for all γ) implies $\bigwedge_\gamma \varphi_\gamma \notin I$. And we call an ideal J a *c-ideal,* when J can be represented in the form $\Pi_1^\infty I_n$ where $I_1 \supset I_2 \supset I_3 \supset \ldots$, and I_n $(n = 1, 2, 3, \ldots)$ are open ideals. We denote by $\mathfrak{L}(R)$ the collection of all minimum c-ideals of $L(R)$. Then we can show that *for any open ideal I, there exists a point* $x_0 (\in R)$, *at which there exists a number* $a_0 \geqq 0$ *such that*

$$\varphi(x_0) \leqq a_0 \ (\varphi \in L(R)) \textit{ implies } \varphi \in I.$$

For assume that the assertion is false. Then, for every point $x (\in R)$, we can find a function $\varphi_x (\in L(R))$ such that

$$\varphi_x(x) = 0, \quad \varphi_x \notin I.$$

Since I is an open ideal, $0 = \bigwedge_{x \in R} \varphi_x \notin I$; hence $I = \phi$, which is impossible.

2. Now we take such a point x_0 for I, and denote by α_0 the supremum of such numbers a_0 at x_0. We remark that if $f(x) > \alpha$, *and* $f(x_0)$ *is continuous,* then $f \notin I$.

For suppose that $f \in I$. Let β_0 be a number such that $f(x_0) > \beta_0 > \alpha$. By the definition of α_0 there exists a function $\psi (\in L(R))$ such that $\psi(x_0) = \beta_0$, $\psi \notin I$. Let $\psi = \inf_\gamma g_\gamma$, where g_γ are continuous. Since $\psi(x_0) < f(x_0)$, $g_\gamma(x_0) < f(x_0)$ for a certain γ. Hence in a certin aneighbourhood $V(x_0)$ of x_0, $\psi(x) \leqq g_\gamma(x) < f(x)$. Let $\psi(x) \leqq A$ $(x \in R)$. Then there exists a continuous function h on R such that

$$h(x_0) = 0, \quad h(x) = A, \ (x \notin V(x_0)) \qquad 0 \leqq h(x) \leqq A.$$

Since $h \in I$, it must be $f \bigvee h \in I$. But $\psi \leqq f \bigvee h$, and $\psi \notin I$, contrary to the fact that I is an ideal.

3. Let J be any minimum c-ideal of $L(R)$ $(J \in \mathfrak{L}(R))$, then $J = \Pi_1^\infty I_n$ where $I_1 \supset I_2 \supset I_3 \supset \ldots$, and I_n are open ideals. We denote by x_n the above considered x_0 for I_n, and by α_n the α_0 for I_n, then wae cn conclude that $x_1 = x_2 = x_3 = \ldots$.

For suppose, for instance, that $x_1 \neq x_2$. We may construct a continuous function f on R such that

$$f(x_2) = 0, \quad f(x_1) > \alpha, \qquad 0 \leq f(x) \leq A.$$

Then $f \in I_2$, and from the above mentioned remark $f \notin I$, but this contradicts the fact that $I_2 \subset I_1$. Therefore it must be $x_1 = x_2 = \dots$.

We denote this point by x_0.

4. Now we denote by $J(x_0)$ the totality of functions of $L(R)$, which vanish at x_0, and by $I_\alpha(x_0)$ the totality of $L(R)$ such that $f(x_0) < \alpha$. Then $I_\alpha(x_0)$ is an open ideal, and $J(x_0) = \Pi_1^\infty I_{\frac{1}{n}}(x_0)$; hence $J(x_0)$ is a c-ideal. Since $J(x) \subset \Pi_1^\infty I_n = J$, and J is minimum c-ideal, it must be $J = J(x_0)$. Conversely, let $J(x_0) = \{\varphi \mid \varphi(x_0) = 0, \varphi \in L(R)\}$. Suppose that $J(x_0) \supset J$, where J is a c-ideal, then as we have shown above, there exists a $J(x_1)$ such that $J(x_1) \subset J \subset J(x_0)$. Hence it must be $x_0 = x_1$, and hence $J(x_0) = J$, which means that $J(x_0)$ is a minimum c-ideal.

5. Thus we have obtained a one-to-one correspondence between $\mathfrak{L}(R)$ and R. We denote this correspondence by $\mathfrak{L}$. Now we shall introduce a topology in $\mathfrak{L}(R)$ by closure as follows.

Let $\mathfrak{L}(R) \supset \mathfrak{L}(A)$, then we define that $J_0\,(\in \mathfrak{L}(R))$ is a point of the closure of $\mathfrak{L}(A)$: $J_0 \in \overline{\mathfrak{L}(A)}$, when and only when

$$\{ \prod_{J \in \mathfrak{L}(A)} J, \ J_0 \} \neq L(R).\text{[3]}$$

Then $J(x_0) \in \overline{\mathfrak{L}(A)}$, *when and only when* $x_0 \in \bar{A}$.

For let $x_0 \notin \bar{A}$, then we may construct a continuous function f such that.

$$f(x_0) = \alpha + \varepsilon, \quad f(x) = 0, \ (x \in \bar{A}), \qquad 0 \leq f(x) \leq \alpha + \varepsilon$$

Suppose that $f(x) > \alpha$ in a certain nbd (= neighbourhood) $V(x_0)$ of x_0. We construct a continuous function g such that

$$g(x_0) = 0, \quad g(x) = \alpha. \ (x \notin V(x_0)), \qquad 0 \leq g(x) \leq \alpha.$$

Then $f \in \prod_{J \in \mathfrak{L}(A)} J$, and $g \in J(x)$; hence $\alpha \leq f \cup g \in \{ \prod_{J \in \mathfrak{L}(A)} J, \ J(x) \}$. Since α is an arbitrary positive number, and all functions of $L(R)$ are

[3]) We denote by {I, J} the ideal which is generated by I and J.

bounded, it must be

$$\{ \prod_{J \in \mathfrak{L}(A)} J,\ J(x_0) \} = L(R), \text{ i.e. } J(x_0) \notin \overline{\mathfrak{L}(A)}.$$

Conversely, let $x_0 \in \overline{A}$, and $\varphi \in \{ \prod_{x \in A} J(x),\ J(x_0) \}$, then there exist two functions φ_1 and φ_2 such that

$$\varphi_1 \in \prod_{x \in A} J(x),\ \varphi_2 \in J(x_0), \text{ and } \varphi \leqq \varphi_1 \vee \varphi_2.$$

Let ε be an arbitrary small positive number. Since $\varphi_2(x_0) = 0$, and $\varphi_2(x)$ is an infimum of some continuous functions, there exists a nbd $U(x_0)$ of x_0, in which $\varphi_2(x)$ is less than ε.

Let $x \in A \cdot U(x_0)$, then, since $\varphi_1(x) = 0$,

$$\varphi(x) \leqq \text{Max}\ (\varphi_1(x), \varphi_2(x)) = \varphi_2(x) < \varepsilon.$$

This fact shows that $\varphi(x)$ may take an arbitrarily small value; hence $\{ \prod_{x \in A} J(x),\ J(x_0) \} \neq L(R)$, i.e. $J(x_0) \in \overline{\mathfrak{L}(A)}$. Therefore $\mathfrak{L}$ is a homeomorphism between $\mathfrak{L}(R)$ and R.

6. Now let $L(R_1)$ and $L(R_2)$ be isomorphic, then from this isomorphism follows the homeomorphism between the spaces $\mathfrak{L}(R_1)$ and $\mathfrak{L}(R_2)$, this last homeomorphism implies the homeomorphism between the spaces R_1 and R_2. Thus Theorem 1 is established.

§ 2. Let R be a general uniform space, and $\{\mathfrak{M}_x\}$ be the uniformity of R.[4] We say that two subsets A and B of R are *u-separated*, when and only when there exists a $\mathfrak{M}_x$ (of $\{\mathfrak{M}_x\}$) such that

$$S(A, \mathfrak{M}_x) \cdot B = \phi.^{4)}$$

Now we can show that the uniformity of a totally bounded uniform space R may be defined by the notion of "u-separation".

Lemma 1. *In order that an open covering $\mathfrak{M}$ of R is a covering of the uniformity $\{\mathfrak{M}_x \mid x \in \mathfrak{X}\}$ of R, it is necessary and sufficient that there exists an open covering $\mathfrak{M}_0$ such that*

(1) *$\mathfrak{M}_0$ possesses a finite subcovering,*

(2) *for every $M_0 \in \mathfrak{M}_0$, there exists $M \in \mathfrak{M}$ such that M_0 and M^c are u-separated.*[5]

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[4] Cf. J. W. Tukey, Convergence and uniformity in topology. (1940).

[5] We denote by M^c the complement of M.

Proof. Suppose that $\mathfrak{M} \in \{\mathfrak{M}_x\}$, then there exists a star-refinement $\mathfrak{M}_x$ in $\{\mathfrak{M}_x\}$, i. e. $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, $\mathfrak{M}_x^* < \mathfrak{M}$.[6] Since R is totally bounded, $\mathfrak{M}_x$ possesses a finite subcovering, and, for an arbitrary $M_x \in \mathfrak{M}_x$, we may choose $M \in \mathfrak{M}$ such that $S(M_x, \mathfrak{M}_x) \subset M$. Then M_x and M^c arec learly u-separated.

Converse'y, suppose that $\mathfrak{M}$ possesses a covering $\mathfrak{M}_0$ with the properties 1) and 2), then $\mathfrak{M} \in \{\mathfrak{M}_x\}$. Assume that the assertion is false, then for every $\mathfrak{M}_x \in \{\mathfrak{M}_x\}$, $\mathfrak{M}_x^{\triangle} \not< \mathfrak{M}$ holds. Hence to every x (of $\mathfrak{X}$) corresponds a point $\varphi(x)$ of R such that

$$S(\varphi(x), \mathfrak{M}_x) \not\subset M \text{ (for all } M \in \mathfrak{M}).$$

Then $\varphi(x \mid \mathfrak{X})$ is a function on the directed system $\mathfrak{X}$. Since $\mathfrak{M}_0$ possesses a finite subcovering, there exists a M_0 $(\in \mathfrak{M}_0)$, in which $\varphi(x)$ is cofina'.[6] But, $S(\varphi(x), \mathfrak{M}_x) \cdot M^c \neq \phi$ for every $M \in \mathfrak{M}$; hence M_0 and M^c are not u-separated, contrary to the assumption. Therefore $\mathfrak{M}$ must be an element of $\{\mathfrak{M}_x\}$, and the Lemma 1 is proved.

Next, let R be a metric space, then we can define the uniformity of R making use of the notion of "u-separation" as in the case of totally bounded unitorm spaec.

Lemma 2. *In order that an open covering $\mathfrak{M}$ of R is a covering of $\{\mathfrak{M}_x\}$ it is necessary and sufficient that there exist two open coverings $\mathfrak{M}_1$ and $\mathfrak{M}_2$ such that*

1) *for every $M_1 \in \mathfrak{M}_1$, there exists an $M \in \mathfrak{M}$ such that M_1 and M^c are u-separated,*

2) $\mathfrak{M}_2^{\triangle\triangle} < \mathfrak{M}_1$,

3) *for every sequence of points $\{a_i\}$ such that*

$$S(a_n, \mathfrak{M}_2) \cdot S(a_m, \mathfrak{M}_2) = \phi \quad (n \neq m),$$

(i) *if $\{b_j\}$ and $\{c_k\}$ are two subsets of $\{a_i\}$, and $\{b_j\} \cdot \{c_k\} = \phi$, then $\{b_j\}$ and $\sum_k S(c_k, \mathfrak{M}_2)$ are u-separated.*

(ii) *$\{a_i\}$ and $\prod_i S^c(a_i, \mathfrak{M}_2)$ are u-separated.*

Proof. By $S_\varepsilon(a)$, we mean the set of all points with the distance less than ε from a.

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6) Cf. J. W. Tukey, loc. cit.

1. Let $\mathfrak{M} \in \{\mathfrak{M}_x\}$, then we may choose $\mathfrak{M}_1$ and $\mathfrak{M}_2$ from $\{\mathfrak{M}_x\}$ such that

$$\mathfrak{M}_1{}^* < \mathfrak{M}, \quad \mathfrak{M}_2{}^{\triangle\triangle} < \mathfrak{M}_1$$

then the above conditions 1), 2), 3) hold.

2. Conversely, suppose that $\mathfrak{M}$ possesses refinements $\mathfrak{M}_1$ and $\mathfrak{M}_2$ with the above properties 1), 2), 3) then $\mathfrak{M} \in \{\mathfrak{M}_x\}$. For assume that the assertion is false. Then, for a sequence of positive numebrs $\varepsilon_n \to 0$, we obtain a sequence of points $\{a_n\}$ such that $S_{\varepsilon_n}(a_n) \not\subset M$ (for all $M \in \mathfrak{M}$). We remark that by the condition 1) $\{a_n\}$ *cannot be cofinal in any element* M_1 *of* $\mathfrak{M}_1$.

Next, *there exists for* a_1 *only a finite number of* a_n *such that*

$$S(a_1, \mathfrak{M}_2) \cdot S(a_n, \mathfrak{M}_2) \neq \phi.$$

For, suppose that there exists an infinite number of such a_n, then, since $\mathfrak{M}_2{}^{\triangle\triangle} < \mathfrak{M}_1$, such a_n would be contained in one and the same element M_1 ($\in \mathfrak{M}_1$), which contradicts the above mentioned remark.

Therefore we can find an n_2 such that

$$S(a_1, \mathfrak{M}_2) \cdot S(a_n, \mathfrak{M}_2) = \phi \quad (n \geqq n_2).$$

3. In the same way we see that there exist for a_{n_2} only a finite number ot a_n such that

$$S(a_{n_2}, \mathfrak{M}_2) \cdot S(a_n, \mathfrak{M}_2) \neq \phi.$$

Therefore we can find an n_3 ($> n_2$) such that

$$S(a_{n_2}, \mathfrak{M}_2) \cdot S(a_n, \mathfrak{M}_2) = \phi \quad (n > n_3).$$

Repeating the above processes we obtain a sequence of integers $n_1 < n_2 < \ldots < n_k < \ldots$ such that

$$S(a_{n_k}, \mathfrak{M}_2) \cdot S(a_n, \mathfrak{M}_2) = \phi \quad (n \geqq n_k).$$

For simplicity we rewrite $a_{n_1}, a_{n_2}, \ldots$ and $\varepsilon_{n_1}, \varepsilon_{n_2}, \ldots$ as $a_1, a_2, \ldots$ and $\varepsilon_1, \varepsilon_2, \ldots$ respectively, then for this $\{a_n\}$ the condition 3) holds.

4. We then show that *there exist an infinite number of n such that*

$$(\alpha) \quad S_{\varepsilon_n}(a_n) \cdot S(a_m, \mathfrak{M}_2) = \phi \quad (\textit{for all } m \neq n).$$

For, assume the contrary, then we can find an integer N such that, for each $n > N$ there exists an m_n such that

$(\beta)\quad S_{\varepsilon_n}(a_n)\cdot S(a_{m_n},\ \mathfrak{M}_2)\neq\phi.$

The sequence $\{m_n\}$ cannot cantain a bounded subsequence $\{m_{n(k)}\}$, for otherwise we may assume without loss of generality that $m_{n(k)}<n_h$ for every pair $\{h,\ k\}$, and hence by 3) (i) $\{a_{n(k)}\}$ and $\sum_k S(a_{m_{n(k)}},\ \mathfrak{M}_2)$ are u-separated, which is easily seen to contradict the last inequality (β).

Therefore, we can choose an increasing sequence $\{n(k)\}$ such that

$$m_{n(k)}>n\ (k-1),\ n\ (k)>m_{n(k-1)}\ (k=2,\ 3\ldots).$$

Then by (β) $\{a_{n(k)}\}$ and $\sum_k S(a_{m_{n(k)}},\ \mathfrak{M})$ are not u-separated, while on the other hand by 3) (i) they must be u-seaprated. This contradiction assures the validity of the proposition (α).

5. We have therefore $S_{\varepsilon_n}(a_n)\subset\prod_{m\neq n}S^c(a_m,\ \mathfrak{M}_2)$ for an infinite number of n, and hence for such n

$$S_{\varepsilon_n}(a_n).\quad \Pi_{m=1}^{\infty}\ S^c(a_m,\mathfrak{M}_2)=S_{\varepsilon_n}(a_n)\cdot S^c(a_n,\mathfrak{M}_2)\neq\phi$$

(We note that $\mathfrak{M}_2^{\Delta\Delta}<\mathfrak{M}_1<\mathfrak{M}$). Therefore $\{a_n\}$ and $\Pi_{m=1}^{\infty}\ S^c(a_m,\mathfrak{M}_2)$ are not u-separatd, which contradicts 3) (ii). From this we can conclude that the lemma is valid.

Now let $L_u(R)$ be the collection of all function $\varphi(x)$ such that

(1) $\varphi(x)$ is a bounded function on R,

(2) $\varphi(x)\geqq 0$,

(3) $\varphi(x)$ is uniformly continuous except at a certain finite number of points $x_1, x_2, \ldots, x_n$.

(4) $\varphi(x_i)>\varphi(x)$ in a certain nbd $U_i(x_i)$ of $x_i\ (i=1,\ 2,\ldots,n)$. If we define the order in $L_u(R)$ as usual, $L_u(R)$ forms a lattice. We have then the following

Theorem 2. *Let R_1 and R_2 be two metric spaces or totally bounded uniform spaces. In order that R_1 and R_2 are uniformly homeomorphic. it is necessary and sufficient that the lattices $L_u(R_1)$ and $L_u(R_2)$ are isomorphic.*

Proof. Since the necessity is obvious, we shall prove only the sufficiency.

We denote by $\mathfrak{L}_u(R)$ the collection of all minimum c-idea's of $L_u(R)$. We introduce in $\mathfrak{L}_u(R)$ a topology in the same way as in §1. Then, by using uniformly continuous functions in place of continuous functions, we can prove similarly as in § 1 that R and $\mathfrak{L}_u(R)$ are homeomorphic. (To a point x_0 $(\in R)$ corresponds $J(x_0) = \{f \mid f(x_0) = 0,\ f \in L_u(R)\}$). We denote this homeomorphism by $\mathfrak{L}_u$.

Now we introduce the notion of u-separation in $\mathfrak{L}_u(R)$ as follows.

Two subsets $\mathfrak{L}_u(A)$ and $\mathfrak{L}_u(B)$ of $\mathfrak{L}_u(R)$ will be called u-separated, if and only if

$$\{\prod_{J \in \mathfrak{L}_u(A)} J,\ \prod_{J \in \mathfrak{L}_u(B)} J\} = L_u(R).$$

Then $\mathfrak{L}_u(A)$ *and* $\mathfrak{L}_u(B)$ *are u-separated if and only if* A *and* B *are u-separated in* R.

For let A and B be u-separated, then there exist two open sets U and V such that $A \subset U$, $B \subset V$, $U \cdot V = \phi$, where A and U^c as well as B and V^c are u-separated. Therefore we may construct uniformly continuous functions f and g such that

$$\begin{aligned} f(x) &= 0 \quad (x \in A), \\ &= \alpha \quad (x \in U^c), \end{aligned} \qquad 0 \leq f(x) \leq \alpha,$$

$$\begin{aligned} g(x) &= 0 \quad (x \in B), \\ &= \alpha \quad (x \in V^c), \end{aligned} \qquad 0 \leq g(x) \leq \alpha.$$

Since $f \in \prod_{J \in \mathfrak{L}_u(A)} J$ and $g \in \prod_{J \in \mathfrak{L}_u(B)} J$, it must be

$$\alpha = f \vee g \in \{\prod_{\mathfrak{L}_u(A)} J,\ \prod_{\mathfrak{L}_u(B)} J\}.$$

Since α is an arbitrary positive number, this shows that

$$\{\prod_{\mathfrak{L}_u(A)} J,\ \prod_{\mathfrak{L}_u(B)} J\} = L_u(R),$$

that is, $\mathfrak{L}_u(A)$ and $\mathfrak{L}_u(B)$ are u-separated.

Conversely, let A and B be not u-separated. Let φ be any element of $\{\prod_{\mathfrak{L}_u(A)} J,\ \prod_{\mathfrak{L}_u(B)} J\}$, then there must be φ_1 and φ_2 such that

$$\varphi \leq \varphi_1 \vee \varphi_2, \quad \varphi_1 \in \prod_{\mathfrak{L}_u(A)} J, \quad \varphi_2 \in \prod_{\mathfrak{L}_u(B)} J.$$

We denote the excepted points of φ_1 and φ_2 by $a_1, a_2, \ldots, a_n$. Since φ_1 and φ_2 are uniformly coutinuous on $R - \sum_1^n a_i$, we can choose for any positive number ε an $\mathfrak{M}_x$ such that

$$| \varphi_1(a) - \varphi_1(b) | < \varepsilon, \quad | \varphi_2(a) - \varphi_2(b) | < \varepsilon$$

for $a \in S(b, \mathfrak{M}_x)$, $a, b \notin \sum a_i$, and such that $a_i \notin S(a_j, \mathfrak{M}_x)$ for $i \neq j$.

Now, since A and B are not u-separated, there exist $a \in A$ and $b \in B$ such that $\mathfrak{M}_x \ni M \ni a, b$. Let $A \cdot B = \phi$, then $a \neq b$ and a and b cannot be excepted points at the same time; for instance a is not an excepted point of φ_2. Since b is not an excepted point of φ, from the second of the last inequality we have $\varphi_2(a) < \varphi_2(b) + \varepsilon = \varepsilon$, and hence

$$\varphi(a) \leqq \varphi_1(a) \vee \varphi_2(a) = 0 \vee \varphi_2(a) < \varepsilon.$$

Hence $\varphi(x)$ can take an arbitrarily small value.

Since this fact is obvious when $A \cdot B = \phi$, we conclude in all cases that

$$\{ \prod_{\mathfrak{L}_u(A)} J, \prod_{\mathfrak{L}_u(B)} J \} \neq L_u(R),$$

that is, $\mathfrak{L}_u(A)$ and $\mathfrak{L}_u(B)$ are not u-separated.

Now it is easy to prove Theorem 2.

Suppose that R is a metric space or a totally bounded uniform space. Since in $\mathfrak{L}_u(R)$ the notion of u-separation is introduced, we can introduce a uniformity in $\mathfrak{L}_u(R)$, by the above mentioned lemmas. Then, since the u-separation of A and B is equivalent to that of $\mathfrak{L}_u(A)$ and $\mathfrak{L}_u(B)$, R and $\mathfrak{L}_u(R)$ are uniformly homeomorphic.

Now, let $\mathfrak{L}_u(R_1)$ and $\mathfrak{L}_u(R_2)$ be isomorphic, then $\mathfrak{L}_u(R_1)$ and $\mathfrak{L}_u(R_2)$ are uniformly homeomorphic; hence R_1 and R_2 are uniformly homeomorphic. Thus the proof of Theorem 2. is complete.

Now, let R be a completely regular topological space. We introduce the weak topology in the ring $C(R)$ of all continuous functions defined on R, i.e., for a certain $f \in C(R)$, we choose a finite system of points $a, \dots, a_n (\in R)$ and nbds U_i of $f(a_i)$ $(i = 1. 2. \dots, n)$, then the set $\{g \mid g(a_i) \in U_i \ (i = 1, 2, \dots n), \ g \in C(R)\}$ is called a nbd of f in $C(R)$. It is obvious that $C(R)$ forms a topological ring. Then we get the following.

Theorem 3. *In order that two completely regular spaces R_1 and R_2 are homeomorphic, it is necessary and sufficient that $C(R)$ and $C(R_2)$ are continuously isomorphic.*

Proof. Since the necessity is obvious, we prove only the sufficiency.

Let R be a completely regular space. We denote by $\mathfrak{C}(R)$ the collection of all closed maximum ideals of $C(R)$, then it is obvious that

$$I(a) = \{f \mid f(a) = 0\} \in \mathfrak{C}(R).$$

Conversely consider any ideal I of $\mathfrak{C}(R)$.

1. Put $F_{f,1/n} = \{x \mid x \in R, \ |f(x)| \leq 1/n\}$ $(f \in I)$, then the intersection of any finite number of them is non-vacuous, i. e.

$$F_{f_1,1/n} \cdot F_{f_2,1/n_2} \cdot \ldots \cdot F_{f_p,1/n_p} \neq \phi.$$

For, let Min $(1/n_i) = 1/n$, $f = f_1^2 + f_2^2 + \ldots + f_p^2 \in I$, then, since $f \leqq 1/n^2$ implies $f_i^2 \leqq 1/n^2$ and $|f_i| \leqq 1/n$, we have

$$F_{f,1/n^2} \subset \Pi_{i=1}^{p} F_{f_i,1/n} \subset \Pi_{i=1}^{p} F_{f_i,1/n_i}$$

Now if $|f(x)| > 1/n^2$ (for any $x \in R$), it would be $I = R$ which is impossible, hence $F_{f,1/n^2} \neq \phi$, and it follows that $\Pi_{i=1}^{p} F_{f_i,1/n_i} \neq \phi$. Accordingly $\{F_{f,1/n} \mid f \in I,\ n = 1, 2, \ldots\} = \mathfrak{F}$ forms a filter. We remark that on this filter, all functions of I tend to zero.

2. Next we can prove that $\mathfrak{F}$ *has a cluster point*. For, suppose that $\mathfrak{F}$ has no cluster point. Then for any point x of R, there exist a nbd $U_0(x)$ of x and $F_{f,1/n}$ such that $U_0(x) \cdot F_{f,1/n} = \phi$. Now, for every nbd $U(x)$ contained in $U_0(x)$, we construct a continuous function $\varphi_{U(x)}(x)$ such that

$$\begin{aligned} &\varphi_{U(x)}(x) = 1, \\ &\varphi_{U(x)}(a) = 0 \quad (a \in U^c(x)), \end{aligned} \qquad 0 \leqq \varphi_{U(x)} \leqq 1.$$

Then $\varphi_{U(x)} \in I$. (For, if $\varphi_{U(x)} \notin I$, Since I is maximum, it would be

$$\{\varphi_{U(x)}, I\} = C(R).$$

On the other hand, if $f \in \{\varphi_{U(x)}, I\}$, f may be represented in the form $\Psi \cdot \varphi_{U(x)} + g$ $(g \in I)$. Therefore f must tend to zero on $\mathfrak{F}$, which is a contradiction. Hence it must be $\varphi_{U(x)} \in I$.)

Let $a_1, a_2, \ldots, a_n$ be any finite system of points of R. We construct as above n functions $\varphi_{U(a_1)}, \ldots, \varphi_{U(a_n)}$, where $U(a_1), \ldots, U(a_n)$ are so chosen that $a_i \notin U(a_j)$ $(i \neq j)$.

Then $\varphi_{U(a_1)} + \dots + \varphi_{U(a_n)} = \varphi \in I$,

$$\varphi(a_i) = 1 \quad (i = 1, 2, \dots, n).$$

Hence every nbd of 1 (a point of $C(R)$), meets I, i.e.

$$1 \in \bar{I} = I.$$

Hence $I = R$, which is a contradiction. Thus $\mathfrak{F}$ has a cluster point a.

3. We have therefore $I \subset I(a) = \{f \mid f(a) = 0\}$. Since I is maximum, we have

$$I = I(a).$$

Thus we have obtained a one-to-one correspondence between R and $\mathfrak{C}(R)$. We introduce now in $\mathfrak{C}(R)$ a topology in the same way as in $\mathfrak{L}(R)$ in the proof of Theorem 1, then the above correspondence is a homeomorphism. Hence a continuous isomorphism between $C(R_1)$ and $C(R\ )$ implies a homeomorphism between $\mathfrak{C}(R_1)$ and $\mathfrak{C}(R_2)$, and hence a homeomorphism between R_1 and R_2. Thus the proof of Theorem 3 is complete.

In the case of a metric space or of a totally bounded uniform apace R, we denote by $U(R)$ the topological ring of all bounded uniformly continuous functions, the topology of $U(R)$ being the weak topology, we can prove in a similar way the following.

Theorem 4. *In order that R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $U(R_1)$ and $U(R_2)$ are continuously isomorphic.*

§ 3. From now on we concern ourselves especially with a complete metric space R. We consider the lattice of all bounded uniformly continuous functions defined on R, which are $\geqq 0$. We regard this lattice as having positive integers as operators and denote it by $L(R, \mathfrak{i})$.

Theorem 5. *In order that R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $L(R_1, \mathfrak{i})$ and $L(R_2, \mathfrak{i})$ are operator isomorphic.*

Proof. Since the necessity is obvious, we prove only the sufficiency.

We mean by an *open cut* a subset I of $L(R_2, \mathfrak{i})$ such that

$$f \in I,\ f \geqq g \text{ imply } g \in I,$$
$$f_\gamma \notin I \text{ (for all } \gamma) \text{ imply } \bigwedge_\gamma f_\gamma \notin I \text{ (if } \bigwedge_\gamma f_\gamma \text{ exists).}$$

Further we mean by a c-*ideal* J a maxinum operator ideal) in $L(R, \dot{\mathrm{I}})$ such that $J = \Pi_1^\infty I_n$, where $I_1 \supset I_2 \supset, \ldots,$ and I_n are open cuts.

1. Let

$$I(a) = \{f \mid f(a) = 0\},\ a \in R,$$
$$J_n(a) = \{f \mid \exists x:\ x \in S_{1/n}(a),\ f(x) < 1/n\}.$$

To see that $J_n(a)$ *is an open cut*, we prove that: if $f_\gamma \notin J_n(a)$ (for all γ) and $f = \bigwedge_\gamma f_\gamma$ has meaning, then $f \notin J_n(a)$. For, suppose on the contrary that $f \in J_n(a)$, then there would exist x_0 such that

$$f(x_0) < 1/n,\ x_0 \in S_{1/n}(a).$$

Since f is continuous, it must be $f(x) < 1/n$ in a certain nbd $U(x_0)$ $(\subset S_{1/n}(a))$ of x_0. We construct here a function g such that

$$g(x_0) = \alpha \quad (f(x_0) < \alpha < 1/n),$$
$$g(x) = 0 \quad (x \in U^c(x_0)),$$
$$0 \leqq g(x) \leqq \alpha,\ g \in L(R, \dot{\mathrm{I}}).$$

Then $f \vee g \in L(R, \dot{\mathrm{I}})$, $f \vee g(x_0) = \alpha > f(x_0)$, i.e. $f \vee g > f$. Take any f_γ, then, since $f_\gamma \notin J_n(a)$, we get $f_\gamma(x) \geqq 1/n > g(x)$ $(x \in U(x_0) \subset S_{1/n}(a))$. Therefore $f_\gamma \geqq f \vee g$, *i.e.* $f \vee g$ is a lower bound of $\{f_\gamma\}$, which contradicts the fact that f is the infimum of $\{f_\gamma\}$. Thus we have $f \notin J_n(a)$. Therefore $J_n(a)$ is an open cut.

2. It is clear that

$$I(a) = \Pi_1^\infty J_n(a).$$

and $I(a)$ is a maximum operator ideal. Hence $I(a)$ is a c-ideal.

Conversely let J be any c-ideal. Then J may be represented in the form $J = \Pi_1^\infty I_n$, where $I_1 \supset I_2 \supset, \ldots, I_n$ are open cuts. $(n = 1, 2, \ldots,)$.

For I_n there exists an open set U such that, if there exists a point x of U at which $f(x)$ vanishes, then $f \in I_n$. For otherwise, there would exist for each open set U of R a point x_U and a functon f_U $(\in L(R, \dot{\mathrm{I}}))$ such that

$$x_U \in U,\ f_U(x_U) = 0,\ f_U \notin I_n.$$

Since $\{x_U\}$ is dense in R, it must be $\bigwedge_U f_U = 0 \notin I_n$, which is impossible.

3. We denote by U_n the sum of all open sets U, which have the above mentioned property about I_n. Then it is clear that $U_1 \supset U_2 \supset \dots$.

Now we can show that *$\{U_n\}$ is a Cauchy filter.* To this end we remark first that *there do not exist sequences $\{a_n\}$ and $\{b_n\}$ such that*

$$a_n,\ b_n \in U_n,\ \textit{and that}\ \{a_n\}\ \textit{and}\ \{b_n\}\ \textit{are u-separated.}$$

For let $\{a_n\}$ and $\{b_n\}$ be u-separated, where $a_n, b_n \in U_n$, then there exist open sets U and V such that

$$\{a_n\} \subset U,\ \{b_n\} \subset V,\ U\ \ V = \phi.$$

$\{a_n\}$ and U^c as well as $\{b_n\}$ and V^c are u-separated. We construct uuiformly continuous functions f and g such that

$$\begin{aligned} &f(a_n) = 0\ (n = 1, 2, \dots), \quad && 0 \leqq f(x) \leqq 1, \\ &f(x) = 1\ (x \in U^c), && \\ &g(b_n) = 0\ (n = 1, 2, \dots), && 0 \leqq g(x) \leqq 1. \\ &g(x) = 1\ (x \in V^c), && \end{aligned}$$

Then, since $f, g \in \Pi_1^\infty I_n = J$, we have $1 = f \bigvee g \in J$. Hence $J = L(R, \mathfrak{i})$, which is impossible.

Now assume that U_n is not a Cauchy filter, and, that, for a certain $\varepsilon > 0$, each U_n is contained in no $S_\varepsilon(a)$ $(a \in R)$. Then there would exist a_1, $b_1 \in U_1$ such that $S_{\varepsilon/2}(a\ ) \cdot S_{\varepsilon/2}(b_1) = \phi$.

If $S_{\varepsilon/2}(a_1) \cdot U_n \neq \phi$ (for all n), since $S_\varepsilon^c(a_1) \cdot U_n \neq \phi$ (for all n), we can select $\{x_n\}$ and $\{y_n\}$ so that

$$x_n \in S_{\varepsilon/2}(a\ )\ \ U_n \text{ and } y_n \in S_\varepsilon^c(a_1)\ \ U_n.$$

Then $\{x_n\}$ and $\{y_n\}$ are u-separated, which contradicts the above mentioned remark. Hence there exists an n_2 such that

$$U_{n_2} \cdot S_{\varepsilon/2}(a\ ) = \phi,\ U_{n_2} \cdot S_{\varepsilon/2}(b_1) = \phi.$$

We choose further a_2, $b_2 \in U_{n_2}$ so that

$$S_{\varepsilon/2}(a_2)\ S_{\varepsilon/2}(b_2) = \phi.$$

We can obtain successively in the same way a sequence of pairs of points a_1, b_1 ; $a\ , b_2$; $a\ , b_3$; ... such that

$$a_n \notin S_{\varepsilon/2}(b_m) \quad \text{(for all } n, m)$$

i. e. $\{a_n\}$ and $\{b_n\}$ are u-separated, where $a_n, b_n \in U_n$. But this contradicts the above mentioned remark. Thus $\{U_n\}$ is a Cauchy filter.

4. Since R is complete, $\{U_n\}$ has a limit point a.
If we set $A_n = \{f \mid \exists\, x \in U_n : f(x) = 0\}$, then it is clear that

$$\Pi_1^\infty A_n \subset I(a).$$

Further we can show that $J = \Pi_1^\infty I_n \subset I(a)$. Assume that there exists a function f such that $f(a) \neq 0$, $f \in J$. Since $f(x)$ is continuous, there exists a nbd $U(a)$ of a, in which $f(x) > \varepsilon > 0$. We choose a nbd $U_0(a)$ so that $U_0(a)$ and $U^c(a)$ are u-separated, and construct a uniformy continuous function g such that

$$g(x) = 0 \ (x \in U_0(a)), \quad g(x) = \varepsilon \ (x \in U^c(a)), \qquad 0 \leqq g(x) \leqq \varepsilon.$$

Since a is a limit point of $\{U_n\}$, it must be

$$U_0(a) \cdot U_n \neq \phi \quad \text{(for all } n).$$

Hence $g \in \Pi_1^\infty A_n \subset J$. Hence $f \vee g \in J$, $f \vee g \geqq \varepsilon$. Since J is an operator ideal and $L(R, \mathfrak{i})$ consists of bounded functions, it must be $J = L(R, \mathfrak{i})$, which is a contradiction. Hence $J \subset I(a)$. But, since J is maximum, the last inclusion becomes an identity: $J = I(a)$.

5. If we denote by $\mathfrak{L}(R, \mathfrak{i})$ the set of all c-ideals, the above argument shows that there is a one-to-one correspondence between R and $\mathfrak{L}(R, \mathfrak{i})$.

When we introduce a uniformity in $\mathfrak{L}(R, \mathfrak{i})$ in the same way as in the case of Theorem 3, this correspondence becomes a uniform homeomorphism. Hence an operator isomorphism between $L(R_1, \mathfrak{i})$ and $L(R_2, \mathfrak{i})$ generates a uniform homeomorphism between $\mathfrak{L}(R_1, \mathfrak{i})$ and $\mathfrak{L}(R_2, \mathfrak{i})$, and this in turn generates a uniform homeomorphism between R_1 and R_2. Thus the proof of Theorem 5 is complete.

Next we consider the topological ring of all bounded uniformly continuous functions defined on R, whose topology is the strong one, and denote it by $U_s(R)$.

Theorem 6. *In order that R_1 and R_2 are uniformly homeomorphic,* *it is necessary and sufficient, that* $U_s(R)$ *and* $U_s(R)$ *are continuously isomorphic.*

Proof. Since the necessity is obvious, we prove only the sufficiency. Let R be a complete metric space. We denote by $\mathfrak{U}_s(R)$ the collection of all ideals I of $U_s(R)$ such that

1) I is algebraically a maximum ideal,

2) I is a principal closed ideal.

(A closed ideal I is called principal, when it is generated by an element.)

1. We shall show that $I(a) = \{f \mid f(a) = 0, f \in U_s(R)\} \in \mathfrak{U}_s(R)$.

It is clear that $I(a)$ is an algebraical maximum ideal.

Further *$I(a)$ is generated by $\rho(a, x) = f(x) \in I$ (ρ = distance).* To see this we define, for an arbitrary $g \in I(a)$, a sequence of functions $g_n(x)$ by

$$g_n(x) = g(x) \qquad (\rho(x, N_n) \geqq 1/n),$$
$$g_n(x) = n\,\rho(x, N_n) \cdot g(x) \qquad (0 < \rho(x, N_n) \leqq 1/n),$$
$$g_n(x) = 0 \qquad (x \in N_n),$$

where $\qquad N_n = \{x \mid \rho(a, x) \leqq 1/n\}.$

Then it is easily verified that $g_n(x)$ is bounded and uniformly continuous, and hence $g_n \in I(a)$.

Next we construct a sequence of functions $h_n(x)$ such that

$$h_n(x) = g_n(x)/f(x) \qquad (x \notin N_n),$$
$$= 0 \qquad (x \in N_n),$$

then h_n is obviously uniform continuous and $g_n = h_n \cdot f$ converges to g in $U_s(R)$.

Hence $I(a)$ is generated by an element f.

2. Conversely let I be any ideal of $\mathfrak{U}_s(R)$ and suppose that f is the only generator of I. Then f must tend to zero on a certain sequence $\{a_p\}$.

Now we sall show that *every function of I tends to zero on $\{a_p\}$.*

Let $g \in I$, and $\{g_n \cdot f\}$ converges to g in $U_s(R)$.

For an arbitrary positive number ε, we choose n and p_0 such that

$$| g_n f(x) - g(x) | < \varepsilon/2 \,(x \in R), \quad | g_n f(a_p) | < \varepsilon/2 \,(p \geqq p_0),$$

then for $p \geqq p_0$

$$| g(a_p) | \leqq | g(a_p) - g_n f(a_p) | + | g_n f(a_p) | < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

i. e. g tends to zero on $\{a_p\}$.

3. We can see that $\{a_p\}$ *has a Cauchy subsequence.* Assume the contrary, then we can select two u-separated subsequences $\{b_n\}$ and $\{c_n\}$ of $\{a_p\}$ in the same way as in the case of Theorem 5.

If we set $I\{b_n\} = \{f \mid f$ tends to zero on $\{b_n\}\}$, then $I\{b_n\}$ is an ideal and $I \subset I\{b_n\}$. And if we construct a bounded uniformly continuous function $f(x)$ such that

$$f(b_n) = 0$$
$$f(c_n) = 1 \quad (n = 1, 2, \ldots),$$

then $f \in I\{b_n\}$ and $f \notin I$. Hence $I \neq I\{b_n\}$, which contradicts the fact that I is maximum.

4. Hence $\{a_p\}$ has a Cauchy subsequence, and so a cluster point from the completeness of R. Hence every function of I must vanish at a, i. e.

$$I \subset I(a) \text{ or } I = I(a), \ I \text{ being maximum.}$$

Thus we get a one-to-one correspondence between R and $\mathfrak{U}_s(R)$, and, introducing a uniformity in the usual way, we further get a uniform homeomorphism between R and $\mathfrak{U}_s(R)$.

Thus a continuous isomorphism between $U_s(R_1)$ and $U_s(R_2)$ generates a uniform homeomorphism between $\mathfrak{U}_s(R_1)$ and $\mathfrak{U}_s(R_2)$, and this in turn generates a uniform homeomorphism between R_1 and R_2, and the proof of Theorem 6 is complete.

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