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Classification of Topological Fibre Bundles.

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1. A fibre bundle A over B with fibre F is a join of a finite number of its portions, each of which is a product space of F and an open subset of B. These portions are joined according to a *certain* law, and it is our main concern to know the topological structure of A from the knowledge of those of B and F, and of the law of joining. But this is, in general, impossible, for the definition of a fibre bundle contains indeed the existence of such a law, but nothing to specify it.

There is, however, another view point, where a fibre bundle is generated by a B-parameter motion of F in some space. This was suggested in the recent work of N. E. Steenrod [1], classifying sphere-bundles over a complex. In order to define B-parameter motions strictly, we have first to define the continuity of many valued functions, which is itself an interesting subject.

The author has investigated along the above line, and introduced the concept of F-continuity, which is recognized useful and convenient because of its intuitivity.

In § 2 the classification theorem of topological fibre bundles are formulated in an analogous form to those of N. E. Steenrod. In §§ 3-6 the proofs thereof are given. In §7 a generalization of the Feldbau's classification theorem is deduced as an application of our theorems. In § 8 homotopy groups with respect to many valued functions are discussed.

2. Let f be a many valued function of a topological space B into another topological space Ω.

f is *continuous* if the following conditions are satisfied: For each $b_0 \in B$ there exist a neighborhood V (*-nbd.) of b_0 and a family $\{\sigma_b\}$ of homeomorphisms such that,

*) It is a pleasure to record here a debt of gratitude to Professor A. KOMATU for his kindness in reading the original manuscript.

(1) σ_b is defined for $b \in V$ as a parameter,

(2) σ_b is a homeomorphism of $f(b_0)$ with $f(b)$; σ_{b_0} = the identity,

(3) The correspondence

$$f(b) \times V \ni (y, b) \leftrightarrow (\sigma_b y, b) \in \mathfrak{G}(f, V) = [\text{the graph of } f \text{ over } V]$$

is a homeomorphism.

If B is connected — a constant assumption throughout this paper — $f(b)$ is homeomorphic with a definite topological space F for every $b \in B$, and we have reason to call such a function an "*F-continuous* function".

A fibre bundle over B with fibre F is *topological* if the group G of automorphisms of F appearing in the definition of fibre bundles (see, Steenrod [1]) contains all automorphisms.

THEOREM I. *For B compact, the graph $\mathfrak{G}(f, B)$ of an F-continuous function f of B into Ω is a topological fibre bundle over B with projection π: $\mathfrak{G}(f, B) \ni (y, b) \to b \in B$, and with fibre F. If f_0 is F-homotopic*[1] *to f_1 ($f_0 \simeq f_1$), then $\mathfrak{G}(f_0, B)$ and $\mathfrak{G}(f_1, B)$ are equivalent fibre bundles*[2].

THEOREM II. *Let B be compact, and f_0, f_1 two F-continuous functions of B into the Hilbert fundamental cube E^∞.*

If $\mathfrak{G}(f_0, B)$ and $\mathfrak{G}(f_1, B)$ are equivalent fibre bundles, then f_0 and f_1 are F-homotopic to each other.

THEOREM III. (Classification Theorem) *If B is compact separable, and F separable normal, then the set of all topological fibre bundles over B with fibre F is in one-to-one correspondence with the set of all F-homotopy classes of F-continuous functions of B into E^∞.*

3. *Proof of* THEOREM I. The first part of the theorem is obvious; we shall prove the second. By assumption there exists an F-continuous function $f(b, t)$ of $B \times I$ (I is the unit interval) into Ω such that, $f(b, t) \equiv f_0(b)$ and $f(b, 1) \equiv f_1(b)$.

Since $B \times I$ is compact, we may select a finite covering $\{V\}$ consisting of $*$-nbd's. Further, we may find a positive number ε such that,

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1) I, e., there exists an F-continuous function h of $B \times I$ into Ω such that $h(b, 0) = f_0(b)$, $h(b, 1) = f_1(b)$.

2) See [1].

for every interval I' of length less than ε and for any $b \in B$ $b \times I'$ is contained in some member of the covering $\{V\}$. Divide the interval I by a finite number of points $0 = t_0 < t_1 < \dots < t_n = 1$, such that, $0 < t_{i+1} - t_i < \varepsilon$.

Obviously it is sufficient to prove, for each i, that $\mathfrak{G}(f_{t_i}, B)$ is equivalent to $\mathfrak{G}(f_{t_{i+1}}, B)$, where $f_t(b) \equiv f(b, t)$, and, in turn, it is sufficient to prove this for the original I itself, which is now assumed to possess the property for I'.

We select, using the compactness of B, a finite system of pairs of open sets $\{W_i, U_i\}$ $(i = 1, 2, \dots, m)$ such that, $\overline{W}_i \subset U_i, \bigcup_{i=1}^{m} W = B$, $U_i \times I \subset$ some V.

By the Urysohn's lemma, there exists a continuous real valued function $u_i(b)$ defined on B such that $u_i = 1$ for b in W_i, $u_i = 0$ outside U_i, and $0 \leq u_i \leq 1$.

Define

$$\tau_j(b) = \max_{i \leq j} u_i(b) \qquad \text{for} \qquad j > 0,$$
$$\tau_0(b) = 0.$$

Clearly τ_j is continuous, $\tau_j(b) \leq \tau_{j+1}(b)$ and $\tau_m(b) \equiv 1$. Furthermore, it is easily seen that the closure of the set

$$T_j = \{(b, t) \mid \tau_j(b) < t \leq \tau_{j+1}(b)\}$$

is contained in some V.

Now, putting $g_j(b) = f(b, \tau_j(b))$, we shall show that the consecutive terms of the following sequence of fibre bundles

$$\mathfrak{G}(f_0, B) = \mathfrak{G}(g_0, B), \mathfrak{G}(g_1, B), \dots, \mathfrak{G}(g_m, B) = \mathfrak{G}(f_1, B)$$

are equivalent.

As previously mentioned, T_j is contained in a $*$-nbd V, and therefore we may find a family of homeomorphisms

$$\sigma(b, t) \text{ of } F \text{ with } f(b, t), \text{ for } (b, t) \in \overline{T}_j,$$

such that

$$(y, (b, t)) \leftrightarrow (\sigma(b, t)\, y, (b, t))$$

is a homeomorphism.

Define

$$\chi(b) = \sigma(b, \tau_{j+1}(b))(\sigma(b, \tau_j(b)))^{-1} \text{ if } \tau_j(b) < \tau_{j+1}(b),$$
$$\chi(b) = \text{the identity} \qquad \text{if } \tau_j(b) = \tau_{j+1}(b),$$

then the correspondence

$$(y, b) \leftrightarrow (\chi(b)\, y, b) \text{ for } y \in f(b, \tau_j(b)) = g_j(b)$$

is easily seen to be a homeomorphism giving an equivalence between

$\mathfrak{G}(g_j, B)$ and $\mathfrak{G}(g_{j+}, B)$.

4. *The deformation* ζ_t. The Hilbert fundamental cube may be given as the direct product of countably infinite number of unit intervals

$$E^\infty = I \times I \times I \times \ldots = \{x \mid x = (x_1, x_2, x_3, \ldots), x_i \in I\}.$$

For a sequence of integers $1 \leq i_1 < i_2 < i_3 < \ldots$, let E^∞_* be the subset of E^∞ consisting of all points of the form $x = (x_j)$, $x_j = 0$ unless $j =$ some i_k of the above sequence. Define a deformation ζ_t by

$$\zeta_1(x_1, x_2, \ldots) = (0, \ldots, 0, \overset{i_1}{x_1}\, 0, \ldots, 0, \overset{i_2}{x_2}\, 0, \ldots) \in E^\infty_*$$
$$\zeta_t(x) = (1-t)\, x + t\, \zeta_1(x).\ ^{3)}$$

Then it has the property that: $\zeta_t(x) = \zeta_t(x')$ for some t, implies $x = x'$.

LEMMA 4.1. Any F-continuous function f of B into E^∞ is homotopic to an F-continuous function f' such that $f'(B) \subset E^\infty_*$

5. *Proof of* THEOREM II. Let $E_1^\infty = \{x \mid x_{2k} = 0\ (k = 1, 2, \ldots)\}$ and $E_2^\infty = \{x \mid x_{2k-1} = 0\ (k = 1, 2, \ldots)\}$.

By the lemma of the preceding § there exist f_0' and f_1' such that, $f_0 \simeq f_0'$, $f_0'(B) \subset E_1^\infty$; $f_1 \simeq f_1'$, $f_1'(B) \subset E_2^\infty$.[4] Consequently, by THEOREM I, $\mathfrak{G}(f_0, B)$, $\mathfrak{G}(f_1, B)$ are respectively equivalent to $\mathfrak{G}(f_0', B)$, $\mathfrak{G}(f_1', B)$, and, remembering that $\mathfrak{G}(f_0, B)$ and $\mathfrak{G}(f_1, B)$ are equivalent by assumption, we see that $\mathfrak{G}(f_0', B)$ and $\mathfrak{G}(f_1', B)$ are equivalent.

Let the last equivalence be defined by

$$\{\chi(b) \mid \chi(b) \text{ is a homeomorphism of } f_0'(b) \text{ with } f_1'(b)\}.$$

Then the family of functions

$$f_t'(b) = \{x' \mid x' = (1-t)\, x + t\, (\chi(b)\, x), x \in f_0'(b)\}$$

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3) + denotes vector summation.

4) $\simeq$ means "homotopic" to.

defines the homotopy of f_0' to f_1'.

6. *Proof of* THEOREM III. Given a topological fibre bundle $\mathfrak{G}$ over B with fibre F and projection π. Since B is compact separable, and F separable normal, $\mathfrak{G}$ is separable normal, and is regarded as a subset of the Hilbert fundamental cube E^∞. Let $f = \pi^{-1}$, then it is easily verified that, f is F-continuous and $\mathfrak{G}$ is equivalent to $\mathfrak{G}(f, B)$. Combining this with THEOREM I and THEOREM II, we get the proof of THEOREM III.

7. *A generalization of Feldbau's classification theorem.* Let $\Theta(F, \Omega)$ be the set of all homeorphisms of F into Ω. A function g of B into $\Theta(F, \Omega)$ is *continuous* if $F \times B \ni (y, b) \leftrightarrow (g(b)y, b) \in \mathfrak{G}(g, B)$ is a homeomorphism. Let $\mathfrak{A}(F)$ be the set of all automorphisms of F (homeomorphisms of F onto itself). The continuity of a function of B into $\mathfrak{A}(F)$ is analogously defined as above. If F is a subset of Ω, $\mathfrak{A}(F)$ is a subset of $\Theta(F, \Omega)$. Although they are not topologized, the homotopy of two continuous functions of B into any of them (denoted by X) and the following related concepts may be well defined: $\tilde{\pi}_B(X) =$ [the set of all homotopy classes of continuous functions of B into X]; $\pi_B(X) =$ [the set of all homotopy classes of continuous functions of B into X mapping the reference point b_0 of B into the reference point $1 \in \mathfrak{A}(F)$]; $\pi_n(X) = \pi_{S_n}(X) =$ [the n-th homotopy group of X]; $\pi_n(\Theta(F, \Omega), \mathfrak{A}(F))$; $(X)_\sigma =$ [the (arcwise connected) component of σ in X]; [connectedness of X]; etc.

Let ρ be the operation associating with each element σ of $\Theta(F, \Omega)$ an element $\sigma F \in \mathfrak{F}(\Omega) =$ [the set of all subsets of Ω homeomorphic with F]. Then it induces an operation $\hat{\rho}$ mapping the function space $[\Theta(F, \Omega)]^B$ into the set of F-continuous functions of B into Ω. An F-continuous function belonging to the image set $\hat{\rho}([\Theta(F, \Omega)]^B)$ will be called *trivial.* An F-continuous function is then characterized by saying that it is *locally trivial.* A function f of B into $\mathfrak{F}(\Omega)$ is *continuous* if it is F-continuous (when regarded as a many valued function of B into Ω). Then we may define, in the same way as above, the following concepts: $\tilde{\pi}_B^F(\Omega) =$ [the set of all F-homotopy classes of F-continuous functions of B into Ω]; $\tilde{\pi}_{B,B'}^F(\Omega) =$ [the set of all F-homotopy classes of continuous functions f of B into Ω such that, for every pair of points $b_1', b_2' \in B' <$

B, $f(b_1') = f(b_2')$]; $\pi_n^F(\Omega) =$ [the n-th F-homotopy group of Ω]; [F-connectedness of Ω] $=$ [the connectedness of $\mathfrak{F}(\Omega)$]; etc.

LEMMA 7.1. If B is compact nnd contractible, then any F-continuous function defined on B is trivial.

Proof. Let θ_t be a contraction of B into a point $b_0 \in B$, then it is easily seen that, for any F-continuous function f defined on B, $f(\theta_t b)$ gives an F-homotopy of $f = f\theta_0$ to $f\theta$. By THEOREM I, the fibre bundle $\mathfrak{G}(f\theta, B)$ is equivalent to the fibre bund'e $\mathfrak{G}(f\theta, B)$, which is easily seen to be a product bundle.[5] This shows, in turn, that $\mathfrak{G}(f, B)$ itself is a product bundle, from which the assertion of the lemma follows immediately.

LEMMA 7.2. $\tilde{\pi}_B(\Theta(F, E^\infty))$ contains only one element.

The proof is contained in the proof of THEOREM II.

COROLLARY 7.3. E^∞ is F-connected.

Now, we define the decomposition-sets $[\tilde{\pi}_B(\mathfrak{A}(F))]$, $[\pi_B(\mathfrak{A}(F))]$ respectively of $\tilde{\pi}_B(\mathfrak{A}(F))$, $\pi_B(\mathfrak{A}(F))$. Let f, g be two elements of $[\mathfrak{A}]^B$. We say that they are specially equivalent ($f \overset{s}{\sim} g$), if, for some $\sigma, \tau \in \mathfrak{A}$ and for every $b \in B$, $f(b) = \sigma g(b) \tau$. Then the following two conditions are equivalent:

(1) f is homotopic to an element specially equivalent to g,

(2) f is specially equivalent to an element homotopic to g.

It is easily seen that, the above conditions satisfy the usual three conditions for equivalence, and we may divide the set $\tilde{\pi}_B(\mathfrak{A})$ into equivalence classes, the totality of which is denoted by $[\tilde{\pi}_B(\mathfrak{A})]$. The last notation is appropriate, because it torms a decomposition-set of $\tilde{\pi}_B(\mathfrak{A})$. In an entirely same way, we obtain $[\pi_B(\mathfrak{A})]$, with the only difference, that we define, in this case, the special equivalence of f, $g \in [\mathfrak{A}, 1]^{(B, b_0)}$ as: $f \overset{s}{\sim} g$. if and only if, for some $\sigma \in \mathfrak{A}$ and for every $b \in B$, $f(b) = \sigma g(b) \sigma^{-1}$.

LEMMA 7.4. $[\tilde{\pi}_B(\mathfrak{A}(F))] \approx [\pi_B(\mathfrak{A}(F))]$.

Proof. Let f and g represent the same element of $[\pi_B]$, then, for some $\sigma \in \mathfrak{A}$, f and $\sigma g \sigma^{-1}$ represent the same element of π_B, and hence of $\tilde{\pi}_B$. This implies that f and g represent the same element of $[\pi_B]$. Let f re-

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5) See [1].

presents an element of $[\tilde{\pi}_E]$, then $f' = (f(b))^{-1} f$ represents an element of $[\pi_B]$ as well as the same element of $[\tilde{\pi}_E]$ as f does. Let f and g represent twe elements of $[\pi_B]$, while they represent the same element of $[\tilde{\pi}_E]$, then for some $\sigma, \tau \in \mathfrak{A}$ f and $\sigma f \tau$ represent the same element of π_B, *i. e.* there exists a function $h_t(b) = h(b, t) \in [\mathfrak{A}]^{B\times I}$ with $h_0 = f$ and $h_1 = \sigma g \tau$. Let $h_t' = (h_t(b))^{-1} h_t$, then h_t' represents an element of π_B and $h_0' = h_0 = f$, $h_1' = (h_1(b_0))^{-1} h_1 = (\sigma g(b_0)\tau)^{-1}(\sigma g \tau) = \tau^{-1} g \tau$. Hence f, g represent the same element of $[\pi_B]$.

LEMMA 7.5. Let $\hat{\rho} g_0 = \hat{\rho} g_1 = f$ represents an element of $\tilde{\pi}^F_{B,B'}(\Omega)$, and put $\varphi_i(b') = (g_i(b))^{-1} g_i(b')$ for $b', b_0 \in B'$. Then φ_0 and φ_1 represent the same element of $[\pi_{B'}(\mathfrak{A})]$, provided that B is contractible.

Proof. Put $\psi(b) = (g_1(b))^{-1} g_0(b) \in [\mathfrak{A}]^B$. Then $g_0(b) = g_1(b)\psi(b)$, and $\varphi_0(b') = (\psi(b))^{-1} \varphi_1(b')\psi(b')$. Put $\chi_t(b') = (\varphi(b_0))^{-1}\varphi_1(b')\psi(\theta_t b')$, where θ_t is a contraction of B into b_0. Then $\chi_t(b_0) = 1$, and $\chi_0(b') = \varphi_0(b')$, $\chi_1(b') = (\psi(b_0))^{-1}\varphi_1(b')\psi(b_0)$. Hence the lemma follows.

Let B be compact and contractible. We define a mapping λ of $\tilde{\pi}^F_{B,B'}(\Omega)$ into $[\pi_{B'}(\mathfrak{A})]$ under the following prescription: For $\alpha \in \tilde{\pi}^F_{B,B'}(\Omega)$, take a representative f of α, choose $g \in [\Theta(F, \Omega)]^B$ with $\hat{\rho} g = f$ by LEMMA 7.1, put $\varphi(b')) = (g(b_0)^{-1} g(b')$ for $b' \in B'$, finally let $\lambda\alpha$ be the element of $[\pi_{B'}(\mathfrak{A})]$ represented by φ. Then $\lambda\alpha$ is determined only by α and independent of the choices of f and g. For, its independence of the choice of g is assured by LEMMA 7.5, and that of the choice of f may be verified directly.

LAMMA 7.6. If $\Omega = E^\infty$, B' is compact and B is a cone over B', then λ is one-to-one.

Proof. Let f, f_1 repesent two elements of $\tilde{\pi}^F_{B,B'}(E^\infty)$. Since E^∞ is F-connected, using the property of B', f_i' may be found representing the same element of $\tilde{\pi}^F_{B,B'}$ as f_i and such that the values on B' is constnt F. Choose $g_i \in [\Theta(F, E^\infty)]^B$ such that $\hat{\rho} g_i = f'_i$ and $g_i(b) = 1 \in \mathfrak{A}$. Then $g_i \mid B'$ represents the element $\lambda\{f_i\}$ of $[\pi_{B'}(\mathfrak{A})]$. Assume that $\lambda\{f_0\} = \lambda\{f_1\}$, *i. e.*, for some $\sigma \in \mathfrak{A}$, $g_1 \mid B'$ is homotopic to $\sigma(g_1 \mid B')\sigma^{-1}$ in $\pi_{B'}(\mathfrak{A})$. Then there exists $h \in [\mathfrak{A}]^{B'\times I}$ such that, $h(b', 0) = g_0(b')$, $h(b', 1) = \sigma g \sigma^{-1}$, for every $b' \in B'$. Let $w(t)$ be a curve in $\Theta(F, E^\infty)$ joining 1 to σ^{-1}, and put $k(b', t) = w(t) h(b', t)$, $k(b, 0) = g_0(b)$, $k(b, 1) = g_1(b)\sigma^{-1}$, where $q' \in B'$ and $b \in B$. Obviously k represents a continuous function of the boundary of $B \times I$ into $\Theta(F, E^{\infty})$, and by LEMMA 7.2, can be extended to a function $\bar{k} \in [\Theta(F, F^{\infty})]^{B\times I}$. Then $\hat{\rho}\,\bar{k}$ gives a homotopy of $f_0' = \hat{\rho}\, g_0 = \hat{\rho}\, k_0$ to $\hat{\rho}\,\bar{k}_1 = \hat{\rho}\,(g_1\,\sigma^{-1}) = \hat{\rho}\, g_1 = f_1'$. Hence f_0, f_1 represent the same element of $\tilde{\pi}^F_{B,B'}$.

LEMMA 7.7. If $\Omega = E^{\infty}$, B' is compact, and B is a cone over B', then λ is onto.

Proof. Let φ represents an element of $[\pi_{B'}(\mathfrak{A}).]$ Then it can be extended to a funbtion $g \in \Theta\,[(F, E^{\infty})]^{B}$, by LEMMA 7.2. and the fact that B is a cone over B'. Then $\hat{\rho}\, g$ represents an element of $\tilde{\pi}^F_{B,B'}$, which is mapped under λ into the element of $[\pi_{B'}]$ represented by φ.

Combining LEMMA's 7.4, 7.6, 7.7, we obtain

THEOREM IV. *If B' is compact and B a cone over B', the following one-to-one correspondence holds*:

$$\tilde{\pi}^F_{B,B'}(E^{\infty}) \approx [\tilde{\pi}_{B'}(\mathfrak{A}(F))].$$

The suspension $< B' >$ over a topological space B' is a topological space which is obtained from $B' \times I$ by identifying the subsets $B' \times (0)$ and $B' \times (1)$ into points v_0 and v_1 respectively. If B' is compact separable, the cone B as well as the suspension $< B' >$ over B' are also compact separable. In such a case the set $\tilde{\pi}^F_{B,B'}(\Omega)$ is obviously in one-to-one correspondence with the set $\tilde{\pi}^F_{< B' >}(\Omega)$. Thus we have

THEOREM V. *Let B' be compact separable, $< B' >$ the suspension over B'. If F is separable normal, then the set of all topological fibre bundles over $< B' >$ with fibre F is in one-to-one correspondence with the set of all equivalence classes of continuous functions of B' into $\mathfrak{A}(F)$.*

This theorem contains as its particular case the

FELDBAU'S CLASSIFICATION THEOREM. *If F is separable normal, then the set of all topological fibre bundles over S^{n+1} with fibre F is in one-to-one correspondence with the set of all equivalence classes of continuous functions of S^n into $\mathfrak{A}(F)$ $(n \geq 0)$.*

8. *F-homotopy groups.* In the particular cases where $B' = S^n$ $(n \geq 1)$, the relation of THEOREM IV is obtained by the following familiar steps :

LEMMA 8.1. The homomorphism sequence of homotopy groups

$$\to \pi_{n+1}(\Theta(F, \Omega)) \xrightarrow{\alpha} \pi_{n+1}(\Theta(F, \Omega), \mathfrak{A}(F)) \xrightarrow{\beta}$$
$$\pi_n(\mathfrak{A}(F)) \xrightarrow{\gamma} \pi_n(\Theta(F, \Omega)) \longrightarrow$$

is exact, where α, β, γ are relativisation operator, homotopy boundary operator, injection operator, respectively.

LEMMA 8.2. $\pi_{n+1}(\Theta(F, \Omega), \mathfrak{A}(F)) \approx \pi^F_{n+1}(\Omega)$.

LEMMA 8.3. $\pi_k(\Theta(F, E^\infty)) = 0, (k = 1, 2, \ldots)$.

THEOREM VI. $\pi^F_{n+1}(E^\infty) \approx \pi_n(\mathfrak{A}(F)) \quad (n \geqq 1)$

Remark. LEMMA 8.1. states a well-known fact. The proof of LEMMA 8.2. is essentially contained in the observations of the preceding §. We notice that $\Theta(F, \Omega)$ behaves as if it were a fibre bundle over $\mathfrak{F}(\Omega)$ with fibre $\mathfrak{A}(F)$. LEMMA 8.3. is a consequence of LEMMA 7.2. THEOREM VI is obtained from LEMMA's 8.1.-3. using a property of exact sequence. THEOREM VI does not exclude the case $n = 0$, when we define $\pi_0(\mathfrak{A}(F)) = \mathfrak{A}(F)/\mathfrak{A}_0(F)$, where $\mathfrak{A}_0(F)$ is the component of the identity in $\mathfrak{A}(F)$. The proof thereof is essentially the same as the case $n \geqq 1$. Thus

THEOREM VI'. $\pi^F_1(E^\infty) \approx \mathfrak{A}(F)/\mathfrak{A}_0(F) = \pi_0(\mathfrak{A}(F))$.

S. Eilenberg has shown that the fundamental group of a space may be regarded as an operator group of the homotopy groups of the space. This circumstance is not altered when we replace the ordinary homotopy groups by our F-homotopy groups. In fact, we proceed as follows: Given $\alpha \in \pi^F_{n+1}(E^\infty)$ and $\zeta \in \pi^F_1(E^\infty)$, take representatives f and w respectively of α and ζ, define $h_t(b) = h(b, t) \in [\mathfrak{F}(E^\infty)]^{E^{n+1}\times(o) + \dot{E}^{n+1}\times I}$ by $h(b, 0) = f(b)$, $h(b', t) = w(t)$ for $b \in B$, $b' \in B'$, extend h to a function $\bar{h} \in [\mathfrak{F}(E^\infty)]^{E^{n+1}\times I}$, let the element of $\pi^F_{n+1}(E^\infty)$ represented by $\bar{h}_1(b)$ be denoted by α^ζ. That α^ζ is determined only by α and ζ may be easily verified. Thus we obtain a homomorphism of $\pi^F_1(E^\infty)$ into the group of automorphisms of $\pi^F_{n+1}(E^\infty)$, that is, we may regard $\pi^F_1(E^+)$ as an operator group of $\pi^F_{n+1}(E^\infty)$. As for $\pi_n(\mathfrak{A}(F))$, we take $\pi_0(\mathfrak{A}(F))$ instead of $\pi_1(\mathfrak{A}(F))$, and proceed as follows: Given $\beta \in \pi_n(\mathfrak{A}(F))$ and $\eta \in \pi_0(\mathfrak{A}$

(F)), take representatives g and σ respectively of β and η, put $g'(b) = \sigma g(b) \sigma^{-1}$, let the element of $\pi_n(\mathfrak{A}(F))$ represented by g' be β^{η}. Then β^{η} is determined only by β and η, and in the same way as above, we may regard $\pi_0(\mathfrak{A}(F))$ as an operator group of $\pi_n(\mathfrak{A}(F))$.

We now show that the isomorphisms of THEOREM VI, and THEOREM VI' are operator isomorphisms.

Let $\alpha, \zeta, f, w, \bar{h}$ have the same meaning as above. Choose a function $k_t(b) = k(b, t) \in [\Theta(F, E^{\infty})]^{E^{n+1} \times I}$ with $\hat{\rho} k = h$, $k(0, 0) = 1$, and put $\chi_t(b') = (k_t(0))^{-1} k_t(b')$ for $b' \in \dot{E}^{n+1}$. Then it gives a homotopy in $\pi_n(\mathfrak{A}(F))$ of χ_0 to $\chi_1 = (k_1(0))^{-1}(k_1 \mid \dot{E}^{n+1})$ But, by definition, $\alpha = \{f\}$, $\zeta = \{w\}$, $\omega_n(\alpha^{\zeta}) = \{(k_1 \mid \dot{E}^{n+1}) \sigma^{-1}\}$ ($\sigma = k_1(0)$, $\omega_1 \zeta = \{\sigma\}$; and $\omega_n \alpha = \{\chi_0\} = \{\sigma^{-1}(k_1 \mid E^{n+1}\}$, according to the above homotopy). Therefore $(\omega_n \alpha)^{\omega_1 \zeta} = \omega_n(\alpha^{\zeta})$, which is to be proved. Thus we have

THEOREM VII.

$$\pi_{n+1}^{F}(E^{\infty}) \approx \pi_n(\mathfrak{A}(F)) \quad (n \geq 0)$$

is an operator isomorphism.

This reveals a deeper content of THEOREM IV, in a particular case when B' is an n-sphere.

9. *Remark.* The group $\pi_k(B)$ and $\pi_B^F(E^{\infty})$ may be regarded as forming a pair with respect to $\pi_k^F(E^{\infty})$ in the following way: Let φ, f represent $\zeta \in \pi_k(B)$, $\alpha \in \pi_B^F(E^{\infty})$, respectively. Then the element $\zeta \circ \alpha$ of $\pi_k^F(E^{\infty})$ represented by $f \varphi$ is independent of special choices of φ and f.

Define $\alpha_k(\zeta) = \zeta \cdot \alpha$, for $\zeta \in \pi_k(B)$ and $\alpha \in \pi_B^F(E^{\infty})$. Then α_k seems to play an important role in determining the structure of the fibre bundle represented by α.

LITERATURE

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