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Relations between Homotopy and Homology. I.

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1. INTRODUCTION.

This paper is a continuation of the author's earlier investigation [1], studying the problem of essential dimensions [1]) of continuous transformations using the method of homology with local coefficients [2]. The exact homology sequence, recently clarified by J. L. KELLEY and E. PITCHER [3], can be applied to this method and give many new results some of which are already obtained by S. EILENBERG and S. MAC LANE [4], L. PONTRJAGIN [5] and G. W. WHITEHEAD [6].

Let K^n be the n-section of a complex K, then we have the following exact sequence with respect to the homotopy groups

$$\pi_m(K^{n-1}) \xrightarrow{i} \pi_m(K^n) \xrightarrow{r} \pi_m(K^n \bmod K^{n-1}) \xrightarrow{\partial t}$$
$$\pi_{m-1}(K^{n-1}) \xrightarrow{i} \pi_{m-1}(K^n)$$

The kernel-images in $\pi_m(K^n)$, $\pi_m(K^n \bmod K^{n-1})$, $\pi_{m-1}(K^{n-1})$ of this sequence are essentially the same as the groups $\nu_m(K^n)$, $\mu_m(K^n)$, $\lambda_{m-1}(K^{n-1})$, respectively, which were introduced by the author in [1].

2. THE CASE OF SIMPLY CONNECTED COMPLEX.

THEOREM 1. *Let α_n be the number of n-simplexes of a simply connected complex K. Then the relative homotopy group $\pi_n(K^n \bmod K^{n-1})$ $(n>2)$ is isomorphic with the weak direct sum (I, α_n) of α_n integer groups.*

PROOF. The proof is similar to that of theorem 2. 1, [1].

COROLLARY 1. 1.

$$\pi_n(K^n) \approx \mu_n(K^n) + \nu_n(K^n) .$$

PROOF. The group $\mu_n(K^n) \approx \pi_n(K^n)/\nu_n(K^n)$ is a subgroup of the

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1) For this definition see [1]. The essential dimension of a continuous mapping f of M in K is the least dimension of the image sets $g(M)$, where g is any continuous mapping of the same homotopy class with f.

free abelian group $\pi_n(K^n \bmod K^{n-1})$, therefore $\mu_n(K^n)$ is a direct component of $\pi_n(K^n)$.

COROLLARY 1. 2. $\lambda_n(K^n)$ is isomorphic with the direct sum of the subgroup $\lambda_n(K^n) \bigwedge \nu_n(K^n)$ and the subgroup isomorphic with $\lambda_n(K^n)/\lambda_n(K^n) \bigwedge \nu_n(K^n)$.

For $\lambda_n(K^n)/\lambda_n(K^n) \bigwedge \nu_n(K^n)$ is a module, being isomorphic with a subgroup of $\pi_n(K^n \bmod K^{n-1})$.

COROLLARY 1. 3. The n-chain group with integer coefficients $L^n(K, I)$ of K is isomorphic with $\pi_n(K^n \bmod K^{n-1})$.

THEOREM. 2. *Let ∂_l be the homology boundary operator of $L^n(K, I)$ $(n > 3)$, and ∂_t the homotopy boundary operator, then there holds the relation*

$$\partial_l = r\,\partial_t$$

PROOF. It is sufficient to prove the case of one simplex $1.\sigma^n \in L^n(K, I)$, for $\partial_l, r, \partial_t$ are all homomorphic mappings of abelian groups.

Let
$$\partial_l(\sigma^n) = \sum_i{}' \sigma_i^{n-1},$$
$$\partial_t(\sigma^n) = \alpha \in \lambda_{n-1}(K^{n-1}),$$

where α is a homotopy class of the continuous mapping of an $(n-1)$-sphere S^{n-1} on the sphere $\partial_l(\sigma^n) = \sum_i{}' \sigma_i^{n-1}$ with mapping-degree $+1$. Then

$$r(\alpha) = \sum{}' \sigma_i^{n-1}, \text{ i. e. } \partial_l = r\,\partial_t.$$

A chain $c^n \in \pi_n(K^n \bmod K^{n-1})$ is a cycle, when $r\,\partial_t(c^n) = 0$, and is a spherical cycle, when $\partial_t(c^n) = 0$. A homology-boundary is a spherical cycle and the spherical homology group $\sum{}^n(K)$ is defined as the factor group of the group $\mu_n(K^n)$ of spherical cycles by the homology boundary $r(\lambda_n(K^n))$.

COROLLARY 2. 1. $\sum{}^n(K) \approx \pi_n(K)/\nu_n(K) \approx \mu_n(K)$.

PROOF. The group of boundaries is $r(\lambda_n(K^n)) \approx \lambda_n(K^n)/\lambda_n(K^n) \bigwedge \nu_n(K^n) = B^n(K)$. Therefore $\sum{}^n(K)$ is isomorphic with

$$\mu_n(K^n)/r(\lambda_n(K^n)) \approx \pi_n(K^n)/\nu_n(K^n)/\lambda_n(K^n)/\lambda_n(K^n) \bigwedge \nu_n(K^n).$$

The last term of the above sequence of groups is easily verified to be isomorphic with $\pi_n(K)/\nu_n(K) \approx \mu_n(K)$.

COROLLARY 2. 2. $H^n(K)/\sum{}^n(K) \approx \lambda_{n-1}(K^{n-1}) \bigwedge \nu_{n-1}(K^{n-1})$.

LEMMA 2. 1. $\nu_n(K^n) \approx \pi_n(K^{n-1})/\lambda_n(K^{n-1})$.

COROLLARY 2. 3. If $\pi_i(K) = 0$ $(0 \leq i < n)$, then

$$H^n(K, I) \approx \sum^n(K) \approx \pi_n(K),$$
$$H^{n+1}(K, I) \approx \sum^{n+1}(K) \approx \pi_{n+1}(K)/\nu_{n+1}(K).$$

PROOF. By the result of W. HUREWICZ any compact set of K^{n-1} is homotopic to zero in K^n. Therefore $\pi_n(K^{n-1}) \approx \lambda_n(K^{n-1})$. And so by LEMMA 2. 1, $\nu_n(K^n) = 0$, i. e. $\nu_n(K) = 0$. This proves the theorem by COROLLARIES 2. 1, 2. 2.

If we apply the Freudenthal's theory of "Einhängung" to the group $\nu_{n+1}(K) \approx \nu_{n+1}(K^{n+1})/\nu_{n+1}(K^{n+1}) \cap \lambda_{n+1}(K^{n+1})$, we can deduce the results of G. W. WHITEHEAD. For instance we get the following relations:

If $\pi_i(K) = 0$ $(0 < i < n)$,

$$\pi_n(K^n)/2\pi_n(K^n) \approx \pi_{n+1}(K^n),$$
$$\pi_n(K^n)/(\lambda_n(K^n),\ 2\pi_n(K^n)) \approx \pi_{n+1}(K^n)/\lambda_{n+1}(K^n),$$
$$\pi_n(K)/2\pi_n(K) \approx \nu_{n+1}(K^{n+1}).$$

3. THR CASE WHEN K IS NOT SIMPLY CONNECTED.

Let $\bar{K}$ be the universal covering complex of K and $\bar{K}^n$ the n-section of $\bar{K}$. $\bar{K}^n$ $(n > 1)$ is the universal covering complex of K^n. Let $\mathfrak{F} = \{x_\alpha\}$ be the fundamental group of K, then the n-simplex of $\bar{K}$ are represented in the form $\{x_\alpha \sigma_i^n\}$, where $\{\sigma_i^n\}$ are n-simplexes of K. The mapping u: $x_\alpha\sigma_i^n \rightarrow \sigma_i^n$ is the covering mapping of $\bar{K}$ onto K. Remembering that the homotopy groups of a complex are isomorphic with those of the covering complex, we can easily verify that the following two sequences

$$\pi_{n+1}(\bar{K}^{n+1} \bmod \bar{K}^n) \rightarrow \pi_n(\bar{K}^n) \rightarrow \pi_n(\bar{K}^n \bmod \bar{K}^{n-1}),$$
$$\pi_{n+1}(K^{n+1} \bmod K^n) \rightarrow \pi_n(K^n) \rightarrow \pi_n(K^n \bmod K^{n-1})$$

are equivalent as homomorphism sequences. In particular we have

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[2]) After this paper was submitted for publication, I have read G. W. WHITEHEAD'S paper [6] that recently came to Japan. Although the proof is only sketched, it seems to me that his method is different from that of mine. I could not read the paper of H. HOPF: Über die Bettischen Gruppen, die zu einer beliebigen Gruppen gehören. Comment. Math. Helv. 17, 1944.

$$\lambda_n(\bar{K}^n) \approx \lambda_n(K^n),$$
$$\mu_n(\bar{K}^n) \approx \mu_n(K^n),$$
$$\nu_n(\bar{K}^n) \approx \nu_n(K^n).$$

As is shown in § 2, $\pi_n(\bar{K}^n \bmod \bar{K}^{n-1})$ is isomorphic with the chain group $L^n(\bar{K}, I)$, and its elements can be represented in the form $\sum a\, x_\alpha \sigma_i^n$, where a's are integers. Clearly the elements of the form $\sum a.\ 1\ \sigma^n$, where 1 is the unit element of $\mathfrak{F}$, form a subgroup of $L^n(\bar{K}, I)$ which is isomorphic with the chain group $L^n(K, I)$. We suppose that $L^n(K, I)$ is imbedded in $\pi_n(\bar{K}^n \bmod \bar{K}^{n-1}) \approx L_n(\bar{K}, I)$ by the above isomorphism.

We remark that $L_n(K, I)$ is a direct summand of $\pi_n(\bar{K}^n \bmod \bar{K}^{n-1})$ and the natural homomorphism of the latter group onto the former is induced by the covering mapping $u:\ x_\alpha \sigma_i^n \to \sigma_i^n$. We denote by Γ^n the kernel of the last homomorphism.

Then we have the following important

THEOREM 3. *The homology boundary operator* ∂_l *of* $L^n(K, I)$ $(n > 3)$ *can be decomposed into 3 successive operators, i. e.*

$$\partial_l = u\, r\, \partial_t .$$

PROOF. It is sufficient to prove the case of one simplex σ^n.

Let

$$\partial_l(\sigma^n) = \sum \sigma_i^{n-1}$$
$$\partial_t(\sigma^n) = \alpha \in \lambda_{n-1}(K^{n-1}) \approx \lambda_{n-1}(\bar{K}^{n-1}),$$

where α is the homotopy class of continuous mapping f of S^{n-1} on the $(n-1)$-sphere $\sum_i \sigma_i^{n-1}$ of K^{n-1} with mapping degree $+1$, or the mapping $\bar{f}$ of S^{n-1} on an $(n-1)$-sphere $\sum_i x_\alpha \sigma_i^{n-1}$ of $\bar{K}^{n-1}$. The mapping f is equal to the mapping $u\bar{f}$. The image sphere $\sum_i x_\alpha \sigma_i^{n-1}$ is invariant by the relativisation r, as in theorem 2 and by the covering mapping u it reduces to the sphere $\sum_i \sigma_i^{n-1}$, i. e. $\partial_l(\sigma^n)$. Therefore for every chain c^n of $L^n(K, I)$

$$\partial_l(c^n) = ur\,\partial_t(c^n).$$

A chain $c^n \in L^n(K, I) \subset \pi_n(K^n \bmod K^{n-1})$ is called spherical, when it satisfies $\partial_t(c^n + \gamma^n) = 0$ for some $\gamma^n \in \Gamma^n$, and is called simple, when it satisfies $r\,\partial_t\,(c^n + \gamma^n) = 0$ for some $\gamma^n \ni \Gamma^n$. Then we see easily that c^n is a spherical cycle or a simple cycle if and only if it is an image under u of a spherical cycle or a cycle of $\bar{K}$, respectively.

THEOREM 4. *Homology boundaries are spherical.*

PROOF. Let c^n be the boundary of a chain c^{n+1}, that is, $\partial_t\,(c^{n+1}) = u\,r\,\partial_t\,(c^{n+1}) = c^n$ or $r\,\partial_t\,(c^{n+1}) = c^n + \gamma^n$ for some $\gamma^n \in \Gamma^n$. Using relation $\partial_t\,r = 0$, we have then $\partial_t\,(c^n + \gamma^n) = \partial_t\,r\,\partial_t\,(c^{n+1}) = 0$.

By this theorem we can define the spherical homology group $\Sigma^n\,(K, I)$ and the simple homology group $\Theta^n\,(K, I)$ of K as subgroups of $H^n\,(K, I)$.

THEOREM 5.

$$\Sigma^n\,(K,\ I) \approx \Sigma^n\,(\bar{K},\ I)/\Sigma^n\,(\bar{K},\ I) \cap \Gamma^n\,,$$
$$\Theta^n\,(K,\ I) \approx H^n\,(\bar{K},\ I)/H^n\,(\bar{K},\ I) \cap \Gamma^n.$$

PROOF. We shall prove only the former relation. The proof of the latter is similar.

Let c^n be the homology boundary of c^{n+1} and d^n, d^{n+1}, respectively, the image chains $u\,(c^n)$, $u\,(c^{n+1})$ in K. Then for a suitable element $\gamma^{n+1} \in \pi_{n+1}\,(\bar{K}^{n+1} \bmod K^n)$

$$c^{n+1} = d^{n+1} + \gamma^{n+1}\,,$$
$$u\,r\,\partial_t\,(d^{n+1}) = u\,r\,\partial_t\,(c^{n+1} - \gamma^{n+1})$$
$$= u\,r\,\partial_t\,(c^{n+1}) - u\,r\,\partial_t\,(\gamma^{n+1}) = u\,(c^n) = d^n.$$

Hence the mapping u defines a homomorphism of $\Sigma^n\,(\bar{K}, I)$ in $\Sigma^n\,(K, I)$.

Let d^n be a spherical cycle in K. With a suitable γ^n the sum $\gamma^n + d^n = c^n$ is a spherical cycle in $\bar{K}$, i. e.

$$\partial_t\,(\gamma^n + d^n) = 0\,,$$

and $u\,(c^n) = d^n$. Hence $u\,(\Sigma^n\,(\bar{K}, I)) = \Sigma^n\,(K, I)$.

Let d^n be a boundary in K and c^n the original element $u^{-1}\,(d^n)$ in $\Sigma^n\,(\bar{K}, I)$. These conditions are written

$$c^n = d^n + \gamma^n,\ \gamma^n \in L\,(K^n, I)\,,$$
$$(1) \qquad \partial_t\,(c^n) = 0,$$
$$(2) \qquad u r\,\partial_t\,(d^{n+1}) = d^n, \quad d^{n+1} \in L^{n+1}\,(K^{n+1}, I)\,.$$

From (2) for a suitable γ'^n

$$r\,\partial_t\,(d^{n+1}) = d^n + \gamma'^n\,,$$

hence (3) $\partial_t (d^n + \gamma'^n) = 0.$

From (1) and (3)

$$\partial_t (\gamma^n - \gamma'^n) = 0, \text{ i.e. } \gamma^n - \gamma'^n \in \Sigma^n (\bar{K}, I) \wedge \Gamma^n ,$$

and

$$c^n = r\, \partial_t (d^{n+1}) + (\gamma^n - \gamma'^n).$$

Therefore the original element $c^n = u^{-1} (d^n)$ is contained in the subgroup $\Sigma^n (K, I) \wedge \Gamma^n$ of $\Sigma^n (\bar{K}, I)$.

LITERATURE.

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