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The extension of topological groups [1]

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1. For given two topological groups $\mathfrak{N}$ and $\mathfrak{A}$, an abstract group $\mathfrak{G}$ which contains a normal subgroup $\mathfrak{N}'$ such that $\mathfrak{N}'$ and $\mathfrak{G}/\mathfrak{N}'$ are algebraically isomorphic with $\mathfrak{N}$ and $\mathfrak{A}$ respectively will be called an *algebraic extension of* $\mathfrak{N}$ *by* $\mathfrak{A}$; while a topological group $\mathfrak{G}$ containing a normal subgroup $\mathfrak{N}'$ such that $\mathfrak{N}'$ and $\mathfrak{G}/\mathfrak{N}'$ are topologically isomorphic (that is, algebraically isomorphic and also homeomorphic) with $\mathfrak{N}$ and $\mathfrak{A}$ respectively will be called a *topological extension of* $\mathfrak{N}$ *by* $\mathfrak{A}$. The problem of algebraic extensions was first formulated and solved by O. Schreier,[2] and afterwards in another way by K. Shoda.[3]

The fundamental theorem of O. Schreier is as follows:

Theorem 1. *Let* $\mathfrak{N}$ *and* $\mathfrak{A}$ *be two abstract groups, and* $A_{\mathfrak{N}}$ *the group formed by all automorphisms of* $\mathfrak{N}$. *If a mapping* $a \to T_a$ *from* $\mathfrak{A}$ *into* $A_{\mathfrak{N}}$ *and a mapping* $(a, b) \to C_{a,b}$ *from* $\mathfrak{A} \times \mathfrak{A}$ *into* $\mathfrak{N}$ *satisfy the following conditions*:

$$e_1) \qquad T_a T_b(A) = C_{a,b}\, T_{ab}(A)\, C_{a,b}^{-1} \qquad (A \in \mathfrak{N})$$

$$e_2) \qquad C_{a,b}\, C_{ab,c} = T_a(C_{b,c})\, C_{a,bc}$$

then an algebraic extension $\mathfrak{G}$ *of* $\mathfrak{N}$ *by* $\mathfrak{A}$ *may be determined as follows*: $\mathfrak{G}$ *is the set of symbols* AS_a, *where* A *and* a *run through* $\mathfrak{N}$ *and* $\mathfrak{A}$ *respectively, and the group operation is defined by* $(AS_a)\cdot(BS_b) = A\,T_a(B)\,C_{a,b}\,S_{ab}$. *Conversely every algebraic extension of* $\mathfrak{N}$ *by* $\mathfrak{A}$ *may be obtained in such a way.*

In this paper, provided every togological group satisfies the first axiom of countability, we shall introduce into an algebraic extension $\mathfrak{G}$ a topology with regard to which $\mathfrak{G}$ is a topoloical extension. Moreover,

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1) The writer is grateful to Prof. K. Shoda, who gave an impulse to the present paper.

2) O. Schreier: Über die Erweiterung von Gruppen, Monatschefte für Math. u. Physik, 34 (1926) 321-346.

3) K. Shoda, Über die Schreiersche Erweiterungstheorie. Proc. Acad. Tokyo (1943) 518-519.

a group of extensions will be defined for abelian groups, and researched in connexion with their character groups.

2. Henceforth, we shall assume that every topological group appearing in this section satisfies the first axiom of countability.

Theorem 2. *Let $\mathfrak{N}$ and $\mathfrak{A}$ be two topological groups, and $\mathfrak{G}$ a topological extension of $\mathfrak{N}$ by $\mathfrak{A}$. Then we can find a system of representatives $\mathfrak{S} = \{S_a\}$ of $\mathfrak{G}$ mod $\mathfrak{N}$ which satisfies the following conditions:*

If we denote $C_{a,b} = S_a S_b S_{ab}^{-1}$ and $T_a(A) = S_a^{-1} A S_a$, then,

e_3) *$C_{e,a} = C_{a,e} = E$, $C_{a,a^{-1}} = E$, where e and E are the identities of $\mathfrak{A}$ and $\mathfrak{G}$ (or $\mathfrak{N}$) respectively.*

e_4) *The mapping $(a, b) \to C_{a,b}$ is continuous at (e, e).*

e_5) *The mapping $b \to D_{a,b} = C_{a\,b} C_{ab,a^{-1}}$ is continuous at e for each a.*

e_6) *For any element A of $\mathfrak{N}$ and its neighborhood (abbreviated "nbd") $U_{\mathfrak{N}}(A)$ in $\mathfrak{N}$ there exist, a nbd of the identity $V_{\mathfrak{A}}$ (abbreviated "e —nbd") of $\mathfrak{A}$ and a nbd $U'_{\mathfrak{N}}(A)$ of A, such that for any $a \in V_{\mathfrak{A}}$, $T_a(U_{\mathfrak{N}}'(A)) \subset U_{\mathfrak{N}}(A)$.*

e_7) *A sequence $\{A_i S_{a_i}\}$ of elements from $\mathfrak{G}$ converges to E if and only if $\{A_i\}$ and $\{a_i\}$ converge to E and e respectively.*

Proof. Let f be the natural homomorphic mapping from $\mathfrak{G}$ onto $\mathfrak{A}$. We can find a complete system $\Sigma = \{W_n\}$ of e-nbd of $\mathfrak{G}$,

$$\mathfrak{G} = W_0 \supset W_1 \supset \ldots \supset W_n \supset \ldots$$

such that $W_n^{-1} = W_n$ and if $m \neq n$ $f(W_m) \neq f(W_n)$. Let $\mathfrak{S} = \{S_a\}$ be a system of representatives such that if $a \ni f(W_n) - (W_{n+1})$, then $S_a \in W_n - W_{n+1}$ and $S_a^{-1} = S_a^{-1}$, $S_e = E$. Then for any $a \in f(W_n)$, $S_a \in W_n$. It can be shown as follows that this satisfies the conditions e_3)—e_7).

e_3) Since $S_e = E$, $C_{e,a} = C_{a,e} = E$, and since $S_{a^{-1}} = S_a^{-1}$, $C_{a,a^{-1}} = E$.

e_4) For any e-nbd $U_{\mathfrak{N}}$ of $\mathfrak{N}$, there exists W_n such that $W_n \frown \mathfrak{N} \subseteq U_{\mathfrak{N}}$. Suppose that $W_m^3 \subset W_n$ and $f(W_l)^2 \subset f(W_m)$. Then $a, b \in f(W_l)$ implies $C_{a,b} \in U_{\mathfrak{N}}$; since $C_{a,b} = S_a S_b S_{ab}^{-1} \subset W_m^3 \frown \mathfrak{N} \subset W_n \frown \mathfrak{N} \subset U_{\mathfrak{N}}$. Hence $(a, b) \to C_{a,b}$ is continuous at (e, e).

e_6) For any nbd $U_{\mathfrak{N}}(A)$ of A in $\mathfrak{N}$, there exists a nbd $W(A)$ of A

in $\mathfrak{G}$ satisfying $W(A) \cap \mathfrak{N} \subset U_{\mathfrak{N}}(A)$. Let W_n and $W'(A)$ be a e-nbd and a nbd of A such that $W_n W'(A) W_n^{-1} \subset W(A)$. Put $U_{\mathfrak{N}}'(A) = \mathfrak{N} \cap W'(A)$ and $V_{\mathfrak{A}} = f(W_n)$. Then $a \in V_{\mathfrak{A}}$ implies $T_a(U_{\mathfrak{N}}'(A)) \subset U_{\mathfrak{N}}(A)$.

e_7) Let $V_{S(\mathfrak{A})}$ be the set of representatives corresponding to elements of $V_{S(\mathfrak{A})}$ and Σ^* the system of subsets of $\mathfrak{G}$ written in the forms $U_{\mathfrak{N}} V_{S(\mathfrak{A})}$, where $U_{\mathfrak{N}}$ and $V_{\mathfrak{A}}$ are e-nbd of $\mathfrak{N}$ and $\mathfrak{A}$ respectively. First of all we show that for any W_n from Σ, there exists a subset W^* from Σ^* satisfying $W^* \subset W_n$, and conversely. For any W_n, there exists W_m such that $W_m^2 \subset W_n$. Set $U_{\mathfrak{N}} = W_m \cap \mathfrak{N}$ and $V_{\mathfrak{A}} = f(W_m)$. Then, since $U_{\mathfrak{N}} \subset W_m$ and $V_{S(\mathfrak{A})} \subset W_m$, $W^* = U_{\mathfrak{N}} V_{S(\mathfrak{A})} \subset W_n$. Conversely, let $W^* = U_{\mathfrak{N}} V_{S(\mathfrak{A})}$ be an arbitrary subset from Σ^*. There exists W_n such that $W_n \cap \mathfrak{N} \subset U_{\mathfrak{N}}$ and $f(W_n) \subset V_{S(\mathfrak{A})}$. Suppose that $W_m W_m^{-1} \subset W_n$, and $W_m = \bigcup_{c \in f(W_n)} U_{\mathfrak{N}}^{(c)} S_c$ where $U_{\mathfrak{N}}^{(c)}$ is an e-nbd of $\mathfrak{N}$. Since $U_{\mathfrak{N}}^{(c)} = U_{\mathfrak{N}}^{(c)} S_c S_c^{-1} \subset W_m W_m^{-1} \cap \mathfrak{N} \subset W_n \cap \mathfrak{N} \subset U_{\mathfrak{N}}$, we get $W_n \subset W^*$. Now, it is almost evident that a sequence $\{A_i S_{a_i}\}$ converges to E if and only if for any W^* there exists an integer i_0 such that $i > i_0$ implies $A_i S_{a_i} \in W^*$, that is, $\{A_i\}$ and $\{a_i\}$ converges to E and e respectively.

e_5) For any a, $U_{\mathfrak{N}}$, and $V_{S(\mathfrak{A})}$, there exist W_n and W_m such that $S_a W_m S_a^{-1} \subset W_n \subset W^* = U_{\mathfrak{N}} V_{S(\mathfrak{A})}$. If $b \in f(W_m)$, then $S_a S_b S_a^{-1} = D_{a,b} S_{aba^{-1}} \in W^*$. That is, $D_{a,b} \in U_{\mathfrak{N}}$. Hence $b \to D_{a,b}$ is continuous at e, q.e.d.

Theorem 3. *Let $\{C_{a,b}\}$ and $\{T_a\}$ satisfy the conditions e_1)—e_6). Then the algebraic extension $\mathfrak{G}$ defined by them becomes a topological extension when we introduce a topology into $\mathfrak{G}$ in such a way that $\{A_i S_{a_i}\}$ converges to E if and only if $\{A_i\}$ and $\{a_i\}$ converge to E and e respectively.*

Proof. Let $\mathfrak{S} = \{S_a\}$ be a system of representatives of $\mathfrak{G}$ mod $\mathfrak{N}$ such that $S_a A S_a^{-1} = T_a(A)$ and $C_{a,b} = S_a S_b S_{ab}^{-1}$ and Σ^* the sysem of subsets with forms $U_{\mathfrak{N}} V_{S(\mathfrak{A})}$ where $U_{\mathfrak{N}}$ and $V_{\mathfrak{A}}$ are e-nbd of $\mathfrak{N}$ and $\mathfrak{A}$ respectively. Then it may be easily shown that Σ^* satisfies the following conditions:

a) $\bigcap_{W^* \in \Sigma^*} W^* = E$.

b) For any two subsets W_1^*, W_2^* from Σ^*, there exists another subset W_3^* from Σ^* satisfying $W_3^* \subset W_1^* \cap W_2^*$.

c) For any subset W_1^*, there exists another subset W_2^* from Σ^* satisfying $W_2^* W_2^{*-1} \subset W_1^*$.

d) For any subset W_1^* from Σ^* and any element G from $\mathfrak{G}$, there exists a subset W_2^* from Σ^* such that $G W_2^* G^{-1} \subset W_1^*$.

If we define a topology of $\mathfrak{G}$ by Σ^*, that is, $\{G_n\}$ converges to G if and only if for any $W^* \in \Sigma^*$, there exists n_0 such that $n > n_0$ implies $G_n G^{-1} \in W^*$, then $\mathfrak{G}$ becomes a topological group, and furthermore a topological extension of $\mathfrak{N}$ by $\mathfrak{A}$. In this case, evidently $\{A_i S_{a_i}\}$ converges to E if and only if A_i and a_i converge to E and e respectively. Accordingly our assertion has been proved.

Combining Theorem 2 and Theorem 3, we have

Theorem 4. *For two given topological groups $\mathfrak{N}$ and $\mathfrak{A}$ satisfying the first axiom of countability, every extension of $\mathfrak{N}$ by $\mathfrak{A}$ satisfying the same axiom may be determined in the sense of Theorem 3 by $\{C_{a,b}\}$ and $\{T_a\}$ which satisfy the conidtions $e_1) - e_6)$.*

3. For an algebraic extension $\mathfrak{G}$ of $\mathfrak{N}$ by $\mathfrak{A}$, there may exist many different topologies with regard to which $\mathfrak{G}$ becomes a topological extension. But such a topology is determined by some system of representatives of $\mathfrak{G}$ mod $\mathfrak{N}$, and concerning the relation between the topologies wrich are given by the systems $\mathfrak{S} = \{S_a\}$ and $\mathfrak{S}^* = \{S_a^*\}$, we get

Theorem 5. *Two topologies given by $\mathfrak{S}$ and $\mathfrak{S}^*$ coincide with each other if and only if $a \to A_a = S_a^* S_a^{-1}$ is continuous at e.*

Proof. Set $\Sigma = \{W = U_{\mathfrak{N}} V_{S(\mathfrak{A})}\}$ and $\Sigma^* = \{W^* = U_{\mathfrak{N}} V_{S^*(\mathfrak{A})}\}$.

1) Suoppose that for any W from Σ there exists W^* from Σ^* satisfying $W^* \subset W$. We shall verify that $a \to A_a = S_a^* S_a^{-1}$ is continuous at e under this supposition.

Let $U_{\mathfrak{N}}$ be an arbitrary e-nbd of $\mathfrak{N}$. For $W = U_{\mathfrak{N}} V_{S(\mathfrak{A})}$, there exists $W' = U_{\mathfrak{N}}' V'_{S(\mathfrak{A})}$ such that $W' W'^{-1} \subset W$, and for such W' there exists $W^* = U_{\mathfrak{N}}'' V''_{S^*(\mathfrak{A})}$ satisfying $W^* \subset W'$. Then since S_a and S_a^* are contained in W', $a \in V_{\mathfrak{A}}''$ implies $A_a = S_a^* S_a^{-1} \in U_{\mathfrak{N}}$. That is, $a \to A_a$ is continuous at e.

2) Suppose that $a \to A_a = S_a^* S_a^{-1}$ is continuous at e. Let $W =$

$U_{\mathfrak{N}} V_{S(\mathfrak{A})}$ be an arbitrary subset from Σ, and $U_{\mathfrak{N}}'$ an e-nbd safissfying $U_{\mathfrak{N}}'^2 \subset U_{\mathfrak{N}}$. Then there exist an e-nbd $V_{\mathfrak{A}}'$ of $\mathfrak{A}$ such that $V_{\mathfrak{A}}' \subset V_{\mathfrak{A}}$ and if $a \varepsilon V_{\mathfrak{A}}'$ then $A_a = S_a^* S_a^{-1} \varepsilon U_{\mathfrak{N}}'$. Since $a \varepsilon V_{\mathfrak{A}}'$ implies $S_a^* \varepsilon U_{\mathfrak{N}}' S_a$, $W^* = U_{\mathfrak{N}}' V_{S^*(\mathfrak{A})} \subset U_{\mathfrak{N}}'^2 V_{S(\mathfrak{A})} \subset W$. On the other hand, if $a \rightarrow A_a = S_a^* S_a^{-1}$ is continuous at e then $a \rightarrow A_a^{-1} = S_a S_a^{*-1}$ is also continuous at e, therefore for any $W^* \varepsilon \Sigma^*$ there exists $W \varepsilon \Sigma$ such that $W \subset W^*$. Hence from our supposition that $a \rightarrow A_a$ is continuous at e, we can conclude that $\mathfrak{S}$ and $\mathfrak{S}^*$ give the same topology to $\mathfrak{G}$, q.e.d.

In this proof, it is remarkable that if one of the topologies of $\mathfrak{G}$ introduced by $\mathfrak{S}$ and $\mathfrak{S}^*$ is weaker than the other, then from 1), $a \rightarrow A_a$ is continuous at e, and hence from 2) the two topologies are the same. This fact may be formulated as follows:

Theorem 6. *Suppose that $\mathfrak{G}$ is algebraically isomorphic to $\mathfrak{G}'$, and also by this isomorphic mapping f from $\mathfrak{G}$ onto $\mathfrak{G}'$, $\mathfrak{G}$ is continuoucly mapped onto $\mathfrak{G}'$. Suppose further that by the mapping f, a closed normal subgroup $\mathfrak{N}$ and the facter group $\mathfrak{G}/\mathfrak{N}$ is homeomorphic with $f(\mathfrak{N})$ and $\mathfrak{G}'/f(\mathfrak{N})$ respectively. Then $\mathfrak{G}$ is homeomorphic with $\mathfrak{G}'$.*

4. Let $\mathfrak{G}$ be an extension of $\mathfrak{N}^*$ by $\mathfrak{A}$, and f the natural homomorphic mapping from $\mathfrak{G}$ onto $\mathfrak{A}$ with the kernel $\mathfrak{N}$. Let further φ be an isomorphic mapping from $\mathfrak{N}$ onto $\mathfrak{N}^*$. Then $(\mathfrak{G}, f, \varphi)$ will be called a type of extension of $\mathfrak{N}^*$ by $\mathfrak{A}$, and we shall define that two types $(\mathfrak{G}, f, \varphi)$ and $(\mathfrak{G}', f', \varphi')$ are the same if and only if there exists a topologically isomorphic mapping π from $\mathfrak{G}$ onto $\mathfrak{G}'$ such that $\pi(\mathfrak{N}) = \mathfrak{N}'$ (where $\mathfrak{N}$ and $\mathfrak{N}'$ are the kernels of f and f' respectively), for any G from $\mathfrak{G}$ $f'(\pi(G)) = f(G)$, and for any A from $\mathfrak{N}$ $\varphi'(\pi(A)) = \varphi(A)$.

Given a type $(\mathfrak{G}, f, \varphi)$ of extension of $\mathfrak{N}^*$ by $\mathfrak{A}$, and let $\mathfrak{S} = \{S_a\}$ be a system of representatives defining the topology of $\mathfrak{G}$, and set $C_{a,b}^* = \varphi(S_a S_b S_{ab}^{-1})$, $T_a^*(A^*) = \varphi(S_a \varphi^{-1}(A^*) S_a^{-1}) (A^* \varepsilon \mathfrak{N}^*)$. Then the system $\{C_{a,b}^*, T_a^*\}$ satisfies the conditions $e_1) - e_6)$. Conversely any system $\{C_{a,b}^*\}$ satisfying $e_1) - e_6)$ determines a type of extension. But two different systems may determine the same type, and about this we have easily the following theorem:

Theorem 7. $\{C_{a,b}^*, T_a^*\}$ *and* $\{D_{a,b}^*, R_a^*\}$ *determine the same type* *if and only if there exists a mapping* $a \to A_a^*$ *from* $\mathfrak{A}$ *into* $\mathfrak{N}^*$ *satisfying the following conditions:*

1) $a \to A_a^*$ *is continuous at* e.

2) $R_a^*(A^*) = A_a^* T_a^*(A^*) A_a^{*-1}$.

3) $D_{a,b}^* = A_a^* T_a^*(A_b^*) C_{a,b}^* A_{a,b}^{*-1}$.

Now let $\mathfrak{N}^*$ and $\mathfrak{A}$ be commutative and locally compact groups satisfying the second axiom of countability. Every extension of $\mathfrak{N}$ by $\mathfrak{A}$ is also locally compact and satisfies the second axiom of countability. Hereafter, we shall restrict ourselves to commutative extensions.

Let $(\mathfrak{G}, f, \varphi)$ and $(\mathfrak{G}', f', \varphi')$ be two types of extensions of $\mathfrak{N}^*$ by $\mathfrak{A}$. In $\mathfrak{G} \times \mathfrak{G}'$ the set of all elements GG' satisfying $f(G) = f'(G')$ forms a closed subgroup $\mathfrak{H}$, and the set of all elements NN'^{-1} of $\mathfrak{N} \times \mathfrak{N}'$ such that $\varphi(N) = \varphi'(N')$ forms also a closed subgroup $\mathfrak{M}$. Set $\mathfrak{H}^* = \mathfrak{H}/\mathfrak{M}$. Then by the mapping $\varphi^* : \mathfrak{M}N \to \varphi(N)$ and $f^* : \mathfrak{M}\mathfrak{N}GG' \to f(G) = f(G')$, $\mathfrak{M}\mathfrak{N}/\mathfrak{M}$ and $\mathfrak{H}/\mathfrak{M}\mathfrak{N}$ are isomorphic to $\mathfrak{N}^*$ and $\mathfrak{A}$ respectively.

Hence $(\mathfrak{H}^*, f^*, \varphi^*)$ is a type of extension of $\mathfrak{N}^*$ by $\mathfrak{A}$. Moreover if $(\mathfrak{G}, f, \varphi)$ and $(\mathfrak{G}', f', \varphi')$ are determined by $\{C_{a,b}^*\}$ and $\{D_{a,b}^*\}$ respectively, then $(\mathfrak{H}^*, f^*, \varphi^*)$ is determined by $\{C_{a,b}^* D_{a,b}^*\}$. It will be easily seen that by the definition of product $(\mathfrak{H}^*, f^*, \varphi^*) = (\mathfrak{G}, f, \varphi) \circ (\mathfrak{G}', f', \varphi')$, the set of all different types forms an abstract group. It will be called the group of extensions of $\mathfrak{N}^*$ by $\mathfrak{A}$, and denoted by $\varepsilon(\mathfrak{N}^*, \mathfrak{A})$.

5. Let $(\mathfrak{G}, f, \varphi)$ be a type of commutative extension of $\mathfrak{N}^*$ by $\mathfrak{A}$, where $\mathfrak{N}^*$ and $\mathfrak{A}$ are both commutative and locally compact topological groups satisfying the second axiom of countability. We shall denote a character group by $\chi(\ )$. $\chi(\mathfrak{G})$ may be regarded as an extension of $\chi(\mathfrak{A})$ by $\chi(\mathfrak{N})$. Let $\mathfrak{N}$ be the kernel of f and Φ the subgroup of $\chi(\mathfrak{G})$ corresponding to $\mathfrak{N}$. Let further $\mathfrak{S} = \{S_a\}$ be a system of representatives of $\mathfrak{G}$ mod $\mathfrak{N}$. Then, for any α from Φ, $\overline{\varphi}(\alpha) : a \to S_a{}^\alpha$ is a character of $\mathfrak{A}$, and $\overline{\varphi}$ is a topologically isomorphic mapping from Φ onto $\chi(\mathfrak{G})$. On the other hand, for any α from $\chi(\mathfrak{G})$, $\overline{f}(\alpha) : N^* \to (\varphi^{-1}(N\ ))^\alpha$ is a character of $\mathfrak{N}^*$, and $\overline{f}$ is an open homomorphic mapping from $\chi(\mathfrak{G})$ onto $\chi(\mathfrak{N}^*)$ whose kernel is Φ. Accordingly, $(\chi(\mathfrak{G}), \overline{f}, \overline{\varphi})$ is a type of

extension of $\chi(\mathfrak{N}^*)$ by $\chi(\mathfrak{A})$.

Theorem 8. *The mapping* $(\mathfrak{G}, f, \varphi) \to (\chi(\mathfrak{G}), \bar{f}, \bar{\varphi})$ *is an ismorphic mapping of* $\varepsilon(\mathfrak{N}^* \mathfrak{A})$ *onto* $\varepsilon(\chi(\mathfrak{A}), \chi(\mathfrak{N}^*))$.

Proof. Obviously, it gives a one to one correspondence between them.

We shall describe the group opesations of character groups by additions. Let $(\mathfrak{G}, f, \varphi)$ and $(\mathfrak{G}', f', \varphi')$ be two types of extensions of $\mathfrak{N}^*$ by $\mathfrak{A}$, $\mathfrak{N}$ and $\mathfrak{N}'$ the kernels of f and f', and Φ, Φ' the subgroups of $\chi(\mathfrak{G})$ and $\chi(\mathfrak{G}')$ correspoding to $\mathfrak{N}$ and $\mathfrak{N}'$ respectively. Let further $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{H}}$ be the subgroups of $\chi(\mathfrak{G} \times \mathfrak{G}')$ corresponding to the subgroups $\mathfrak{H}$ and $\mathfrak{M}$ of $\mathfrak{G} \times \mathfrak{G}'$ defined in section 4. In order that an element $\alpha + \alpha'$ from $\chi(\mathfrak{G}) + \chi(\mathfrak{G}')(= \chi(\mathfrak{G} \times \mathfrak{G}'))$ is contained in $\overline{\mathfrak{M}}$ it is necessary and sufficient that $f(G) = f'(G)$ implies $G^{\alpha} = -G'^{\alpha'}$, that is, $\alpha \varepsilon \Phi$, $\alpha' \varepsilon \Phi'$ and $\bar{\varphi}(\alpha) = -\bar{\varphi}'(\alpha')$. Hence $\overline{\mathfrak{M}}$ consists of all elements $\alpha - \alpha'$ satisfying $\alpha \varepsilon \Phi$, $\alpha' \varepsilon \Phi'$, and $\bar{\varphi}(\alpha) = -\bar{\varphi}'(\alpha')$. On the other hand, $\alpha + \alpha'$ is contained in $\overline{\mathfrak{H}}$ if and only if $\varphi(N) = \varphi'(N')$ implies $N^{\alpha} = N'^{\alpha'}$, and hence $\overline{\mathfrak{H}}$ consists of all elements $\alpha + \alpha'$ satisfying $\bar{f}(\alpha) = \bar{f}'(\alpha)$. Accordingly, the type corresponding to $(\mathfrak{G}, f, \varphi) \circ (\mathfrak{G}', f', \varphi')$ is $(\chi(\mathfrak{G}), \bar{f}, \bar{\varphi}) \circ (\chi(\mathfrak{G}'), \bar{f}', \bar{\varphi}')$. q.e.d.

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