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WAVE AND INVERSE WAVE OPERATORS FOR THE QUADRATIC NONLINEAR SCHRÖDINGER EQUATIONS IN 3D

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Abstract

Our purpose in this paper is to prove existence of wave operator $\mathcal{W}_+$ or inverse wave operator $\widetilde{\mathcal{W}}_+$ for nonlinear Schrödinger equations with quadratic nonlinearities in three space dimensions. Our results show that the mapping $\mathcal{W}_+\widetilde{\mathcal{W}}_+$ is well defined and are improvement of results on the range of inverse wave operator obtained in [6].

1. Introduction

In this paper, we study asymptotic properties of small solutions for nonlinear Schrödinger equations with quadratic nonlinearities in three space dimensions:

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda u^2 + \mu\overline{u}^2, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^3, \tag{1.1}$$

where $\overline{u}$ is the complex conjugate of u and $\lambda,\ \mu \in \mathbf{C}$.

The purposes of this paper are twofold. One of them is to show that the inverse wave operator for (1.1) can be defined in a suitable Banach space. In order to do it, we consider the initial value problem:

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = \lambda u^2 + \mu\overline{u}^2, & (t, x) \in \mathbf{R} \times \mathbf{R}^3, \\ u(0, x) = u_0(x), & x \in \mathbf{R}^3 \end{cases} \tag{1.2}$$

and we find a unique global solution to (1.2) under the conditions that the initial function is small in a Banach space $\mathbf{Y}$, and $\{\mathcal{U}(-t)u(t)\}_{t\geq 0}$ is a Cauchy sequence in a Banach space $\mathbf{X}$ ($\mathbf{Y}$ and $\mathbf{X}$ will be defined in the theorem below precisely), namely

$$\lim_{t,s\to\infty} \|\mathcal{U}(-t)u(t) - \mathcal{U}(-s)u(s)\|_{\mathbf{X}} = 0, \tag{1.3}$$

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where the free Schrödinger evolution group $\mathcal{U}(t)$ is given by

$$\mathcal{U}(t)\psi = \frac{1}{(2\pi it)^{3/2}} \int e^{i|x-y|^2/(2t)}\psi(y)\,dy.$$

By (1.3) we see that there exists a unique $\phi_+ \in \mathbf{X}$ such that

$$\lim_{t\to\infty} \|\mathcal{U}(-t)u(t) - \phi_+\|_{\mathbf{X}} = 0$$

which means that the operator $\widetilde{\mathcal{W}}_+ : u_0 \mapsto \phi_+$ is well defined. We call $\widetilde{\mathcal{W}}_+$ the inverse wave operator and denote a range of $\widetilde{\mathcal{W}}_+$ by $R(\widetilde{\mathcal{W}}_+)$ and a domain of $\widetilde{\mathcal{W}}_+$ by $D(\widetilde{\mathcal{W}}_+)$. Then we see that $\mathbf{Y} \supset D(\widetilde{\mathcal{W}}_+)$ and $\mathbf{X} \subset R(\widetilde{\mathcal{W}}_+)$. Another purpose is to consider the final states problem for (1.1)

$$\text{(1.4)} \qquad \begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = \lambda u^2 + \mu\overline{u}^2, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^3, \\ \mathcal{U}(-\infty)u(\infty) = u_+ \end{cases}$$

which is written as the integral equation

$$\text{(1.5)} \qquad u(t) = \mathcal{U}(t)u_+ + i\int_t^\infty \mathcal{U}(t-\tau)(\lambda u^2 + \mu\overline{u}^2)(\tau)\,d\tau, \quad (t, x) \in \mathbf{R} \times \mathbf{R}^3,$$

for a given $u_+ \in \mathbf{X}_1$ which is small and we prove existence of a unique global solution $u(t) \in \mathbf{C}([1, \infty); \mathbf{Y}_1)$ of (1.5) for $t \geq 1$ under the condition that

$$\lim_{t\to\infty} \|u(t) - \mathcal{U}(t)u_+\|_{\mathbf{Y}_1} = 0,$$

where Banach spaces $\mathbf{X}_1$ and $\mathbf{Y}_1$ will be defined in the theorem below precisely. Then the operator $\mathcal{W}_+ : u_+ \mapsto u(1)$ is well defined and we call $\mathcal{W}_+$ the wave operator. It is easy to see that $\mathbf{X}_1 \supset D(\mathcal{W}_+)$ and $\mathbf{Y}_1 \subset R(\mathcal{W}_+)$. If we can show the above two existence results under the condition $\mathbf{X}_1 = \mathbf{X}$, we have $D(\mathcal{W}_+) \subset \mathbf{X}_1 = \mathbf{X} \subset R(\widetilde{\mathcal{W}}_+)$ which implies that the operator $\mathcal{W}_+\widetilde{\mathcal{W}}_+$ is well defined as the mapping from the neighborhood of the origin of $\mathbf{Y}$ into $\mathbf{Y}_1$. This is the main result. However, existence of the scattering operator $\widetilde{\mathcal{W}}_+\mathcal{W}_+$ is still an open problem.

Many works have been devoted to study of global existence and asymptotic behavior of solutions to quadratic nonlinear Schrödinger equations (see e.g., [4], [5], [6], [10], [11], [13] and [14]). In [6], global existence and asymptotic behavior in time of small solutions to the initial value problem (1.2) was studied when the initial condition $u_0 \in \mathbf{Y} \equiv \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2}$ and small, where the weighted Sobolev space $\mathbf{H}_p^{m,k}$ is defined by

$$\mathbf{H}_p^{m,k} = \{\phi \in \mathcal{S}' : \|\phi\|_{\mathbf{H}_p^{m,k}} = \|(1 + |x|^2)^{k/2}(1 - \Delta)^{m/2}\phi\|_{\mathbf{L}^p} < \infty\},$$

with $m, k \in \mathbf{R}$, $1 \le p \le \infty$. For simplicity, we denote $\mathbf{H}^{m,k} = \mathbf{H}_2^{m,k}$ and $\|\cdot\|_{\mathbf{H}^{m,k}} = \|\cdot\|_{\mathbf{H}_2^{m,k}}$. They obtained $\mathbf{L}^\infty$ time decay estimate of solutions to (1.2):

$$\|u(t)\|_{\mathbf{L}^\infty} \le Ct^{-3/2} \quad \text{for} \quad t > 1 \tag{1.6}$$

and the following existence theorem: for any small $u_0 \in \mathbf{Y}$, there exists a unique final state $\phi_+ \in \mathbf{X} \equiv \mathbf{L}^2 \cap \mathcal{F}\mathbf{L}^\infty$ satisfying the estimate

$$\|u(t) - \mathcal{U}(t)\phi_+\|_{\mathbf{L}^2} \le Ct^{-1/2} \quad \text{for} \quad t > 1.$$

Order of time decay was improved by [10] as

$$\|u(t) - \mathcal{U}(t)\phi_+\|_{\mathbf{L}^2} \le Ct^{-5/4} \quad \text{for} \quad t > 1, \tag{1.7}$$

where $\mathcal{F}\psi$ is the Fourier transform of ψ defined by

$$\mathcal{F}\psi \equiv \hat{\psi} = \frac{1}{(2\pi)^{3/2}} \int e^{-ix\xi}\, \psi(x)\, dx$$

and

$$\mathcal{F}^{-1}\psi = \frac{1}{(2\pi)^{3/2}} \int e^{ix\xi}\, \psi(\xi)\, d\xi$$

is the inverse Fourier transform of ψ. The function space $\mathcal{F}\mathbf{L}^\infty$ is defined by

$$\mathcal{F}\mathbf{L}^\infty = \{\phi \in \mathcal{S}' : \mathcal{F}\phi \in \mathbf{L}^\infty\}.$$

Thus from the results in [6], we can define the inverse wave operator $\tilde{\mathcal{W}}_+ : \mathbf{Y} \to \mathbf{X}$. In [5], sharp time asymptotics of solutions around the final states of the equation (1.5) was obtained. In particular, time asymptotics of solutions from below was studied. Also a wave operator $\mathcal{W}_+ : u_+ \mapsto u(1) \in \mathbf{L}^2$ was constructed for the final function $u_+ \in \mathbf{X}_1 \equiv \mathbf{H}^{0,3} \cap \mathbf{H}^{3,0} \cap \mathbf{H}_1^{2,0}$. However it is impossible to define the operator $\mathcal{W}_+\tilde{\mathcal{W}}_+$, since $\mathbf{X}_1 \subsetneqq \mathbf{X}$.

It is important to define the scattering operator $\tilde{\mathcal{W}}_+\mathcal{W}_+$ for scattering theory. There is a large amount of literature on the scattering problem for Schrödinger equations with nonlinearities satisfying the gauge invariant condition (see e.g., [1], [2], [3], [7] and [12]). Asymptotic completeness is shown if we prove $\mathbf{Y} = \mathbf{X} = \mathbf{X}_1 = \mathbf{Y}_1$. However it is still unsolved for nonlinear Schrödinger equations with quadratic nonlinearities even in the case of three space dimensions. The asymptotic completeness with the gauge invariant quadratic nonlinearity $\lambda|u|u$ was shown in [1] and [12] for $\lambda > 0$ and for $\lambda < 0$ if the data are small.

Before stating our main result, we introduce some notations and function spaces which are used in this paper. We have the following identity

$$\mathcal{U}(t) = M(t)\mathcal{D}(t)\mathcal{F}M(t), \tag{1.8}$$

where,

$$M(t) = e^{i|x|^2/(2t)}, \quad (\mathcal{D}(t)\phi)(x) = (it)^{-3/2}\phi\left(\frac{x}{t}\right)$$

and the method of factorization was used by N. Hayashi and T. Ozawa [9] to study scattering problem. We also define the following function space:

$$\mathbf{Z}_T = \{\phi \in \mathbf{C}([T, \infty); \mathbf{L}^2) : \|\phi\|_{\mathbf{Z}_T} < \infty\},$$

where

$$\|\phi\|_{\mathbf{Z}_T} = \sup_{t\in[T,\infty)} (t^{3/4}\|\phi(t)\|_{\mathbf{L}^4} + t^{1/2}\|\phi(t)\|_{\mathbf{L}^2}).$$

For simplicity, we write $\langle x\rangle = (1 + |x|^2)^{1/2}$.

We state our main results

Theorem 1.1. *Let $u_0 \in \mathbf{Y} \equiv \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2}$ and $\varepsilon = \|u_0\|_{\mathbf{Y}}$. Then there exists an $\varepsilon > 0$ such that* (1.2) *has a unique global solution $u \in \mathbf{C}(\mathbf{R}; \mathbf{Y})$ which satisfies*

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon^{1/2}\langle t\rangle^{-3/2}, \quad \|u(t)\|_{\mathbf{L}^2} \le C\varepsilon^{1/2}.$$

Moreover, there exists a unique final state $\phi_+ \in \mathbf{X} \equiv \mathbf{H}^{1,1}$ such that

$$\|\mathcal{U}(-t)u(t) - \phi_+\|_{\mathbf{X}} \le C\varepsilon t^{-1/4}$$

for $t > 1$. Namely, the inverse wave operator $\widetilde{\mathcal{W}}_+$ is the mapping from the neighborhood of the origin of $\mathbf{Y}$ into $\mathbf{X}$.

Theorem 1.2. *Let $u_+ \in \mathbf{X}_1 \equiv \mathbf{H}^{1,1}$ and the norm $\rho = \|u_+\|_{\mathbf{X}_1}$ be sufficiently small. Then for any positive time $T \ge 1$ there exists a unique solution $u - \mathcal{U}(t)u_+ \in \mathbf{Z}_T$ to* (1.5). *Namely, the wave operator $\mathcal{W}_+$ is the mapping from the neighborhood of the origin of $\mathbf{X_1}$ into $\mathbf{Y_1} \equiv \mathbf{L}^2$.*

By the above theorems we have

Corollary 1.1. *The operator $\mathcal{W}_+\widetilde{\mathcal{W}}_+$ is well defined as the mapping from the neighborhood of the origin of $\mathbf{Y} \equiv \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2}$ into $\mathbf{Y_1} \equiv \mathbf{L}^2$.*

REMARK. Our strategy is to make use of an oscillation property of nonlinearities u^2 and $\overline{u}^2$. However $u\overline{u}$ does not have such property. Therefore, it is difficult to deal with the nonlinearity $u\overline{u}$ by our method.

This paper is organized as follows. In Section 2, we prepare preliminary lemmas. In Section 3, we prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2.

2. Preliminaries

We use the notations

$$\|u\|_{\mathbf{W}} = \|u\|_{\mathbf{H}^{3,0}} + \|\mathcal{J}u\|_{\mathbf{H}^{2,0}} + \langle t\rangle^{-1/2}\|\mathcal{J}^2u\|_{\mathbf{H}^{1,0}} + \langle t\rangle^{3/2}\|u\|_{\mathbf{H}^{1,0}_{\infty}}, \quad \mathcal{J} = x + it\nabla.$$

In this section, we summarize some lemmas and the results obtained in [6] to show Theorem 1.1. We introduce the factorization technique of the free Schrödinger evolution group. It is useful to study the nongauge invariant nonlinearities. We can find the following lemma in the proof of Lemma 3.1 of [8].

Lemma 2.1. *Let $\rho \neq 0$ and $E = e^{it|\xi|^2/2}$. Then*

$$\mathcal{F}\overline{M}\mathcal{F}^{-1}\mathcal{D}\left(\frac{1}{t}\right)\overline{M} = i^{3/2}\mathcal{D}(\rho)E^{\rho^2-\rho}\mathcal{F}\overline{M}^{1/\rho}\mathcal{F}^{-1}\mathcal{D}\left(\frac{1}{t}\right)\overline{M}^{\rho}. \tag{2.1}$$

For the proof, see Lemma 2.1 in [10]. In order to prove Theorem 1.1, we prepare the following lemma.

Lemma 2.2. *Let u be a solution of* (1.2), *and put $\alpha_1 = -3/2$, $\alpha_2 = -1/2$, $\mathcal{A}_\alpha = (i\alpha + (t/4)\Delta)^{-1}$. Then the equality*

$$\mathcal{L}\Psi = \sum_{j=1}^{4} I_j \tag{2.2}$$

is valid, where

$$\mathcal{L} = i\partial_t + \frac{1}{2}\Delta, \quad \Psi = \mathcal{J}u - \frac{i}{2}t^2\nabla(\lambda\mathcal{A}_{\alpha_1}u^2 + \mu\mathcal{A}_{\alpha_2}\overline{u}^2),$$

$$I_1 = \lambda u\mathcal{J}u + \mu\overline{u\mathcal{J}u}, \quad I_2 = -it^2\nabla\big(\lambda\mathcal{A}_{\alpha_1}(u\mathcal{L}u) - \mu A_{\alpha_2}\big(\overline{u}\overline{\mathcal{L}u}\big)\big),$$

$$I_3 = \frac{i\lambda}{2}\left(\alpha_1\mathcal{A}_{\alpha_1} + i\left(\frac{1}{2} + \alpha_1\right)\right)t\nabla\mathcal{A}_{\alpha_1}u^2 + \frac{i\mu}{2}\left(\alpha_2\mathcal{A}_{\alpha_2} + i\left(\frac{1}{2} + 3\alpha_2\right)\right)t\nabla\mathcal{A}_{\alpha_2}\overline{u}^2,$$

$$I_4 = \frac{i\lambda}{4}\nabla\mathcal{A}_{\alpha_1}((\mathcal{J}u)^2 - u\mathcal{J}^2u) - \frac{i\mu}{4}\nabla\mathcal{A}_{\alpha_2}\big(\big(\overline{\mathcal{J}u}\big)^2 - \overline{u}\overline{\mathcal{J}^2u}\big).$$

In the proof of Proposition 3.2 of [6], the above lemma was shown. We state the estimates involving the operator $\mathcal{A}_\alpha$ in the following lemma.

Lemma 2.3 ([6], Lemma 2.3). *Let* $1 \le q \le p \le \infty$, $l = 0, 1$, *and* $\mathcal{A}_\alpha = (i\alpha + (t/4)\Delta)^{-1}$.
(i) *Suppose that* $u \in \mathbf{L}^q$ *and that* $(3/2)(1/q - 1/p) + l/2 < 1$. *Then*

$$\|\nabla^l \mathcal{A}_\alpha u\|_{\mathbf{L}^p} \le Ct^{-(3/2)(1/q-1/p)-l/2}\|u\|_{\mathbf{L}^q}$$

for $t > 0$.
(ii) *Suppose* $p \in [2, 6)$ *and* $q \in [2, \infty)$. *Then*

$$\|\nabla^l \mathcal{A}_\alpha uv\|_{\mathbf{L}^2} \le Ct^{-l/2}(\|u\mathcal{J}v\|_{\mathbf{L}^p} + \|v\mathcal{J}u\|_{\mathbf{L}^p} + \langle t\rangle^{1/2}\|uv\|_{\mathbf{L}^p} + \langle t\rangle^{3/4}\|uv\|_{\mathbf{L}^q}),$$
$$\|\nabla^l \mathcal{A}_\alpha u^3\|_{\mathbf{L}^2} \le Ct^{-l/2}(\|u^2\mathcal{J}u\|_{\mathbf{L}^p} + \langle t\rangle^{1/2}\|u^3\|_{\mathbf{L}^p} + \langle t\rangle^{3/4}\|u^3\|_{\mathbf{L}^q})$$

and

$$\|\nabla^l \mathcal{A}_\alpha u^2\overline{v}\|_{\mathbf{L}^2} \le Ct^{-l/2}\big(\big\|u^2\overline{\mathcal{J}v}\big\|_{\mathbf{L}^p} + \|u\overline{v}\mathcal{J}u\|_{\mathbf{L}^p} + \langle t\rangle^{1/2}\|u^2\overline{v}\|_{\mathbf{L}^p} + \langle t\rangle^{3/4}\|u^2\overline{v}\|_{\mathbf{L}^q}\big)$$

for $t > 0$ *provided that the right hand sides are finite.*

Global existence of small solutions for the Cauchy problem (1.2) is obtained in the following proposition.

Proposition 2.1 ([6], Proposition 3.2). *Assume that* $u_0 \in \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2}$ *and* $\|u_0\|_{\mathbf{H}^{3,0}} + \|u_0\|_{\mathbf{H}^{1,2}} = \varepsilon$. *Then there exists an* $\varepsilon > 0$ *such that* (1.2) *has a unique global solution* u *satisfying* $u \in \mathbf{C}([0, \infty); \mathbf{H}^{3,0} \cap \mathbf{H}^{1,2})$ *and*

$$\sup_{t\ge 0}\|u(t)\|_{\mathbf{W}} < \varepsilon^{1/2}.$$

Furthermore, we have the estimates

$$\|I_1\|_{\mathbf{H}^{2,0}} < C\varepsilon\langle t\rangle^{-5/4}, \quad \|I_2\|_{\mathbf{H}^{2,0}} < C\varepsilon\langle t\rangle^{-3/2},$$
$$\|I_3\|_{\mathbf{H}^{2,0}} < C\varepsilon\langle t\rangle^{\theta-3/2}, \quad \|I_4\|_{\mathbf{H}^{2,0}} < C\varepsilon\langle t\rangle^{-5/4},$$

for any $t \in [0, \infty)$ *and small* $\theta > 0$.

3. Proof of Theorem 1.1

By Proposition 2.1, it is sufficient to show that there exists a unique final state $\phi_+ \in \mathbf{X} \equiv \mathbf{H}^{1,1}$ such that

$$\|\mathcal{U}(-t)u(t) - \phi_+\|_{\mathbf{X}} \le C\varepsilon t^{-1/4} \quad \text{for} \quad t > 1. \tag{3.1}$$

By the integral equation associated with (2.2), we obtain

$$\Psi(t) = \mathcal{U}(t)\Psi(0) - i\int_0^t \mathcal{U}(t-\tau)\sum_{j=1}^4 I_j\, d\tau. \tag{3.2}$$

It follows from Proposition 2.1 and (3.2) that

$$\begin{aligned}\|\mathcal{U}(-t)\Psi(t)-\mathcal{U}(-s)\Psi(s)\|_{\mathbf{H}^{1,0}} &\le C\int_s^t \sum_{j=1}^4 \|I_j\|_{\mathbf{H}^{1,0}}\,d\tau \\ &\le C\varepsilon\int_s^t (\tau^{-5/4}+\tau^{-3/2}+\tau^{\theta-3/2})\,d\tau \le C\varepsilon s^{-1/4}\end{aligned} \tag{3.3}$$

for $t > s > 1$ and small $\theta > 0$. By virtue of Lemma 2.3, letting $p \in [2, 6)$ be close to 6 and $1/q = 1/p - 1/6$, we also have

$$\|\nabla \mathcal{A}_\alpha u^2\|_{\mathbf{L}^2} \le Ct^{-1/2}(\|u\mathcal{J}u\|_{\mathbf{L}^p} + t^{1/2}\|u^2\|_{\mathbf{L}^p} + t^{3/4}\|u^2\|_{\mathbf{L}^q}) \le C\varepsilon t^{\theta-5/2}, \tag{3.4}$$

since by Hölder's and Sobolev's inequalities

$$\begin{aligned}&\|u\|_{\mathbf{L}^q} \le \|u\|_{\mathbf{L}^2}^{2/q}\|u\|_{\mathbf{L}^\infty}^{1-2/q} \le C\varepsilon^{1/2}t^{\theta-3/2},\\ &\|u\mathcal{J}u\|_{\mathbf{L}^p} \le C\|u\|_{\mathbf{L}^q}\|\mathcal{J}u\|_{\mathbf{L}^6} \le Ct^{-1}\|u\|_{\mathbf{L}^q}\|\mathcal{J}^2u\|_{\mathbf{L}^2} \le C\varepsilon t^{\theta-2},\\ &\|u^2\|_{\mathbf{L}^p} \le C\|u\|_{\mathbf{L}^q}\|u\|_{\mathbf{L}^6} \le Ct^{-1}\|u\|_{\mathbf{L}^q}\|\mathcal{J}u\|_{\mathbf{L}^2} \le C\varepsilon t^{\theta-5/2},\\ &\|u^2\|_{\mathbf{L}^q} \le \|u\|_{\mathbf{L}^q}\|u\|_{\mathbf{L}^\infty} \le C\varepsilon t^{\theta-3}.\end{aligned}$$

In the same way as in the proof of (3.4), by the identity $\mathcal{A}_\alpha \overline{u}^2 = \overline{\mathcal{A}_{-\alpha}u^2}$ we have

$$\|\Delta \mathcal{A}_\alpha u^2\|_{\mathbf{L}^2},\ \|\nabla \mathcal{A}_\alpha \overline{u}^2\|_{\mathbf{H}^{1,0}} \le C\varepsilon t^{\theta-5/2}. \tag{3.5}$$

On the other hand, using (3.3), (3.4), (3.5) and the identity $\mathcal{J} = \mathcal{U}(t)x\mathcal{U}(-t)$, we obtain

$$\begin{aligned}&\|x(\mathcal{U}(-t)u(t)-\mathcal{U}(-s)u(s))\|_{\mathbf{H}^{1,0}}\\ &= \|\mathcal{U}(-t)\mathcal{J}u(t)-\mathcal{U}(-s)\mathcal{J}u(s)\|_{\mathbf{H}^{1,0}}\\ &\le \|\mathcal{U}(-t)\Psi(t)-\mathcal{U}(-s)\Psi(s)\|_{\mathbf{H}^{1,0}} + Ct^2(\|\nabla\mathcal{A}_{\alpha_1}u^2\|_{\mathbf{H}^{1,0}} + \|\nabla\mathcal{A}_{\alpha_2}\overline{u}^2\|_{\mathbf{H}^{1,0}})\\ &\quad + Cs^2(\|\nabla\mathcal{A}_{\alpha_1}u^2\|_{\mathbf{H}^{1,0}} + \|\nabla\mathcal{A}_{\alpha_2}\overline{u}^2\|_{\mathbf{H}^{1,0}})\\ &\le C(\varepsilon s^{-1/4} + \varepsilon t^{\theta-1/2} + \varepsilon s^{\theta-1/2}) \le C\varepsilon s^{-1/4}\end{aligned} \tag{3.6}$$

for $t > s > 1$. Thus $\mathcal{U}(-t)u(t)$ is a Cauchy sequence in $\mathbf{X}$, so there exists a unique final state $\phi_+ = \lim_{t\to\infty}\mathcal{U}(-t)u(t) \in \mathbf{X}$ such that

$$\|\mathcal{U}(-t)u(t) - \phi_+\|_{\mathbf{X}} \le C\varepsilon t^{-1/4} \quad \text{for} \quad t > 1. \tag{3.7}$$

Therefore, we have shown (3.1). This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

We consider the linearlized version of equation (1.5)

$$u(t) = \mathcal{U}(t)u_+ + i\int_t^\infty \mathcal{U}(t-\tau)(\lambda v^2 + \mu\overline{v}^2)(\tau)\,d\tau, \quad (t,x)\in \mathbf{R}\times\mathbf{R}^3, \tag{4.1}$$

in the closed ball in $\mathbf{Z}_T$ with a center $\mathcal{U}(t)u_+$ and a radius 2ρ, where $\rho = \|u_+\|_{\mathbf{X}_1} = \|u_+\|_{\mathbf{H}^{1,1}}$, namely $v-\mathcal{U}(t)u_+\in\mathbf{Z}_T$ and $\|v-\mathcal{U}(t)u_+\|_{\mathbf{Z}_T}\le 2\rho$. For simplicity, we put $f\equiv v-\mathcal{U}(t)u_+$. We can rewrite equation (4.1) as

$$u(t)-\mathcal{U}(t)u_+ = K_1(t)+K_2(t)+K_3(t), \tag{4.2}$$

where

$$K_1(t)=i\lambda\int_t^\infty \mathcal{U}(t-\tau)(\mathcal{U}(\tau)u_+)^2(\tau)\,d\tau, \quad K_2(t)=i\mu\int_t^\infty \mathcal{U}(t-\tau)\overline{(\mathcal{U}(\tau)u_+)}^2(\tau)\,d\tau,$$
$$K_3(t)=i\lambda\int_t^\infty \mathcal{U}(t-\tau)(f^2+2f\mathcal{U}(\tau)u_+)(\tau)\,d\tau + i\mu\int_t^\infty \mathcal{U}(t-\tau)\big(\overline{f}^2+2\overline{f\mathcal{U}(\tau)u_+}\big)(\tau)\,d\tau.$$

We estimate equation (4.2) in $\mathbf{L}^2$. Then we have

$$\|u(t)-\mathcal{U}(t)u_+\|_{\mathbf{L}^2}\le \|K_1(t)\|_{\mathbf{L}^2}+\|K_2(t)\|_{\mathbf{L}^2}+C\rho^2t^{-1/2}, \tag{4.3}$$

since

$$\begin{aligned}\|K_3(t)\|_{\mathbf{L}^2} &\le C\int_t^\infty (\|f\|_{\mathbf{L}^4}^2+\|f\|_{\mathbf{L}^4}\|\mathcal{U}(\tau)u_+\|_{\mathbf{L}^4})\,d\tau\\ &\le C\int_t^\infty \tau^{-3/2}(\rho^2+\rho\|u_+\|_{\mathbf{L}^{4/3}})\,d\tau \le C\rho^2t^{-1/2}.\end{aligned} \tag{4.4}$$

We also estimate equation (4.2) in $\mathbf{L}^4$. By Sobolev's inequality we see that $\mathbf{H}^{1,0}\subset\mathbf{L}^4$. Hence,

$$\|u(t)-\mathcal{U}(t)u_+\|_{\mathbf{L}^4}\le \|K_1(t)\|_{\mathbf{H}^{1,0}}+\|K_2(t)\|_{\mathbf{H}^{1,0}}+\|K_3(t)\|_{\mathbf{L}^4}. \tag{4.5}$$

By Hölder's inequality and the estimate $\|\mathcal{U}(\tau)u_+\|_{\mathbf{L}^4}\le C\tau^{-3/4}\|u_+\|_{\mathbf{L}^{4/3}}$ we have

$$\begin{aligned}\|K_3(t)\|_{\mathbf{L}^4} &\le C\int_t^\infty (\tau-t)^{-3/4}(\|f\|_{\mathbf{L}^2}\|f\|_{\mathbf{L}^4}+\|f\|_{\mathbf{L}^2}\|\mathcal{U}(\tau)u_+\|_{\mathbf{L}^4})\,d\tau\\ &\le C\int_t^\infty (\tau-t)^{-3/4}\tau^{-5/4}(\rho^2+\rho\|u_+\|_{\mathbf{L}^{4/3}})\,d\tau \le C\rho^2t^{-1}.\end{aligned} \tag{4.6}$$

We next consider $\|K_1(t)\|_{\mathbf{H}^{1,0}}$ and $\|K_2(t)\|_{\mathbf{H}^{1,0}}$. First of all, we evaluate $K_1(t)$ in $\mathbf{L}^2$. By Lemma 2.1, we obtain the identity

(4.7)
$$\begin{aligned}
\mathcal{F}\mathcal{U}(-\tau)(\mathcal{U}(\tau)u_+)^2 &= -i\mathcal{F}\overline{M}\mathcal{F}^{-1}\mathcal{D}\left(\frac{1}{\tau}\right)\overline{M}(\mathcal{U}(\tau)u_+)^2 \\
&= i^{1/2}\mathcal{D}(2)E^2\mathcal{F}\overline{M}^{1/2}\mathcal{F}^{-1}\mathcal{D}\left(\frac{1}{\tau}\right)\overline{M}^2(\mathcal{U}(\tau)u_+)^2 \\
&= \tau^{-3/2}\mathcal{D}(2)E^2\mathcal{F}\overline{M}^{1/2}\mathcal{F}^{-1}(\mathcal{F}Mu_+)^2 \\
&= (2i\tau)^{-3/2}e^{i\tau|\xi|^2/4}\widehat{u_+}^2\left(\frac{\xi}{2}\right) + \tau^{-3/2}\mathcal{D}(2)E^2(\mathcal{F}(M-1)u_+)(\mathcal{F}(M+1)u_+) \\
&\quad + \tau^{-3/2}\mathcal{D}(2)E^2\mathcal{F}\big(\overline{M}^{1/2}-1\big)\mathcal{F}^{-1}(\mathcal{F}Mu_+)^2.
\end{aligned}$$

We estimate the first term of the right-hand side of (4.7) by integrating the oscillating function $e^{i\tau|\xi|^2}$ by parts with respect to τ. For the second and the third term of the right-hand side of (4.7), we obtain better time decay by making use of $M-1$ and $\overline{M}^{1/2}-1$. Indeed, by (4.7) we have

$$\|K_1(t)\|_{\mathbf{L}^2} = \left\|\lambda\int_t^\infty \mathcal{F}\mathcal{U}(-\tau)(\mathcal{U}(\tau)u_+)^2(\tau)\,d\tau\right\|_{\mathbf{L}^2} \le R_1 + R_2, \tag{4.8}$$

where

$$\begin{aligned}
R_1 &= C\left\|\int_t^\infty \tau^{-3/2}e^{i\tau|\xi|^2/4}\widehat{u_+}^2\left(\frac{\xi}{2}\right)d\tau\right\|_{\mathbf{L}^2}, \\
R_2 &= C\int_t^\infty \tau^{-3/2}\big(\|(\mathcal{F}(M-1)u_+)(\mathcal{F}(M+1)u_+)\|_{\mathbf{L}^2} + \big\|\mathcal{F}\big(\overline{M}^{1/2}-1\big)\mathcal{F}^{-1}(\mathcal{F}Mu_+)^2\big\|_{\mathbf{L}^2}\big)\,d\tau.
\end{aligned}$$

By the relation $|M-1|, \big|\overline{M}^{1/2}-1\big| \le C\tau^{-1/4}|x|^{1/2}$, Hölder's and Sobolev's inequalities, we find

$$\begin{aligned}
R_2 &\le C\int_t^\infty \tau^{-3/2}\big(\|\mathcal{F}(M-1)u_+\|_{\mathbf{L}^3}\|\mathcal{F}(M+1)u_+\|_{\mathbf{L}^6} + \big\|\big(\overline{M}^{1/2}-1\big)\mathcal{F}^{-1}\big(\mathcal{F}Mu_+\big)^2\big\|_{\mathbf{L}^2}\big)\,d\tau \\
&\le C\int_t^\infty (\tau^{-3/2}\|(-\Delta)^{1/4}\mathcal{F}(M-1)u_+\|_{\mathbf{L}^2}\|\nabla\mathcal{F}(M+1)u_+\|_{\mathbf{L}^2} \\
&\qquad + \tau^{-7/4}\||x|^{1/2}\mathcal{F}^{-1}(\mathcal{F}Mu_+)^2\|_{\mathbf{L}^2})\,d\tau \\
&\le C\int_t^\infty (\tau^{-3/2}\||x|^{1/2}(M-1)u_+\|_{\mathbf{L}^2}\|x(M+1)u_+\|_{\mathbf{L}^2} + \tau^{-7/4}\|\nabla(\mathcal{F}Mu_+)^2\|_{\mathbf{L}^{3/2}})\,d\tau \\
&\le C\int_t^\infty (\tau^{-7/4}\|u_+\|_{\mathbf{H}^{0,1}}^2 + \tau^{-7/4}\|\nabla(\mathcal{F}Mu_+)\|_{\mathbf{L}^2}^2)\,d\tau \le Ct^{-3/4}\|u_+\|_{\mathbf{H}^{0,1}}^2.
\end{aligned}$$

Thus, we obtain

$$\|K_1(t)\|_{\mathbf{L}^2} \le R_1 + Ct^{-3/4}\|u_+\|_{\mathbf{H}^{0,1}}^2. \tag{4.9}$$

We next consider the estimate of R_1. Using the identity

$$e^{i\tau|\xi|^2/4} = \frac{1}{1+i\tau|\xi|^2/4}\partial_\tau(\tau e^{i\tau|\xi|^2/4}),$$

we get

$$\begin{aligned}\tau^{-3/2}e^{i\tau|\xi|^2/4} &= \frac{1}{1+i\tau|\xi|^2/4}\tau^{-3/2}\partial_\tau(\tau e^{i\tau|\xi|^2/4})\\ &= \partial_\tau\left(\frac{1}{1+i\tau|\xi|^2/4}\tau^{-1/2}e^{i\tau|\xi|^2/4}\right)+\frac{3}{2}\tau^{-3/2}e^{i\tau|\xi|^2/4}\frac{1}{1+i\tau|\xi|^2/4}\\ &\quad+\tau^{-1/2}e^{i\tau|\xi|^2/4}\frac{i|\xi|^2}{4}\frac{1}{(1+i\tau|\xi|^2/4)^2}.\end{aligned} \tag{4.10}$$

It follows from Sobolev's inequality and (4.10) that

(4.11)

$$\begin{aligned}R_1 &\le C\left\|t^{-1/2}\frac{1}{1+t|\xi|^2}\widehat{u_+}^2\left(\frac{\xi}{2}\right)\right\|_{\mathbf{L}^2}\\ &\quad+C\int_t^\infty\left(\left\|\tau^{-3/2}\frac{1}{1+\tau|\xi|^2}\widehat{u_+}^2\left(\frac{\xi}{2}\right)\right\|_{\mathbf{L}^2}+\left\|\tau^{-1/2}|\xi|^2\frac{1}{(1+\tau|\xi|^2)^2}\widehat{u_+}^2\left(\frac{\xi}{2}\right)\right\|_{\mathbf{L}^2}\right)d\tau\\ &\le Ct^{-1/2}\left\|\frac{|\xi|^{1/2}}{1+t|\xi|^2}\right\|_{\mathbf{L}^\infty}\left\|\frac{1}{|\xi|^{1/4}}\widehat{u_+}\right\|_{\mathbf{L}^4}^2\\ &\quad+C\int_t^\infty\left(\tau^{-3/2}\left\|\frac{|\xi|^{1/2}}{1+\tau|\xi|^2}\right\|_{\mathbf{L}^\infty}\left\|\frac{1}{|\xi|^{1/4}}\widehat{u_+}\right\|_{\mathbf{L}^4}^2+\tau^{-1/2}\left\|\frac{|\xi|^{5/2}}{(1+\tau|\xi|^2)^2}\right\|_{\mathbf{L}^\infty}\left\|\frac{1}{|\xi|^{1/4}}\widehat{u_+}\right\|_{\mathbf{L}^4}^2\right)d\tau\\ &\le Ct^{-3/4}\|u_+\|_{\mathbf{H}^{0,1}}^2+C\int_t^\infty\tau^{-7/4}\|u_+\|_{\mathbf{H}^{0,1}}^2\,d\tau\le Ct^{-3/4}\|u_+\|_{\mathbf{H}^{0,1}}^2.\end{aligned}$$

By (4.9) and (4.11) we have

$$\|K_1(t)\|_{\mathbf{L}^2}\le Ct^{-3/4}\|u_+\|_{\mathbf{H}^{0,1}}^2. \tag{4.12}$$

Using Lemma 2.1, we also have the identity

(4.13)

$$\begin{aligned}&\mathcal{F}\mathcal{U}(-\tau)\overline{(\mathcal{U}(\tau)u_+)}^2\\ &= i^{1/2}\mathcal{D}(-2)E^6\mathcal{F}M^{1/2}\mathcal{F}^{-1}\mathcal{D}\left(\frac{1}{\tau}\right)M^2\overline{(\mathcal{U}(\tau)u_+)}^2\\ &= -\tau^{-3/2}\mathcal{D}(-2)E^6\mathcal{F}M^{1/2}\mathcal{F}^{-1}\overline{(\mathcal{F}Mu_+)}^2\\ &= -(-2i\tau)^{-3/2}e^{3i\tau|\xi|^2/4}\widehat{\overline{u_+}}^2\left(-\frac{\xi}{2}\right)-\tau^{-3/2}\mathcal{D}(-2)E^6\left(\overline{\mathcal{F}(M-1)u_+}\right)\left(\overline{\mathcal{F}(M+1)u_+}\right)\\ &\quad-\tau^{-3/2}\mathcal{D}(-2)E^6\mathcal{F}(M^{1/2}-1)\mathcal{F}^{-1}\overline{(\mathcal{F}Mu_+)}^2.\end{aligned}$$

Therefore, in the same way as in the proof of (4.12), we obtain

$$\|K_2(t)\|_{\mathbf{L}^2} \le Ct^{-3/4}\|u_+\|^2_{\mathbf{H}^{0,1}}. \tag{4.14}$$

We next evaluate $\nabla K_1(t)$ and $\nabla K_2(t)$ in $\mathbf{L}^2$. By Lemma 2.1, we have the identities

$$\begin{aligned}
\mathcal{FU}(-\tau)(\mathcal{U}(\tau)u_+)(\mathcal{U}(\tau)w_+) &= \tau^{-3/2}\mathcal{D}(2)E^2\mathcal{F}\overline{M}^{1/2}\mathcal{F}^{-1}(\mathcal{F}Mu_+)(\mathcal{F}Mw_+)\\
&= (2i\tau)^{-3/2}e^{i\tau|\xi|^2/4}\widehat{u_+}\left(\frac{\xi}{2}\right)\widehat{w_+}\left(\frac{\xi}{2}\right)\\
&\quad + \tau^{-3/2}\mathcal{D}(2)E^2(\mathcal{F}Mu_+)(\mathcal{F}(M-1)w_+)\\
&\quad + \tau^{-3/2}\mathcal{D}(2)E^2(\widehat{w_+})(\mathcal{F}(M-1)u_+)\\
&\quad + \tau^{-3/2}\mathcal{D}(2)E^2\mathcal{F}\big(\overline{M}^{1/2}-1\big)\mathcal{F}^{-1}(\mathcal{F}Mu_+)(\mathcal{F}Mw_+)
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\mathcal{FU}(-\tau)\overline{(\mathcal{U}(\tau)u_+)(\mathcal{U}(\tau)w_+)} &= -\tau^{-3/2}\mathcal{D}(-2)E^6\mathcal{F}M^{1/2}\mathcal{F}^{-1}\overline{(\mathcal{F}Mu_+)(\mathcal{F}Mw_+)}\\
&= -(-2i\tau)^{-3/2}e^{3i\tau|\xi|^2/4}\overline{\widehat{u_+}\left(-\frac{\xi}{2}\right)\widehat{w_+}\left(-\frac{\xi}{2}\right)}\\
&\quad - \tau^{-3/2}\mathcal{D}(-2)E^6\overline{(\mathcal{F}Mu_+)(\mathcal{F}(M-1)w_+)}\\
&\quad - \tau^{-3/2}\mathcal{D}(-2)E^6\overline{(\widehat{w_+})(\mathcal{F}(M-1)u_+)}\\
&\quad - \tau^{-3/2}\mathcal{D}(-2)E^6\mathcal{F}(M^{1/2}-1)\mathcal{F}^{-1}\overline{(\mathcal{F}Mu_+)(\mathcal{F}Mw_+)}.
\end{aligned} \tag{4.16}$$

By substituting ∇u_+ for w_+ at (4.15) and (4.16), respectively, and estimating $\nabla K_1(t)$ and $\nabla K_2(t)$ in the same way as in the proof of (4.12) and (4.14), we obtain

$$\|\nabla K_1(t)\|_{\mathbf{L}^2} \le Ct^{-3/4}\|u_+\|^2_{\mathbf{H}^{1,1}}, \quad \|\nabla K_2(t)\|_{\mathbf{L}^2} \le Ct^{-3/4}\|u_+\|^2_{\mathbf{H}^{1,1}}. \tag{4.17}$$

Collecting estimates (4.3), (4.5), (4.12), (4.14) and (4.17), we have

$$\|u(t)-\mathcal{U}(t)u_+\|_{\mathbf{L}^2} \le C\rho^2(t^{-3/4}+t^{-1/2}), \quad \|u(t)-\mathcal{U}(t)u_+\|_{\mathbf{L}^4} \le C\rho^2(t^{-3/4}+t^{-1}).$$

Therefore,

$$\begin{aligned}
t^{1/2}\|u(t)-\mathcal{U}(t)u_+\|_{\mathbf{L}^2} + t^{3/4}\|u(t)-\mathcal{U}(t)u_+\|_{\mathbf{L}^4} &\le C\rho^2(t^{-1/4}+1)\\
&\le C\rho^2(T^{-1/4}+1) \le 2\rho,
\end{aligned} \tag{4.18}$$

from which it follows that $u-\mathcal{U}(t)u_+ \in \mathbf{Z}_T$, for all $T \ge 1$ and sufficiently small ρ. We put

$$u^{(j)}(t) = \mathcal{U}(t)u_+ + i\int_t^\infty \mathcal{U}(t-\tau)\Big(\lambda v^{(j)2} + \mu\overline{v^{(j)}}^2\Big)(\tau)\,d\tau, \tag{4.19}$$

where $v^{(j)} - \mathcal{U}(t)u_+ \in \mathbf{Z}_T$, $j = 1, 2$. Then we have

$$\|v^{(j)}\|_{\mathbf{L}^4} \le \|v^{(j)}(t) - \mathcal{U}(t)u_+\|_{\mathbf{L}^4} + \|\mathcal{U}(t)u_+\|_{\mathbf{L}^4} \le C\rho t^{-3/4}. \tag{4.20}$$

By virtue of (4.20), we obtain

$$\begin{aligned}\|u^{(1)}(t) - u^{(2)}(t)\|_{\mathbf{L}^2} &\le C\int_t^\infty \|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^4}(\|v^{(1)}(t)\|_{\mathbf{L}^4} + \|v^{(2)}(t)\|_{\mathbf{L}^4})\,d\tau \\ &\le C\sup_{t\in[T,\infty)} t^{3/4}\|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^4}\int_t^\infty \rho\tau^{-3/2}\,d\tau \\ &\le C\rho t^{-1/2}\sup_{t\in[T,\infty)} t^{3/4}\|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^4},\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\|u^{(1)}(t) - u^{(2)}(t)\|_{\mathbf{L}^4} &\le C\int_t^\infty (\tau - t)^{-3/4}\|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^2}(\|v^{(1)}(t)\|_{\mathbf{L}^4} + \|v^{(2)}(t)\|_{\mathbf{L}^4})\,d\tau \\ &\le C\sup_{t\in[T,\infty)} t^{1/2}\|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^2}\int_t^\infty \rho(\tau - t)^{-3/4}\tau^{-5/4}\,d\tau \\ &\le C\rho t^{-1}\sup_{t\in[T,\infty)} t^{1/2}\|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^2}.\end{aligned} \tag{4.22}$$

It follows from (4.21) and (4.22) that

$$\begin{aligned}&t^{1/2}\|u^{(1)}(t) - u^{(2)}(t)\|_{\mathbf{L}^2} + t^{3/4}\|u^{(1)}(t) - u^{(2)}(t)\|_{\mathbf{L}^4} \\ &\le C\rho t^{-1/4}\sup_{t\in[T,\infty)} t^{1/2}\|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^2} + C\rho\sup_{t\in[T,\infty)} t^{3/4}\|v^{(1)}(t) - v^{(2)}(t)\|_{\mathbf{L}^4}.\end{aligned} \tag{4.23}$$

For $v^{(j)} - \mathcal{U}(t)u_+ \in \mathbf{Z}_T$ we have estimate (4.23). By contraction mapping principle with (4.18) and (4.23), there exists a unique solution $u - \mathcal{U}(t)u_+ \in \mathbf{Z}_T$ to (1.5) for any positive time $T \ge 1$. This completes the proof of Theorem 1.2. □

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