Wei, D.
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COMPLETENESS OF THE GENERALIZED EIGENFUNCTIONS FOR RELATIVISTIC SCHRÖDINGER OPERATORS I

DABI WEI

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Abstract

Generalized eigenfunctions of the odd-dimensional ($n \geq 3$) relativistic Schrödinger operator $\sqrt{-\Delta}+V(x)$ with $|V(x)| \leq C\langle x\rangle^{-\sigma}$, $\sigma > 1$, are considered. We compute the integral kernels of the boundary values $R^{\pm}(\lambda) = (\sqrt{-\Delta} - (\lambda \pm i0))^{-1}$, and prove that the generalized eigenfunctions $\varphi^{\pm}(x, k) := \varphi_0(x, k) - R^{\mp}(|k|)V\varphi_0(x, k)$ ($\varphi_0(x, k) := e^{ix\cdot k}$) are bounded for $(x, k) \in \mathbb{R}^n \times \{k \mid a \leq |k| \leq b\}$, where $[a, b] \subset (0, \infty) \setminus \sigma_p(H)$. This fact, together with the completeness of the wave operators, enables us to obtain the eigenfunction expansion for the absolutely continuous spectrum.

On considère les fonctions propres généralisées de l'opérateur relativiste de Schrödinger $\sqrt{-\Delta} + V(x)$ où $|V(x)| \leq C\langle x\rangle^{-\sigma}$ en dimension impaire ($n \geq 3$). On calcule les noyaux intégraux associés aux valeurs limites $R^{\pm}(\lambda) = (\sqrt{-\Delta} - (\lambda \pm i0))^{-1}$, et on prouve que les fonctions propres généralisées $\varphi^{\pm}(x, k) := \varphi_0(x, k) - R^{\mp}(|k|)V\varphi_0(x, k)$ ($\varphi_0(x, k) := e^{ix\cdot k}$) sont bornées pour $(x, k) \in \mathbb{R}^n \times \{k \mid a \leq |k| \leq b\}$, où $[a, b] \subset (0, \infty) \setminus \sigma_p(H)$. Ce résultat, associé à la complétude des opérateurs d'onde, nous permet d'obtenir le développement en fonction propres pour le spectre absolument continu.

Introduction

This paper considers the odd-dimensional ($n \geq 3$) relativistic Schrödinger operator

$$H = H_0 + V(x), \quad H_0 = \sqrt{-\Delta}, \quad x \in \mathbb{R}^n$$

with a short range potential $V(x)$.

Throughout the paper we assume that $V(x)$ is a real-valued measurable function on $\mathbb{R}^n$ satisfying

$$|V(x)| \leq C\langle x\rangle^{-\sigma}, \quad \sigma > 1.$$

When we deal with the boundedness and the completeness of the generalized eigenfunctions, σ will be required to satisfy the assumption $\sigma > (n+1)/2$ and n to be an odd integer with $n \geq 3$.

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In general, the Schrödinger operator is written as $-\Delta + V(x)$, $x \in \mathbb{R}^n$. In [6], the completeness of the generalized eigenfunctions for operator $-\Delta + V(x)$ was proved. However, it was considered in 3-dimensional case. In the relativistic case, the Schrödinger operator is written by $\sqrt{-\Delta + m} + V(x)$, $x \in \mathbb{R}^n$, where m is the mass of the particle. But, like the photon, the zero mass particle exists. Then, the relativitic Schrödinger operator is written by $H = \sqrt{-\Delta} + V(x)$, $x \in \mathbb{R}^n$. H is essentially self adjoint on $C_0^\infty(\mathbb{R}^n)$ [23]. And in the paper [24], T. Umeda considered the 3-dimensional case and proved that the generalized eigenfuctions $\varphi^\pm(x, k)$ are bounded for $(x, k) \in \mathbb{R}^3 \times \{k \mid k \in \mathbb{R}^3,\ a \le |k| \le b\}$, $[a, b] \subset (0, \infty) \setminus \sigma_p(H)$. In [25], T. Umeda announced that he will deal with the completeness of the generalized eigenfunctions, although the full proof has not been published yet.

In the present paper, we show the boundedness of generalized eigenfunctions for odd demensions $n \ge 3$. As is seen in the formula of the resolvent kernel of H_0 in Theorem 2.2, our computation is more complicated when $n > 3$ than the case $n = 3$, and the key estimate is Lemma 3.8 based on the L^p-estmate in Lemma 3.6.

From V. Enss's idea (see V. Enss [3]), we obtain that the wave operators $W_\pm$ defined by

$$W_\pm = \lim_{t\to\infty} e^{itH} e^{-itH_0}$$

are complete. Finally, by the idea of H. Kitada [10] and S.T. Kuroda [13], we obtain the completeness of the generalized eigenfunctions as follows. Moreover, we deal with the even dimensions case in [27].

Theorem. *Assume the dimension $n (n \ge 3)$ is an odd integer, $\sigma > (n + 1)/2$, $s > n/2$ and $[a, b] \subset (0, \infty) \setminus \sigma_p(H)$. For $u \in L^{2,s}(\mathbb{R}^n)$, let $\mathcal{F}_\pm$ be defined by*

$$\mathcal{F}_\pm u(k) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) \overline{\varphi^\pm(x, k)}\, dx.$$

Then for an arbitrary $L^{2,s}(\mathbb{R}^n)$-function $f(x)$,

$$E_H([a, b]) f(x) = (2\pi)^{-n/2} \int_{a \le |k| \le b} \mathcal{F}_\pm f(k) \varphi^\pm(x, k)\, dk$$

where E_H is the spectral measure for H.

The plan of the paper. In Section 1, we construct generalized eigenfunctions of $\sqrt{-\Delta} + V(x)$ on R^n. We compute the resolvent kernel of $\sqrt{-\Delta}$ on $\mathbb{R}^n$ in Section 2. Section 3 proves that the generalized eigenfunctions are bounded in the case of odd-dimension $n \ge 3$. We study the asymptotic completeness of wave operators in Section 4. In the last Section 5, we deal with the completeness of the generalized eigenfunctions.

NOTATION. We introduce the notation which will be used in the present paper.

For $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean norm of x and $\langle x \rangle = \sqrt{1+|x|^2}$. The Fourier transform of a function u is denoted by $\mathcal{F}u$ or $\hat{u}$, and is defined by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx.$$

For s and l in $\mathcal{R}$, we define the weighted L^2-space and the weighted Sobolev space by

$$L^{2,s}(\mathbb{R}^n) = \{ f \mid \langle x \rangle^s f \in L^2(\mathbb{R}^n)\}, \quad H^{l,s}(\mathbb{R}^n) = \{ f \mid \langle x \rangle^s \langle D \rangle^l f \in L^2(\mathbb{R}^n)\}$$

respectively, where D stands for $-i\partial/\partial x$ and $\langle D \rangle = \sqrt{1+|D|^2} = \sqrt{1-\Delta}$. The inner products and the norm in $L^{2,s}(\mathbb{R}^n)$ and $H^{l,s}(\mathbb{R}^n)$ are given by

$$(f, g)_{L^{2,s}} = \int_{\mathbb{R}^n} \langle x \rangle^{2s} f(x)\overline{g(x)}\,dx, \quad (f, g)_{H^{l,s}} = \int_{\mathbb{R}^n} \langle x \rangle^{2s} \langle D \rangle^l f(x)\overline{\langle D \rangle^l g(x)}\,dx,$$
$$\| f \|_{L^{2,s}} = \{(f, f)_{L^{2,s}}\}^{1/2}, \quad \| f \|_{H^{l,s}} = \{(f, f)_{H^{l,s}}\}^{1/2},$$

respectively. For $s = 0$ we write

$$(f, g) = (f, g)_{L^{2,0}} = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx, \quad \| f \|_{L^2} = \| f \|_{L^{2,0}}.$$

For a pair of $f \in L^{2,-s}(\mathbb{R}^n)$ and $g \in L^{2,s}(\mathbb{R}^n)$, we also define $(f, g) = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$.

By $C_0^\infty(\mathbb{R}^n)$ we mean the space of C^∞-functions of compact support. By $\mathcal{S}(\mathbb{R}^n)$ we mean the Schwartz space of rapidly decreasing functions, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions.

The operator $\sqrt{-\Delta} e^{ix\cdot k}$ is formally defined by

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} |\xi| \delta(\xi - k)\,d\xi,$$

where $\delta(x)$ is the Dirac's delta function. As the symbol $|\xi|$ of $\sqrt{-\Delta}$ is singular at the origin $\xi = 0$, giving a definite meaning to $\sqrt{-\Delta} e^{ix\cdot k}$ is one of the main tasks in the present paper.

For a pair of Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathbf{B}(\mathcal{H}, \mathcal{K})$ denotes the Banach space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. For a selfadjoint operator H in a Hilbert space, $\sigma(H)$ and $\rho(H)$ denote the spectrum of H and the resolvent set of H, respectively. The point spectrum, the essential spectrum, the continuous spectrum and the absolutely continuous spectrum of H will be denoted by $\sigma_p(H)$, $\sigma_e(H)$, $\sigma_c(H)$, and $\sigma_{ac}(H)$ respectively. E_H denotes the spectral measure for T, and $E_H(\lambda) = E_H((-\infty, \lambda])$, $E_H((a, b]) = E_H(b) - E_H(a)$. The continous subspace and the absolutely continuous subspace of H will be denoted by $\mathcal{H}_c$, $\mathcal{H}_{ac}$, respectively. By $F(t > A)$, $F(t < A)$, $F(t \geq A)$ and $F(t \leq A)$ we mean the characteristic functions of the sets $\{t \mid t > A\}$, $\{t \mid t < A\}$, $\{t \mid t \geq A\}$ and $\{t \mid t \leq A\}$, respectively.

1. Generalized eigenfuction

We construct generalized eigenfunctions of $\sqrt{-\Delta}+V(x)$ on $\mathbb{R}^n$ in this section, and show that they satisfy the equation

$$\varphi^{\pm}(x,k)=\varphi_0(x,k)-R_0^{\mp}(|k|)V\varphi^{\pm}(x,k),$$

where $R_0(z)$ is the resolvent of $H_0=\sqrt{-\Delta}$ defined by

$$R_0(z):=(H_0-z)^{-1}=\mathcal{F}^{-1}(|\xi|-z)^{-1}\mathcal{F},$$

and $\varphi_0(x,k)$ is definded by

$$\varphi_0(x,k)=e^{ix\cdot k}.$$

Similarly $R(z)$ is the resolvent of $H=\sqrt{-\Delta}+V(x)$ on $\mathbb{R}^n$ and we assume that $V(x)$ is a real-valued measurable function on $\mathbb{R}^n$ and satisfies $|V(x)|<C\langle x\rangle^{-\sigma}$ for $\sigma>1$. To show the above equation for eigenfunctions, we use two theorems demonstrated by Ben-Artzi and Nemirovski. (see [2, Section 2 and Theorem 4A])

Theorem 1.1 (Ben-Artzi and Nemirovski). *Let $s>1/2$. Then*
(1) *For any $\lambda>0$, there exist the limits $R_0^{\pm}(\lambda)=\lim_{\mu\downarrow 0}R_0(\lambda\pm i\mu)$ in $B(L^{2,s},H^{1,-s})$.*
(2) *The operator-valued functions $R_0^{\pm}(z)$ defined by*

$$R_0^{\pm}(z)=\begin{cases}R_0(z) & \text{if} \quad z\in\mathbb{C}^{\pm}\\ R_0^{\pm}(\lambda) & \text{if} \quad z=\lambda>0\end{cases}$$

are $B(L^{2,s},H^{1,-s})$-valued continuous functions, where $\mathbb{C}^{+}$ and $\mathbb{C}^{-}$ are the upper and the lower half-planes respectively: $\mathbb{C}^{\pm}=\{z\in\mathbb{C}\mid \pm\mathrm{Im}\,z>0\}$.

Theorem 1.2 (Ben-Artzi and Nemirovski). *Let $s>1/2$ and $\sigma>1$. Then*
(1) *The continuous spectrum $\sigma_c(H)=[0,\infty)$ is absolutely continuous, except possibly for a discrete set of embedded eigenvalues $\sigma_p(H)\cap(0,\infty)$, which can accumulate only at 0 and ∞.*
(2) *For any $\lambda\in(0,\infty)\setminus\sigma_p(H)$, there exist the limits*

$$R^{\pm}(\lambda)=\lim_{\mu\downarrow 0}R(\lambda\pm i\mu)\quad \text{in} \quad B(L^{2,s},H^{1,-s}).$$

(3) *The operator-valued functions $R^{\pm}(z)$ defined by*

$$R^{\pm}(z)=\begin{cases}R(z) & \text{if} \quad z\in\mathbb{C}^{\pm}\\ R^{\pm}(\lambda) & \text{if} \quad z=\lambda\in(0,\infty)\setminus\sigma_p(H)\end{cases}$$

are $B(L^{2,s},H^{1,-s})$-valued continuous functions.

The main results of this section are

Theorem 1.3. *Let $\sigma > (n+1)/2$. If $|k| \in (0,\infty) \setminus \sigma_p(H)$, then the generalized eigenfunctions*

$$\varphi^{\pm}(x,k) := \varphi_0(x,k) - R^{\mp}(|k|)\{V(\,\cdot\,)\varphi_0(\,\cdot\,,k)\}(x)$$

satisfy the equation

$$(\sqrt{-\Delta_x} + V(x))u = |k|u \quad in \quad \mathcal{S}'(\mathbb{R}^n_x)$$

where $\varphi_0(x,k)$ is definded by $\varphi_0(x,k) = e^{ix\cdot k}$.

Theorem 1.4. *Let $\sigma > (n+1)/2$. If $|k| \in (0,\infty) \setminus \sigma_p(H)$ and $n/2 < s < \sigma - 1/2$, then we have*

$$\varphi^{\pm}(x,k) = \varphi_0(x,k) - R_0^{\mp}(|k|)\{V(\,\cdot\,)\varphi^{\pm}(\,\cdot\,,k)\}(x) \quad in \quad L^{2,-s}(\mathbb{R}^n).$$

First, we investigate the propeties of $\varphi_0 = e^{ix\cdot k}$. It is easy to prove the next lemma.

Lemma 1.1. *Let $\sigma > 1$ and $n \geq 1$.*
(1) *If $s < -n/2$, then $\varphi_0(x,k) \in L^{2,s}(\mathbb{R}^n_x)$.*
(2) *If $s < \sigma - n/2$, then $V(x)\varphi_0(x,k) \in L^{2,s}(\mathbb{R}^n_x)$.*
(3) *If $s + t \leq \sigma$, then $V(x) \in B(L^{2,-s}(\mathbb{R}^n_x), L^{2,t}(\mathbb{R}^n_x))$.*

Proof. Using the following formulas, we can get this lemma. If $V(x) \leq C\langle x\rangle^{-\sigma}$, then

$$\begin{aligned}
&\|\varphi_0(x,k)\|_{L^{2,s}} = \|\langle x\rangle^s\|_{L^2},\\
&\|V(x)\varphi_0(x,k)\|_{L^{2,s}} \leq C^2\|\langle x\rangle^{s-\sigma}\|_{L^2},\\
&\|V(x)u\|_{L^{2,t}} \leq C^2\|\langle x\rangle^{s+t-\sigma}u\|_{L^{2,-s}}.
\end{aligned}$$ □

Next, to prove the main Theorem 1.3, we make the next preparation.

Lemma 1.2. *Let $\sigma > (n+1)/2$.*
(1) *For all $k \in \mathbb{R}^n$, $\varphi_0(x,k)$ satisfies the pseudodifferential equation*

$$\sqrt{-\Delta_x}\varphi_0(x,k) = |k|\varphi_0(x,k) \quad in \quad \mathcal{S}'(\mathbb{R}^n_x).$$

(2) *Let $\lambda \in (0,\infty) \setminus \sigma_p(H)$, $s > 1/2$, if $u \in L^{2,s}$ then u satisfies the equation*

$$(\sqrt{-\Delta_x} + V(x) - |k|)R^{\pm}(\lambda)u = u \quad in \quad \mathcal{S}'(\mathbb{R}^n_x).$$

Proof. From Lamma 1.1 (1), we have that $\varphi_0(x, k)$ belongs to $L^{2,s}(\mathbb{R}^n_x)$ for every $s < -n/2$, This fact, together with T. Umeda [23, Theorem 5.8], implies that $\sqrt{-\Delta_x}\varphi_0(x, k)$ makes sense. Then, we can prove (1) similarly to T. Umeda [24, Lemma 8.1]. To prove (2), we see T. Umeda [24, Theorem 7.2 (ii)]. $\square$

We now prove the main Theorem 1.3.

Proof of Theorem 1.3. Using Lemma 1.2 (1) and Lemma 1.2 (2), we get

$$(\sqrt{-\Delta_x} + V(x))\varphi_0 = |k|\varphi_0 + V\varphi_0,$$
$$(\sqrt{-\Delta_x} + V(x))\{R^{\mp}(|k|)\{V(\,\cdot\,)\varphi_0(\,\cdot\,, k)\}(x)\} = |k|\{R^{\mp}(|k|)\{V(\,\cdot\,)\varphi_0(\,\cdot\,, k)\}(x)\}.$$

From the definition of $\varphi^{\pm}$, we have

$$(\sqrt{-\Delta_x} + V(x))\varphi^{\pm} = |k|\varphi_0 - |k|\{R^{\mp}(|k|)\{V(\,\cdot\,)\varphi_0(\,\cdot\,, k)\}(x)\} = |k|\varphi^{\pm}.$$

Then we have the theorem. $\square$

Next, in order to prove Theorem 1.4, we make the next preparation.

Lemma 1.3. *Let $\sigma > 1$. If $1/2 < s < \sigma - 1/2$ and $z \in \mathbb{C}^{\pm} \cup \{(0, \infty) \setminus \sigma_p(H)\}$, then*

$$(I - R^{\pm}(z)V)(I + R_0^{\pm}(z)V) = I \quad \text{on} \quad L^{2,-s}(\mathbb{R}^n),$$
$$(I + R_0^{\pm}(z)V)(I - R^{\pm}(z)V) = I \quad \text{on} \quad L^{2,-s}(\mathbb{R}^n),$$

where $\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and the lower half-planes respectively.

$$\mathbb{C}^{\pm} = \{z \in \mathbb{C} \mid \pm\mathrm{Im}\, z > 0\}.$$

Proof. In view of Lemma 1.1 (3), Theorem 1.1, Theorem 1.2 and Lemma 1.1 (3), we can get Lemma 1.3 similarly to T. Umeda [24, Lemma 8.2] $\square$

Using this lemma, we can prove the main theorem 1.4.

Proof of Theorem 1.4. According to the definition of $\varphi^{\pm}(x, k)$

$$\varphi^{\pm}(x, k) := \varphi_0(x, y) - R^{\mp}(|k|)\{V(\,\cdot\,)\varphi_0(\,\cdot\,, k)\}(x) = \{I - R^{\mp}(|k|)V\}\varphi_0(x, k),$$

and Lemma 1.1 (1), we see that if $n/2 < s$ then $\varphi_0(x, k) \in L^{2,-s}(\mathbb{R}^n_x)$. We use Lemma 1.3, and get

$$\begin{aligned}\{I + R_0^{\mp}(|k|)V\}\varphi^{\pm}(x, k) &= \{I + R_0^{\mp}(|k|)V\}\{I - R^{\mp}(|k|)V\}\varphi_0(x, k)\\ &= \varphi_0(x, k) \quad \text{in} \quad L^{2,-s},\end{aligned}$$

for $|k| \in (0, \infty) \setminus \sigma_p(H)$ and $n/2 < s < \sigma - 1/2$. Then, we obtain

$$\varphi^{\pm}(x, k) = \varphi_0(x, k) - R_0^{\mp}(|k|)V(x)\varphi^{\pm}(x, k) \quad \text{in} \quad L^{2,-s}(\mathbb{R}^n). \qquad \square$$

2. The integral kernel of the resolvents of H_0

This section is devoted to computing the resolvent kernel of $H_0 = \sqrt{-\Delta}$ on $\mathbb{R}^n$, where $n = 2m + 1$, $m \geq 1$ and $m \in \mathbb{N}$. Then we compute the limit of $g_z(x)$ as $\mu \downarrow 0$, where $z = \lambda + i\mu$ and $\lambda > 0$, and study the properties of the integral operators $G_\lambda^{\pm}$. In this section we suppose that (cf. [4, p.269, Formula (46) and (47)])

$$n = 2m + 1, \quad m \geq 1 \text{ and } m \in \mathbb{N}, \tag{1}$$

$$M_z(x) = \int_0^\infty e^{tz} \frac{1}{t^2 + |x|^2}\, dt = \frac{1}{|x|}\{\mathrm{ci}(-|x|z)\sin(|x|z) - \mathrm{si}(-|x|z)\cos(|x|z)\}, \tag{2}$$

$$N_z(x) = \int_0^\infty e^{tz} \frac{t}{t^2 + |x|^2}\, dt = \mathrm{ci}(-|x|z)\cos(|x|z) + \mathrm{si}(-|x|z)\sin(|x|z),$$

$$m_\lambda(x) = \mathrm{ci}(\lambda|x|)\sin(\lambda|x|) + \mathrm{si}(\lambda|x|)\cos(\lambda|x|), \tag{3}$$

$$n_\lambda(x) = \mathrm{ci}(\lambda|x|)\cos(\lambda|x|) - \mathrm{si}(\lambda|x|)\sin(\lambda|x|).$$

Where $\mathrm{ci}(x)$ and $\mathrm{si}(x)$ are definded by

$$\mathrm{ci}(x) = \int_x^\infty \frac{\cos t}{t}\, dt, \quad \mathrm{si}(x) = -\int_x^\infty \frac{\sin t}{t}\, dt, \quad x > 0.$$

We see that $\mathrm{si}(x)$ has an analytic continuation $\mathrm{si}(z)$ (see [4, p.145]),

$$\mathrm{si}(z) = -\frac{\pi}{2} + \sum_{m=0}^\infty \frac{(-1)^m}{(2m+1)!\,(2m+1)} z^{2m+1}. \tag{2.1}$$

The cosine integral function $\mathrm{ci}(x)$ has an analytic continuation $\mathrm{ci}(z)$, which is a many-valued function with a logarithmic branch-point at $z = 0$ (see [4, p.145]). In this paper, we choose the principal branch

$$\mathrm{ci}(z) = -\gamma - \mathrm{Log}\, z - \sum_{m=1}^\infty \frac{(-1)^m}{(2m)!\, 2m} z^{2m}, \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{2.2}$$

where γ is the Euler's constant. The main theorems are

Theorem 2.1. *Let $n \geq 3$, $\mathrm{Re}\, z < 0$. Then*

$$R_0(z)u = G_z u$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, *where*

$$G_z u(x) = \int_{\mathbb{R}^n} g_z(x-y)u(y)\,dy, \quad g_z(x) = \int_0^\infty e^{tz} \frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}\,dt,$$

$$c_n = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right), \quad \Gamma(x) = \int_0^\infty s^{x-1}e^{-s}\,ds. \tag{2.3}$$

Theorem 2.2. *Let* $n = 2m+1$, $m \geq 1$ $(m \in \mathbb{N})$ *and* $s > 1/2$, $u \in L^{2,s}(\mathbb{R}^n)$. *Let* $[a,b] \subset (0,\infty)$ *and* $\lambda \in [a,b]$.

(1) *There exist polynomials* $a_j(\lambda)$, $b_j(\lambda)$, $c_j(\lambda)$, $j = m, m+1, \ldots, 2m$, *such that,*

$$\begin{aligned}
R_0^\pm(\lambda)u(x) &= G_\lambda^\pm u(x) = \int_{\mathbb{R}^n} g_\lambda^\pm(x-y)u(y)\,dy,\\
g_\lambda^\pm(x) &:= \lim_{\mu\downarrow 0} g_{\lambda\pm i\mu}(x) = \{a_{2m}(\lambda) + b_{2m}(e^{\pm i\lambda|x|} + m_\lambda(x))\}|x|^{-2m}\\
&\quad + \sum_{j=m}^{2m-1} a_j(\lambda)|x|^{-j} + \sum_{j=m}^{2m-1} b_j(\lambda)(e^{\pm i\lambda|x|} + m_\lambda(x))|x|^{-j}\\
&\quad + \sum_{j=m}^{2m-1} c_j(\lambda)(e^{\pm i(\lambda|x|+\pi/2)} + n_\lambda(x))|x|^{-j},
\end{aligned}$$

where $R_0^\pm(\lambda) := \lim_{\mu\downarrow 0} R_0(\lambda \pm i\mu)$.

(2) *There exist positive constants* C_{abj} *for* $j = m, m+1, \ldots, 2m$ *such that*

$$|R_0^\pm(\lambda)u(x)| = |G_\lambda^\pm u(x)| \leq \sum_{j=m}^{2m} |D_j u(x)|,$$

$$D_j(\lambda)u(x) := C_{abj}\int_{\mathbb{R}^n} |x-y|^{-j}u(y)\,dy.$$

Let the resolvent of $H_0 = \sqrt{-\Delta}$ be denoted by $R_0(z) := (H_0 - z)^{-1} = \mathcal{F}^{-1}(|\xi| - z)^{-1}\mathcal{F}$. If $\mathrm{Re}(z) < 0$, we take the Laplace transform of $e^{-tH_0} = \mathcal{F}^{-1}e^{-t|\xi|}\mathcal{F}$ to get

$$\int_0^\infty e^{tz}e^{-tH_0}\,dt = (H_0 - z)^{-1} = R_0(z).$$

Lemma 2.1. *If* $t > 0$ *and* $u \in C_0^\infty(\mathbb{R}^n)$, *then*

$$e^{-tH_0}u(x) = \int_{\mathbb{R}^n} P_t(x-y)u(y)\,dy,$$

where

$$P_t(x) = \frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}, \quad c_n = \pi^{-(n+1)/2}\Gamma\left(\frac{n+1}{2}\right), \quad \Gamma(x) = \int_0^\infty s^{x-1}e^{-s}\,ds.$$

Proof. Using the idea of Strichartz [21, p.54], we get

$$\begin{aligned}\mathcal{F}^{-1}\left(e^{-t|\xi|}\right) &= \int_0^\infty \frac{t}{(\pi s)^{1/2}} e^{-st^2} \mathcal{F}^{-1}(e^{-|\xi|^2/4s})\, ds \\ &= \frac{2^{n/2} t}{(t^2+|x|^2)^{(n+1)/2}} \sqrt{\frac{1}{\pi}}\, \Gamma\left(\frac{n+1}{2}\right).\end{aligned}$$

Since the Fourier transform of convolution satisfies $\mathcal{F}(f * g) = (2\pi)^{n/2}\mathcal{F}(f)\mathcal{F}(g)$, we get $e^{-tH_0}u(x) = \mathcal{F}^{-1}e^{-t|\xi|}\mathcal{F}(u(x)) = P_t * u$. □

Lemma 2.2. *If* $\mathrm{Re}(z) < 0$, *then the integral*

$$\int_0^\infty e^{tz}\left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} P_t(x-y)u(y)\,dy\right)\overline{v}(x)\,dx\right\}$$

is absolutely convergent and is equal to $(R_0(z)u, v)_{L^2}$ *for all* $u, v \in C_0^\infty(\mathbb{R}^n)$, *where* $n \in \mathbb{N}$ *and* $n \geq 3$.

Proof. For $n = 3$, see T. Umeda [24, Theorem2.1]. For $n > 3$, if the integration in Lemma 2.2 is absolutely convergent, then

$$\begin{aligned}(R_0(z)u, v)_{L^2} &= \int_0^\infty e^{tz}(e^{-tH_0}u, v)_{L^2}\, dt \\ &= \int_0^\infty e^{tz}\left\{\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} P_t(x-y)u(y)\,dy\right)\overline{v}(x)\,dx\right\} dt.\end{aligned}$$

We consider the t-integration

$$\left|\int_0^\infty e^{tz}P_t(x-y)\,dt\right| \leq \left|\int_0^\infty e^{t(\mathrm{Re}\, z)}\frac{c_n t}{(t^2+|x-y|^2)^{(n+1)/2}}\,dt\right|.$$

Now we put

$$I_n = \int_0^\infty e^{t(\mathrm{Re}\, z)}\frac{c_n t}{(t^2+|x-y|^2)^{(n+1)/2}}\,dt.$$

Since

$$\frac{d}{dt}\left(-\frac{1}{n-1}\frac{1}{(t^2+|x-y|^2)^{(n-1)/2}}\right) = \frac{t}{(t^2+|x-y|^2)^{(n+1)/2}},$$

using the integration by parts, we see that I_n is equal to

$$I_n = -\frac{c_n}{n-1}\frac{1}{|x-y|^{n-1}} + \frac{c_n \operatorname{Re} z}{n-1}\int_0^\infty e^{t \operatorname{Re} z}\frac{1}{(t^2+|x-y|^2)^{(n-1)/2}}.$$

Then

$$|I_n| \le \frac{c_n}{n-1}\frac{1}{|x-y|^{n-1}} + \frac{c_n|\operatorname{Re} z|}{(n-1)|x-y|^{n-1}}\int_0^\infty e^{t \operatorname{Re} z}\,dt = \frac{2c_n}{n-1}\frac{1}{|x-y|^{n-1}}.$$

Thus we get

$$\begin{aligned}
|(R_0(z)u, v)_{L^2}| &\le \iint_{\mathbb{R}^{2n}} |I_n u(y)\overline{v}(x)|\,dx\,dy \\
&= \int_{\mathbb{R}^n} |v(x)|\,dx\left(\int_{|x-y|\ge 1} |I_n u(y)|\,dy + \int_{|x-y|\le 1} |I_n u(y)|\,dy\right) \\
&\le \int_{\mathbb{R}^n} |v(x)|\,dx\left(\frac{2c_n}{n-1}\|u\|_{L^1} + \frac{2c_n}{n-1}\|u\|_{L^\infty}\int_{|y|\le 1}\frac{1}{|y|^{n-1}}\,dy\right) \\
&< \infty.
\end{aligned}$$

Therefore we obtain the lemma. □

Theorem 2.1 is an immediate consequence of Lemmas 2.1 and 2.2.

We continue $g_z(x)$ analytically to the region $C \setminus [0,\infty)$ by using integration by parts.

Lemma 2.3. *If* $\operatorname{Re} z < 0$, *then there exist polynomials* $a_j(z), b_j(z), c_j(z)$, $j = m-1, m, \dots, 2m-1$, *such that*

$$\begin{aligned}
\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^m}\,dt &= b_{m-1}(z)M_z(x)|x|^{-(m-1)} \\
&\quad + \sum_{j=m}^{2m-1}(a_j(z)+b_j(z)M_z(x)+c_j(z)N_z(x))|x|^{-j}.
\end{aligned} \tag{2.4}$$

Proof. We will prove this lemma by induction.

(i) For $m=1$, since $\int_0^\infty e^{tz}1/(t^2+|x|^2)\,dt = M_z(x)$, (2.4) is obviously valid. For $m=2$, noticing that

$$\frac{1}{(t^2+|x|^2)^2} = \frac{1}{|x|^2}\left(\frac{1}{t^2+|x|^2} - \frac{t^2}{(t^2+|x|^2)^2}\right)$$
$$\frac{d}{dt}\left\{-\frac{1}{2}\frac{1}{t^2+|x|^2}\right\} = \frac{t}{(t^2+|x|^2)^2},$$

and using integration by parts, we get

$$\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^2}\,dt = \frac{1}{2}M_z(x)|x|^{-2} - \frac{z}{2}N_z(x)|x|^{-2}.$$

Then (2.4) is valid too.

(ii) Thus we assume that (2.4) is also valid for $m \le l$ where $l \ge 2$ and $l \in \mathbb{N}$. Now we will prove the case $m = l+1$.

For the case $m = l+1$, we have

$$\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^{l+1}}\,dt = |x|^{-2}\left\{\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^{l}}\,dt - \int_0^\infty te^{tz}\frac{t}{(t^2+|x|^2)^{l+1}}\,dt\right\}. \tag{2.5}$$

Noticing that

$$\begin{aligned}\frac{d}{dt}\left\{\frac{1}{-2l}\frac{1}{(t^2+|x|^2)^l}\right\} &= \frac{t}{(t^2+|x|^2)^{l+1}}\\ \frac{d}{dt}\left\{\frac{1}{-2(l-1)}\frac{1}{(t^2+|x|^2)^{l-1}}\right\} &= \frac{t}{(t^2+|x|^2)^{l}},\end{aligned}$$

and $\operatorname{Re} z < 0$, we make integrations by parts. Then we get

$$\begin{aligned}&\int_0^\infty te^{tz}\frac{t}{(t^2+|x|^2)^{l+1}}\,dt\\ &= \frac{1}{2l}\int_0^\infty \frac{d}{dt}(te^{tz})\frac{1}{(t^2+|x|^2)^l}\,dt\\ &= \frac{1}{2l}\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^l}\,dt + \frac{z}{2l}\int_0^\infty e^{tz}\frac{t}{(t^2+|x|^2)^l}\,dt\\ &= \frac{1}{2l}\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^l}\,dt + \frac{z}{4l(l-1)}\left\{|x|^{-(2l-2)} + z\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^{l-1}}\,dt\right\}.\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$\begin{aligned}\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^{l+1}}\,dt = &-\frac{z}{4l(l-1)}|x|^{-2l} + |x|^{-2}\frac{2l-1}{2l}\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^l}\,dt\\ &- |x|^{-2}\frac{z^2}{4l(l-1)}\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^{l-1}}\,dt.\end{aligned} \tag{2.7}$$

Then using assumption of the cases $m = l$ and $m = l-1$, we obtain that (2.4) is valid for $m = l+1$.

Finally, using (i) and (ii), we can finish the proof of (2.4) for any integer $m \ge 1$. □

Then by the definition in the Theorem 2.1 we can compute the resolvent kernel $g_z(x)$. We give the next lemma.

Lemma 2.4. *If* $\operatorname{Re} z < 0$, *there exist polynomials* $a_j(z)$, $b_j(z)$, $c_j(z)$, $j = m - 1, m, \dots, 2m-1$, *such that*

$$g_z(x) = \frac{c_n}{2m}|x|^{-2m} + b_{m-1}(z)M_z(x)|x|^{-(m-1)} + \sum_{j=m}^{2m-1}(a_j(z) + b_j(z)M_z(x) + c_j(z)N_z(x))|x|^{-j},$$

where c_n, $g_z(x)$ *are the same as in* Theorem 2.1.

Proof. From (2.3), noticing that

$$\frac{d}{dt}\left\{-\frac{1}{2m}\frac{1}{(t^2+|x|^2)^m}\right\} = \frac{t}{(t^2+|x|^2)^{m+1}}$$

and making integration by parts, we get

$$\begin{aligned} g_z(x) &= \int_0^\infty e^{tz}\frac{c_n t}{(t^2+|x|^2)^{(n+1)/2}}\,dt = \int_0^\infty e^{tz}\frac{c_n t}{(t^2+|x|^2)^{m+1}}\,dt \\ &= \frac{c_n}{2m}|x|^{-2m} + \frac{c_n z}{2m}\int_0^\infty e^{tz}\frac{1}{(t^2+|x|^2)^m}\,dt. \end{aligned}$$

Thus using Lemma 2.3, we obtain the lemma. □

Making analytic continuation of si(z) and ci(z), we can get the next theorem.

Theorem 2.3. *Let* $n = 2m+1$, $m \geq 1$ ($m \in \mathbb{N}$) *and* $z \in \mathbb{C}\setminus[0,\infty)$. *If* $u \in C_0^\infty(\mathbb{R}^n)$, *then there exist polynomials* $a_j(z)$, $b_j(z)$, $c_j(z)$, $j = m-1, m, \dots, 2m-1$, *such that*

$$\begin{aligned} R_0(z)u(x) &= G_z u(x) := \int_{\mathbb{R}^n} g_z(x-y)u(y)\,dy, \\ g_z(x) &= \frac{c_n}{2m}|x|^{-2m} + b_{m-1}(z)M_z(x)|x|^{-(m-1)} \\ &\quad + \sum_{j=m}^{2m-1}(a_j(z) + b_j(z)M_z(x) + c_j(z)N_z(x))|x|^{-j}. \end{aligned}$$

Proof. From Theorem 2.1 and Lemma 2.4, we get

$$R_0(z)u = G_z u,$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{Re} z < 0$. From (2.1) and (2.2), $(G_z u, v)_{L^2}$ is a holomorphic function of z in $\mathbb{C}\setminus[0,\infty]$ for any test function $v \in \mathcal{S}(\mathbb{R}^n)$. Then $(R_0(z)u, v)_{L^2}$ is also a holomorphic fuction of z in $\mathbb{C}\setminus[0,\infty]$. □

Next, let $z = \lambda + i\mu$ and $\lambda > 0$. We study the limit of $g_z(x)$ as $\mu \downarrow 0$. From (2.1) and (2.2), we get

$$\mathrm{si}(-z) \to -\pi - \mathrm{si}(\lambda), \quad \mathrm{ci}(-z) \to \pm i\pi + \mathrm{ci}(\lambda),$$

as $\mu \downarrow 0$. Then we get

$$\lim_{\mu\downarrow 0} M_{\lambda\pm i\mu}(x) = |x|^{-1}\{e^{\pm i\lambda|x|} + m_\lambda(x)\},$$
$$\lim_{\mu\downarrow 0} N_{\lambda\pm i\mu}(x) = e^{\pm i(\lambda|x|+\pi/2)} + n_\lambda(x).$$

This fact together with Lemma 2.4 yields that there exist polynomials $a_j(\lambda)$, $b_j(\lambda)$, $c_j(\lambda)$, $j = m, m+1, \ldots, 2m$ such that

$$\begin{aligned}
g^{\pm}_{\lambda}(x) &:= \lim_{\mu\downarrow 0} g_{\lambda\pm i\mu}(x) \\
&= \{a_{2m}(\lambda) + b_{2m}(e^{\pm i\lambda|x|} + m_\lambda(x))\}|x|^{-2m} \\
&\quad + \sum_{j=m}^{2m-1} a_j(\lambda)|x|^{-j} + \sum_{j=m}^{2m-1} b_j(\lambda)(e^{\pm i\lambda|x|} + m_\lambda(x))|x|^{-j} \\
&\quad + \sum_{j=m}^{2m-1} c_j(\lambda)(e^{\pm i(\lambda|x|+\pi/2)} + n_\lambda(x))|x|^{-j}.
\end{aligned} \tag{2.8}$$

Checking the properties of $g^{\pm}_{\lambda}(x)$, we get the next lemma.

Lemma 2.5. *Let* $[a, b] \subset (0, \infty)$. *If* $\lambda \in [a, b]$, *then there exist positive constants* C_{abj}, $j = m, m+1, \ldots, 2m$, *such that*

$$|g^{\pm}_{\lambda}(x)| \le \sum_{j=m}^{2m} C_{abj}|x|^{-j}.$$

Proof. It follows from the definition of $\mathrm{ci}(t)$ and $\mathrm{si}(t)$ that

$$|\,\mathrm{ci}(t)| \le \text{const.}\begin{cases} t^{-1} & \text{if} \quad t \ge 1, \\ 1 + |\log t| & \text{if} \quad 0 < t < 1, \end{cases}$$

and the integration by parts yields that $|\,\mathrm{si}(t)| \le \text{const.}(1+|t|)^{-1}$. Since $\lim_{t\downarrow 0} \sin t(1+|\log t|) = 0$, and $|x|^{\delta}(1+|\log(\lambda|x|)|) \to 0$ $(|x| \to 0)$ for all $\delta > 0$, we get $|m_\lambda(x)| \le C_{ab}$, $n_\lambda(x)| \le C_{ab}|x|^{-1}$. This fact, together with $|e^{\pm i\lambda|x|}| = |e^{\pm i(\lambda|x|+\pi/2)}| = 1$ and (2.8), gives the lemma. □

Then, we can give the next theorem.

Theorem 2.4. *Let $n = 2m + 1$, $m \geq 1$ $(m \in \mathbb{N})$ and $\lambda > 0$. If $u \in C_0^\infty(\mathbb{R}^n)$, then there exist polynomials $a_j(\lambda)$, $b_j(\lambda)$, $c_j(\lambda)$, λ for $j = m, m+1, \ldots, 2m$, such that*

$$R_0^\pm(\lambda)u(x) = G_\lambda^\pm u(x), \quad G_\lambda^\pm u(x) := \int_{\mathbb{R}^n} g_\lambda^\pm(x-y)u(y)\,dy,$$

where $R_0^\pm(\lambda) := \lim_{\mu\downarrow 0} R_0(\lambda \pm i\mu)$, and $g_\lambda^\pm(x)$ are defined by (2.8).

Proof. Let u and v belong to $C_0^\infty(\mathbb{R}^n)$. Noticing that if $c > 0$, then there exists a positive constant $C_{\lambda uvc}$ such that

$$\left|g_{\lambda\pm i\mu}(x-y)u(y)\overline{v(x)}\right| \leq C_{\lambda uvc}|x-y|^{2m}|u(y)v(x)|$$

for all $0 \leq |x| < c$, we can prove this theorem similarly to T. Umeda [24, Theorem 4.1]. $\square$

Next, we will consider the action of the resolvent on the functions in $L^{2,s}(\mathbb{R}^n)$ for $s > 1/2$. It follows from Lemma 2.5 that if $[a,b] \subset (0,\infty)$ and $\lambda \in [a,b]$, there exist positive constants C_{abj}, $j = m, m+1, \ldots, 2m$, such that

$$|G_\lambda^\pm u(x)| \leq \sum_{j=m}^{2m} |D_j u(x)|, \quad D_j u(x) := C_{abj}\int_{\mathbb{R}^n} |x-y|^{-j}u(y)\,dy. \tag{2.9}$$

We will consider the properties of D_j. At first, we make the next preparations.

Lemma 2.6. *Let $n \in \mathbb{N}$ and $\Phi(x)$ be defined by*

$$\Phi(x) := \int_{\mathbb{R}^n} \frac{1}{|x-y|^\beta \langle y\rangle^\gamma}\,dy.$$

If $0 < \beta < n$ and $\beta + \gamma > n$, then $\Phi(x)$ is a bounded continuous function satisfying

$$|\Phi(x)| \leq C_{\beta\gamma n}\begin{cases} \langle x\rangle^{-(\beta+\gamma-n)} & \text{if} \quad 0 < \gamma < n, \\ \langle x\rangle^{-\beta}\log(1+\langle x\rangle) & \text{if} \quad \gamma = n, \\ \langle x\rangle^{-\beta} & \text{if} \quad \gamma > n, \end{cases}$$

where $C_{\beta\gamma n}$ is a constant depending on β, γ and n.

For the proof of this lemma, see T. Umeda [24, p.62, Lemma A.1].

Lemma 2.7. *Let $s > 1/2$. If $u(x)$ belongs to $L^{2,s}(\mathbb{R}^n)$, then there exists a positive constant C_{abs} such that $|D_m u(x)| \leq C_{abs}\|u\|_{L^{2,s}}$.*

Proof. Letting $s > 1/2$ and using the definition of $D_m u(x)$ and the Schwarz inequality, we have

$$|D_m u(x)| \le C_{abs} \left\{ \int_{\mathbb{R}^n} \frac{1}{|x-y|^{2m} \langle y \rangle^{2s}} \, dy \right\}^{1/2} \|u\|_{L^{2,s}}.$$

Applying Lemma 2.6 with $\beta = 2m$ and $\gamma = 2s > 1$, we get this lemma. □

Lemma 2.8. *Let $s > n/2$. If $u(x)$ belongs to $L^2(\mathbb{R}^n)$ then there exists a positive constant C_{abjs} such that for all $m+1 \le j \le 2m$, $\|D_j u\|_{L^{2,-s}} \le C_{abjs} \|u\|_{L^2}$.*

Proof. First, letting $u \in L^{2,s}(\mathbb{R}^n)$, we prove that

$$\|D_j u\|_{L^2} \le C_{abs} \|u\|_{L^{2,s}}, \tag{2.10}$$

where C_{abs} is a positive constant. With $B = \{x \mid |x| \le 1\}$ and $E = \{x \mid |x| \ge 1\}$, we decompose $|x|^{-j}$ into two parts

$$|x|^{-j} = h_{Bj}(x) + h_{Ej}(x),$$
$$h_{Bj}(x) := \frac{F(x \le 1)}{|x|^j}, \quad h_{Ej}(x) := \frac{F(x \ge 1)}{|x|^j},$$

where $F(x \le 1)$ and $F(x \ge 1)$ are the characteristic functions of the sets B and E respectively. It is easy to prove that $h_{Bj}(x) \in L^1(\mathbb{R}^n)$, $h_{Ej}(x) \in L^2(\mathbb{R}^n)$ for all $m+1 \le j \le 2m$. Then we can prove (2.10) for all $s > n/2$ similarly to T. Umeda [24, Lemma 5.1 (i)].

Next, let $u \in L^{2,s}(\mathbb{R}^n)$. Then the lemma follows from (2.10) similarly to T. Umeda [24, Lemma 5.1 (ii)]. □

Proof of Theorem 2.2. In view of Theorem 2.4, (2.9), Lemma 2.7 and Lemma 2.8, we obtain the theorem. □

3. Boundedness of the generalized eigenfunctions

In this section, we assume that n, $V(x)$ and k satisfy the following inequalities:

(1) $\quad n = 2m+1 \ (m \in \mathbb{N}) \quad \text{and} \quad m \ge 1,$

(2) $\quad |V(x)| \le C\langle x \rangle^{-\sigma}, \quad \sigma > \dfrac{n+1}{2},$

(3) $\quad k \in \{k \mid a \le |k| \le b\} \quad \text{and} \quad [a, b] \subset (0, \infty) \setminus \sigma_p(H).$

Applying Theorem 1.4, we see that generalized eigenfuction $\varphi^{\pm}(x, k)$ defined by

$$\varphi^{\pm}(x, k) = \varphi_0(x, y) - R^{\mp}(|k|)\{V(\,\cdot\,)\varphi_0(\,\cdot\,, k)\}(x), \tag{3.1}$$

satisfies the equation

$$\varphi^{\pm}(x, k) = \varphi_0(x, k) - R_0^{\mp}(|k|)\{V(\,\cdot\,)\varphi^{\pm}(\,\cdot\,, k)\}(x), \tag{3.2}$$

where $\varphi_0(x, k) = e^{ix\cdot k}$.

In this section, let $\{D_j V(\cdot)\varphi^{\pm}(\cdot, k)\}(x)$ be denoted by $D_j V(x)\varphi^{\pm}(x, k)$. Moreover, let $V(x)D_{j_r} V(x)D_{j_{r-1}} \cdots V(x)D_{j_1} V(x)\varphi^{\pm}(x, k)$ be denoted by

$$\left(\prod_{p=1}^{r} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x, k)\}.$$

The main theorem is

Theorem 3.1. *Let $n = 2m + 1$, $m \geq 1$ $(m, n \in \mathbb{N})$, and $[a, b] \subset (0, \infty) \setminus \sigma(pH)$. Then there exists a constant C_{ab} such that generalized eigenfunctions defined by $\varphi^{\pm}(x, k) := \varphi_0(x, y) - R^{\mp}(|k|)\{V(\,\cdot\,)\varphi_0(\,\cdot\,, k)\}(x)$ satisfy*

$$|\varphi^{\pm}(x, k)| \leq C_{ab},$$

for all $(x, k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$, where $\varphi_0(x, k) = e^{ix\cdot k}$.

First, in order to use Theorem 2.2, we have to prove that $V(x)\varphi^{\pm}(x, k)$ belongs to $L^{2,s}$ for $s > 1/2$.

Lemma 3.1. *If $s > n/2$, then $V(\,\cdot\,)\varphi^{\pm}(\,\cdot\,, k)$ are $L^{2,\sigma-s}(\mathbb{R}^n_x)$-valued continuous fuctions on $\{k \mid |k| \in (0, \infty) \setminus \sigma_p(H)\}$.*

Proof. The lemma follows from Lemma 1.1 and the definition (3.1) similarly to T. Umeda [24, Lemma 9.1]. $\square$

From Lemma 3.1 with $\sigma > m + 1$, Theorem 2.2 and (3.2), we get

$$|\varphi^{\pm}(x, k)| \leq |\varphi_0(x, k)| + |R_0^{\mp}(|k|)\{V(\,\cdot\,)\varphi^{\pm}(\,\cdot\,, k)\}(x)| \leq 1 + \sum_{j=m}^{2m} |D_j V(x)\varphi^{\pm}(x, k)|, \tag{3.3}$$

where D_j are the same operators as those in Theorem 2.2. We now give some lemmas concerning the properties of D_j.

Lemma 3.2. *There exists a positive constant C_{ab}, such that $|D_m V(x)\varphi^{\pm}(x,k)| \leq C_{ab}$, for all $(x,k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$.*

Proof. From Lemma 3.1, we get $\|V(x)\varphi^{\pm}(x,k)\|_{L^{2,s}} \leq C'_{ab}$, for $(x,k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$, where C'_{ab} is a positive constant. This fact, together with Lemma 2.7, gives the lemma. □

Lemma 3.3. *Let $m \leq j \leq 2m$ ($j \in \mathbb{N}$) and C'_{ab} is a positive constant. If $|u(x,k)| \leq C'_{ab}$ for all $(x,k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$, then there exists a positive constant such that*

$$|D_j V(x) u(x,k)| \leq C_{ab},$$

for all $(x,k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$.

Proof. From definition (2.9), the assumption and $|V(y)| \leq C\langle y\rangle^{-\sigma}$, we get

$$\begin{aligned}|D_j V(x) u(x,k)| &\leq C'_{ab} C_{abj} \int_{\mathbb{R}^n} |x-y|^{-j} |V(y)|\, dy \\ &\leq C C'_{ab} C_{abj} \int_{\mathbb{R}^n} |x-y|^{-j} \langle y\rangle^{-\sigma}\, dy.\end{aligned}$$

Since $j \geq m$ and $\sigma > m+1$, we get $j+\sigma > n$. Then applying Lemma 2.6 with $\beta = j$, $\gamma = \sigma$, we obtain the lemma. □

Lemma 3.4. *Let $m+1 \leq j \leq 2m$ ($j \in \mathbb{N}$) and $p > n/(n-j)$. If $u(x,k) \in L^2(\mathbb{R}^n_x) \cap L^p(\mathbb{R}^n_x)$, and $\|u(x,k)\|_{L^2} \leq C'_{ab}$, $\|u(x,k)\|_{L^p} \leq C''_{ab}$, ($C'_{ab}$ and C''_{ab} are positive constants) for all $(x,k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$, then there exists a positive constant C_{ab}, such that*

$$|D_j u(x,k)| \leq C_{ab},$$

for all $(x,k) \in \mathbb{R}^n \times \{a \leq |k| \leq b\}$.

Proof. From definition (2.9), we get

(3.4)
$$\begin{aligned}|D_j u(x)| &\leq C_{abj} \int_{\mathbb{R}^n} |x-y|^{-j} |u(y,k)|\, dy \\ &\leq C_{abj} \int_{|x-y|\leq 1} |x-y|^{-j} |u(y,k)|\, dy + C_{abj} \int_{|x-y|>1} |x-y|^{-j} |u(y,k)|\, dy.\end{aligned}$$

The assumption $\|u(x,k)\|_{L^2} \leq C'_{ab}$, together with the Schwarz inequality, yields

$$\int_{|x-y|>1} |x-y|^{-j} |u(y,k)|\, dy \leq C'_{ab} \left(\int_{|x-y|>1} |x-y|^{-2j}\, dy \right)^{1/2}. \tag{3.5}$$

Since $j \geq m+1$, we have $2j > n$, so that the function of x defined by the integral on the right hand side is bounded. The assumption $\|u(x,k)\|_{L^p} \le C''_{ab}$, together with the Hölder inequality, gives

$$\int_{|x-y|\le 1} |x-y|^{-j}|u(y,k)|\,dy \le C''_{ab}\left(\int_{|x-y|\le 1} (|x-y|^{-j})^{p/(p-1)}\,dy\right)^{(p-1)/p}. \tag{3.6}$$

Since $p > n/(n-j) > 1$ $(m+1 \le j \le 2m)$, we have $jp/(p-1) = j/(1-1/p) < j/(1-(n-j)/n) = n$. So the function of x defined by the integral on the right hand side of (3.6) is bounded. In view of (3.4), (3.5) and (3.6), we obtain the lemma. □

Lemma 3.5. *Let r, $j_p \in \mathbb{N}$ and $s > 1/2$. If $m+1 \le j_p \le 2m$ for $1 \le p \le r$, then*

$$\left(\prod_{p=1}^{r} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\} \in L^{2,s}(\mathbb{R}^n_x)$$

for all $r \in \mathbb{N}$. Moreover, there exits a positive constant C_{ab} such that

$$\left\|\left(\prod_{p=1}^{r} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right\|_{L^{2,s}} \le C_{ab}$$

for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$.

Proof. Applying Lemma 3.1, we see that there exists a positive constant C'_{ab} such that

$$\|V(x)\varphi^{\pm}(x,k)\|_{L^2} \le C'_{ab}. \tag{3.7}$$

For $m+1 \le j_1 \le 2m$, by Lemma 2.8, we have that if $\sigma - 1/2 > t > n/2$, there exists a positive constant C_{abj_1s} such that

$$\|D_{j_1}V(x)\varphi^{\pm}(x,k)\|_{L^{2,-t}} \le C_{abj_1t}\|V(x)\varphi^{\pm}(x,k)\|_{L^2} \le C_{abj_1t}C'_{ab}.$$

Noticing that $|V(x)| < C\langle x\rangle^{-\sigma}$, $\sigma > (n+1)/2$, and $\sigma - t > 1/2$, where C is a positive constant, we get

$$\|V(x)D_{j_1}V(x)\varphi^{\pm}(x,k)\|_{L^{2,\sigma-t}} \le CC_{abj_1s}C'_{ab}.$$

Similarly, we can prove this lemma by induction. □

Lemma 3.6. *Let $0 < \alpha < n$, $1 < p < q < \infty$ and $f \in L^p(R^n)$. Let $I_\alpha f(x)$ be definded by $I_\alpha f(x) := \int_{\mathbb{R}^n} |x-y|^{-n+\alpha} f(y)dy$. If $1/q = 1/p - \alpha/n$, there exists a positive constant C_{pq} such that*

$$\|I_\alpha f\|_{L^q} \le C_{pq}\|f\|_{L^p}.$$

For the proof of the lemma, see [20, p.119].

Lemma 3.7. *Let $r \in \mathbb{N}$. If $m+1 \le j_p \le 2m$ $(1 \le q \le r)$, and $2\sum_{p=1}^{q} j_p > (2q-1)n$ for all $q \le r$, then*

$$\left(\prod_{p=1}^{r} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\} \in L^{2n/\left(2\sum_{p=1}^{r} j_p-(2r-1)n\right)}(\mathbb{R}^n_x)$$

for all $r \le n-1$. Moreover, there exits a positive constant C_{ab} such that

$$\left\|\left(\prod_{p=1}^{r} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right\|_{L^{2n/\{2\sum_{p=1}^{r} j_p-(2r-1)n\}}} \le C_{ab} \tag{3.8}$$

for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$.

Proof. For $r=1$, since $m+1 \le j_1 \le 2m$, we get $0 < 2j_1 - n < n$. Let $\beta = 2$, $\alpha = n - j_1$, $\gamma = 2n/(2j_1-n)$. Then $0 < \beta = 2 < \gamma$, and $1/\gamma = 1/\beta - \alpha/n$. Since $|V(x)| < C\langle x\rangle^{-\sigma} < C$ (C is a positive constant), we apply Lemma 3.6 with $p = \beta$, $q = \gamma$, and we get that there exists a constant $C_{\beta\gamma}$ such that

$$|V(x)D_{j_1}V(x)\varphi^{\pm}(x,k)| \le CC_{abj_1}\int_{\mathbb{R}^n} |x-y|^{-n+\alpha}V(y)\varphi^{\pm}(y,k)\,dy.$$

Therefore we have

$$\|V(x)D_{j_1}V(x)\varphi^{\pm}(x,k)\|_{L^\gamma} \le CC_{abj_1}C_{\beta\gamma}\|V(x)\varphi^{\pm}(x,k)\|_{L^\beta}.$$

This fact together with (3.7) gives (3.8) for $r=1$. Similarly, we can prove this lemma by induction. □

Lemma 3.8. *Let $r \in \mathbb{N}$ and $r \le n$. If $m \le j_p \le 2m$ for all $1 \le p \le r$, then there exists a positive constant C_{ab} such that*

$$\sum_{2\sum_{p=1}^{r} j_p < (2r-1)n} \left|D_{j_r}\left(\prod_{p=1}^{r-1} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right| \le C_{ab}, \tag{3.9}$$

for all $(x,k) \in \mathbb{R}^n \times \{a \le |k| \le b\}$.

Proof. We will prove this lemma by induction.

(i) For $r = 1$, since $m \le j_1 \le 2m$, then $2j_1 < n \Rightarrow j_1 = m$, so

$$\sum_{2j_1<n} D_{j_1} V(x)\varphi^{\pm}(x,k) = D_m V(x)\varphi^{\pm}(x,k).$$

Applying Lemma 3.2, we see that (3.9) is valid for $r = 1$.

(ii) Thus we assume that (3.9) is also valid for $r \le l$ where $l \ge 1$ and $l \in \mathbb{N}$. Now we will prove the case $r = l + 1$.

From the assumption of cases $r \le l$, there exist positive constants C_{abr} such that

$$\sum_{2\sum_{p=1}^{l} j_p<(2l-1)n} \left| D_{j_r}\left(\prod_{p=1}^{r-1} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right| \le C_{abr} \tag{3.10}$$

for $r \le l$.

For the case $r = l + 1$. Let

$$A = \{(j_1, j_2, \dots, j_l) \mid m \le j_p \le 2m \text{ for all } 1 \le p \le l\},$$
$$B = \{(j_1, j_2, \dots, j_l) \mid m+1 \le j_p \le 2m \text{ for all } 1 \le p \le l\},$$
$$C = A \cap \left\{(j_1, j_2, \dots, j_l) \;\middle|\; 2\sum_{p=1}^{q} j_p > (2q-1)n \text{ for } 1 \le q \le l\right\}.$$

Since n is an odd integer, there does not exist $(j_1, j_2, \dots, j_l)$ satisfying $2\sum_{p=1}^{r} j_p = (2l-1)n$, for $r \le l$. Then, we get

$$\begin{aligned}
&\sum_{j_1,j_2,\dots,j_r} \left| D_{j_r}\left(\prod_{p=1}^{r-1} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right| \\
&= \sum_{2\sum_{p=1}^{r} j_p>(2l-1)n} \left| D_{j_r}\left(\prod_{p=1}^{r-1} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right| \\
&\quad + \sum_{2\sum_{p=1}^{r} j_p<(2l-1)n} \left| D_{j_r}\left(\prod_{p=1}^{r-1} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right|
\end{aligned}$$

for all $r \le l$. By this fact together with assumption (3.10) and Lemma 3.3, we get that there exists a positive constant C'_{ab} such that

$$\left| D_{l+1}V(x)\left\{\sum_{A\setminus C} D_{j_l}\left(\prod_{p=1}^{l-1} V(x)D_{j_p}\right)\{V(x)\varphi^{\pm}(x,k)\}\right\}\right| \le C'_{ab}. \tag{3.11}$$

From Lemma 3.2, Lemma 3.5, Lemma 2.7 and Lemma 3.3, we see that there exists a positive constant C''_{ab}, such that

$$\left| D_{l+1} V(x) \left\{ \sum_{A \setminus B} D_{j_l} \left(\prod_{p=1}^{l-1} V(x) D_{j_p} \right) \{ V(x) \varphi^{\pm}(x,k) \} \right\} \right| \leq C''_{ab}. \tag{3.12}$$

For $(j_1, j_2, \ldots, j_l) \in B \cap C$, applying Lemma 3.5 and Lemma 3.7, we see that there exists a positive constant C_{abl} such that

$$\left\| \left(\prod_{p=1}^{l} V(x) D_{j_p} \right) \{ V(x) \varphi^{\pm}(x,k) \} \right\|_{L^2} \leq \left\| \left(\prod_{p=1}^{l} V(x) D_{j_p} \right) \{ V(x) \varphi^{\pm}(x,k) \} \right\|_{L^{2,s}} \leq C_{abl},$$

$$\left\| \left(\prod_{p=1}^{l} V(x) D_{j_p} \right) \{ V(x) \varphi^{\pm}(x,k) \} \right\|_{L^{2n/\{2 \sum_{p=1}^{l} j_p - (2l-1)n\}}} \leq C_{abl},$$

where $s > 1/2$. For $2 \sum_{p=1}^{l+1} j_p < (2l+1)n$, we get $2n \big/ \big(2 \sum_{p=1}^{l} j_p - (2l-1)n \big) > n/(n-(l+1))$. It follows from Lemma 3.4 that there exists a positive constant $C_{ab,l+1}$ such that

$$\left| D_{l+1} \sum_{B \cap C} \left(\prod_{p=1}^{l} V(x) D_{j_p} \right) \{ V(x) \varphi^{\pm}(x,k) \} \right| \leq C_{ab,l+1} \tag{3.13}$$

for $2 \sum_{p=1}^{l+1} j_p < (2l+1)n$. Collecting (3.11), (3.12) and (3.13), we obtain that (3.9) is valid for $r = l+1$.

Finally, using (i) and (ii), we finish the proof of (3.9) for any integer $r \geq 1$. □

In view of the lemmas and (3.3), we will prove the main theorem 3.1.

Proof of Theorem 3.1. From (3.3), we get $|\varphi^{\pm}(x,k)| \leq 1 + \sum_{j_n=m}^{2m} |D_{j_n} V(x) \varphi^{\pm}(x,k)|$. Applying (3.3) again, similarly, we see that there exists a positive constant C'_{ab} such that

$$|\varphi^{\pm}(x,k)| \leq C'_{ab} + \sum_{j_1, j_2, \ldots, j_n} \left| \left(\prod_{p=1}^{n} V(x) D_{j_p} \right) \{ V(x) \varphi^{\pm}(x,k) \} \right|, \tag{3.14}$$

where $m \le j_p \le 2m$ for $1 \le p \le n$. Noticing that $2\sum_{p=1}^{n} j_p < 2n \times 2m = 2n^2 - 2n < (2n-1)n$, we get

$$\begin{aligned}
&\sum_{j_1, j_2, \dots, j_n} \left| \left(\prod_{p=1}^{n} V(x) D_{j_p} \right) \{V(x)\varphi^{\pm}(x, k)\} \right| \\
&= \sum_{2\sum_{p=1}^{n} j_p < (2n-1)n} \left| \left(\prod_{p=1}^{n} V(x) D_{j_p} \right) \{V(x)\varphi^{\pm}(x, k)\} \right|.
\end{aligned}$$

This fact together with Lemma 3.8 with $r = n$ yields that there exists a positive constant C''_{ab} such that

$$\sum_{j_1, j_2, \dots, j_n} \left| \left(\prod_{p=1}^{n} V(x) D_{j_p} \right) \left\{ V(x)\varphi^{\pm}(x, k) \right\} \right| \le C''_{ab}.$$

From this inequality together with (3.14)), we finally have the theorem. $\square$

4. Asymptotic completeness

We investigate the asymptotic completeness of wave operators in this section. We assum that the potential $V(x)$ is a real-valued measurable function on $\mathbb{R}^n$ satisfying

$$|V(x)| \le C\langle x\rangle^{-\sigma}, \quad \sigma > 1. \tag{4.1}$$

Under this assumption, it is obvious that V is a bounded selfadjoint operator in $L^2(\mathbb{R}^n)$, and that $H = H_0 + V$ defines a selfadjoint operator in $L^2(\mathbb{R}^n)$, whose domain is $H^1(\mathbb{R}^n)$ (see T. Umeda [23, Theorem 5.8]). Moreover H is essentially selfadjoint on $C_0^\infty(\mathbb{R}^n)$ (see T. Umeda [23]). Since V is relatively compact with respect to H_0, it follows from Reed-Simon [18, p.113, Corollary 2] that

$$\sigma_e(H) = \sigma_e(H_0) = [0, \infty).$$

In this section, we prove the next main theorem with V. Enss's idea (see V. Enss [3] and H. Isozaki [7]).

Theorem 4.1. *Let* $H_0 = \sqrt{-\Delta}$, $H = H_0 + V(x)$ *and* $V(x)$ *satisfies* (4.1). *Then there exists the limits*

$$W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0},$$

and the asymptotic completeness holds:

$$\mathcal{R}(W_\pm) = \mathcal{H}_{ac}(H).$$

Lemma 4.1. *Let* $H_0 = \sqrt{-\Delta}$, $H = H_0 + V(x)$ *and* $V(x)$ *satisfies* (4.1). *Then there exists the limits*

$$W_\pm = \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}.$$

Proof. The proof of this lemma is similar to H. Kitada [8, p.60, Theorem 6.2]. □

It is obvious that $\mathcal{R}(W_\pm) \subset \mathcal{H}_{ac}(H)$ (see [7, p.70 Lemma 1.2]), then we just need to prove that $\mathcal{H}_{ac}(H) \subset \mathcal{R}(W_\pm)$.

Let $\varphi(t) \in C_0^\infty((a, b))$, $a > 0$, $\rho_\pm(t) \in C_0^\infty(\mathbb{R})$ satisfy $\rho_+(t) + \rho_1(t) = 1$, $\rho_+(t) = 0$ for $t < -1/2$, $\rho_-(t) = 0$ for $t > 1/2$. Let $\chi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $\chi(x) = 0$ for $|x| < 1$, $\chi(x) = 1$ for $|x| > 2$. We put $\omega_x = x/|x|$ and $\omega_\xi = \xi/|\xi|$. Let $p_\pm(x, \xi)$ be defined by

$$p_\pm(x, \xi) = \rho_\pm(\omega_x \cdot \omega_\xi)\chi(x)\varphi(|\xi|),$$

and $P_\pm$ is the psendodifferential operator with symbol $p_\pm(x, \xi)$

$$P_\pm u = (2\pi)^{n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} p_\pm(x, \xi)\hat{u}(\xi)\, d\xi$$

and $P_\pm(A) = \chi(x/A)P_\pm$ $(A > 0)$. Let $F(t > A)$ and $F(t < A)$ be the characteristic functions of the sets $\{t \mid t > A\}$ and $\{t \mid t < A\}$, respectively.

Lemma 4.2. *If* $u \in \mathcal{H}_{ac}(H)$, *then* $e^{-itH}u$ *converges weakly to* 0 *as* $t \to \infty$.

Proof. Let $E_H(\lambda)$ be the spectral measure on H. For every $v \in L^2(\mathbb{R}^n)$, we have

$$(e^{-itH}u, v) = \int_{-\infty}^{\infty} e^{-it\lambda}\, d(E_H(\lambda)u, v).$$

Since $(E_H(\lambda)u, v)$ is absolutely continuous on λ, there exists a function $f(\lambda) \in L^1(\mathbb{R})$, such that

$$(e^{-itH}u, v) = \int_{-\infty}^{\infty} e^{-it\lambda} f(\lambda)\, d\lambda.$$

Lemma 4.2 now follows from Riemann-Lebesgue's lemma. □

Lemma 4.3. *Let* $d > 0$, $s \geq 1$. *Then*

$$\sup_{t>d} \|(1 + t + |x|)^s P_- e^{-itH_0}\langle x\rangle^{-s}\|_{L^2} < \infty, \tag{4.2}$$

$$\sup_{t<-d} \|(1 - t + |x|)^s P_+ e^{-itH_0}\langle x\rangle^{-s}\|_{L^2} < \infty. \tag{4.3}$$

Proof. We will prove (4.2). The proof of (4.3) is similar. Using the interpolation theorem, we just need to prove the cases $s \in \mathbb{N}$. Let $\hat{u}(\xi)$ be the Fourier transform of $u(x)$. The definition of $P_- e^{-itH_0}$ is

$$P_- e^{-itH_0} u = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|)} p_-(x, \xi)\hat{u}(\xi)\, d\xi.$$

Let

$$L = -i|\nabla_\xi(x\cdot\xi - t|\xi|)|^{-2}\nabla_\xi(x\cdot\xi - t|\xi|)\cdot\nabla_\xi.$$

We have $Le^{i(x\cdot\xi - t|\xi|)} = e^{i(x\cdot\xi - t|\xi|)}$. Since supp $p_- \subset \{\omega_x \cdot \omega_\xi < 1/2\}$ and $t > 0$, we get

$$|\nabla_\xi(x\cdot\xi - t|\xi|)|^2 = |x - t\omega_\xi|^2 = |x|^2 + t^2 - 2tx\cdot\omega_\xi > |x|^2 + t^2 - t|x| \geq \frac{1}{2}(|x|^2 + t^2).$$

Noticing that $t > d > 0$, we have that there exists a positive constant C such that,

$$|\nabla_\xi(x\cdot\xi - t|\xi|)| > C(|x| + t + 1). \tag{4.4}$$

Then using integration by parts, we have

$$P_- e^{-itH_0} u = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{i(x\cdot\xi - t|\xi|)} L^*\{p_-(x, \xi)\hat{u}(\xi)\}\, d\xi,$$

where L^* is adjoint operator of L. Notice that supp $p_- \subset \{a < |\xi| < b\}$ and $\sigma > 2$. Then we see that there exists a positive constant C_1 such that $|P_- e^{-itH_0}u| < C_1(1 + t + |x|)^{-1}$. Thus we get $(1 + t + |x|)|P_- e^{-itH_0}u| < C_1$. Then, we use integration by parts again and we get that there exists a positive constant C_2 such that $(1 + t + |x|)^2|P_- e^{-itH_0}u| < C_2$. Similarly, for $s \in \mathbb{N}$, we get $(1+t+|x|)^s|P_- e^{-itH_0}u| < C_s$, where C_s is a positive constant depending on s. Then, we can finish proving this lemma. □

Lemma 4.4. *Let $d > 0$. Then*

$$\sup_{t>d} \|(e^{-itH} - e^{-itH_0})P_+(A)^*\| \to 0, \tag{4.5}$$

$$\sup_{t<-d} \|(e^{-itH} - e^{-itH_0})P_-(A)^*\| \to 0, \tag{4.6}$$

as $A \to \infty$, where $P_\pm(A)^*$ is the adjoint of the operators $P_\pm(A)$, respectively.

Proof. We will prove (4.5). The proof of (4.6) is similar. Noticing that

$$\begin{aligned}\frac{d}{dt}\{e^{itH}e^{-itH_0}\} &= iHe^{itH}e^{-itH_0} - ie^{itH}e^{-itH_0}H_0\\ &= ie^{itH}(H-H_0)e^{-itH_0} = ie^{itH}Ve^{-itH_0},\end{aligned}$$

we have

$$e^{-itH} - e^{-itH_0} = e^{-itH}(I - e^{itH}e^{-itH_0}) = -ie^{-itH}\int_0^t e^{isH}Ve^{-isH_0}\,ds.$$

Since $e^{-i(t-s)H}$ is uniformly bounded in $t-s \in \mathbb{R}$, we have by (4.1)

$$\|(e^{-itH} - e^{-itH_0})P_+(A)^*\| \le C\int_0^t \|\langle x\rangle^{-\sigma}e^{-isH_0}P_+(A)^*\|\,ds. \tag{4.7}$$

Since $P_+(A)^* = P_+^*\chi(x/A)$, we have

$$\begin{aligned}&\|\langle x\rangle^{-\sigma}e^{-isH_0}P_+(A)^*\|\\ &\le \|\langle x\rangle^{-\sigma}e^{-isH_0}P_+(A)^*(1+s+|x|)^{\sigma}\|\,\|(1+s+|x|)^{-\sigma}F(|x|>A)\|\\ &\le C'(1+s+A)^{-\sigma}\|(1+s+|x|)^{\sigma}P_+(A)e^{isH_0}\langle x\rangle^{-\sigma}\|,\end{aligned}$$

where C' is a positive constant. Then applying (4.3) and (4.7) and noticing that $\sigma > 1$, we get this Lemma. □

Lemma 4.5. *If $u \in \mathcal{H}_{ac}(H)$ then $\|P_-e^{-itH}u\|_{L^2} \to 0$, as $t \to \infty$.*

Proof. Let $d > 0$. It follows from Lemma 4.4, for every $\varepsilon > 0$, there exists a constant $A > 0$, such that

$$\sup_{t>d}\left\|\chi\left(\frac{x}{A}\right)P_-(e^{-itH} - e^{-itH_0})u\right\| < \varepsilon. \tag{4.8}$$

Since $u \in L^2(\mathbb{R}^n)$, for every $\varepsilon > 0$ there exists a function $v \in \mathcal{S}(\mathbb{R}^n)$, such that $\|u - v\|_{L^2} < \varepsilon$. Noticing that $P_-e^{-itH_0}$ is uniformly bounded in $t \in \mathbb{R}$, we get $\|P_-e^{-itH_0}(u-v)\|_{L^2} < \varepsilon$, for all t. It follows from Lemma 4.3 that $\|P_-e^{-itH_0}v\|_{L^2} \to 0$, as $t \to \infty$. So, we get

$$\|P_-e^{-itH_0}u\|_{L^2} \to 0, \tag{4.9}$$

as $t \to \infty$. The integral kernel $K_\pm(x,y)$ of the operator $(1-\chi(x/A))P_\pm$ is

$$K_\pm(x,y) = (2\pi)^{2n}\left(1-\chi\left(\frac{x}{A}\right)\right)\chi(x)\int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi}\rho_\pm(\omega_x\cdot\omega_\xi)\varphi(|\xi|)\,d\xi.$$

Noting that $\langle x-y\rangle^{-2}(1-\Delta_\xi)e^{i(x-y)\cdot\xi} = e^{i(x-y)\cdot\xi}$, we make the integration by parts, and get $|K_\pm(x,y)| \le C(1-\chi(x/A))\langle x-y\rangle^{-2n}$. So $K_\pm \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. Then we have that $(1-\chi(x/A))P_\pm$ is a compact operator, and $e^{-itH}u$ converges weakly to 0 as $t \to \infty$ (Lemma 4.2). Then we get

$$\lim_{t\to\infty} \left\| \left(1-\chi\left(\frac{x}{A}\right)\right) P_- e^{-itH} u \right\| = 0. \tag{4.10}$$

Collecting (4.6), (4.9) and (4.10), we get $\|P_- e^{-itH}u\|_{L^2} \to 0$, as $t \to \infty$. □

Lemma 4.6. *If $u \in \mathcal{H}_{ac}(H)$ then $\lim_{t\to\infty} \|e^{-itH}\varphi(H)u - P_+(A)e^{-itH}u\|_{L^2} = 0$, for all $A > 0$.*

Proof. The equation of resolvent is $(H-z)^{-1} - (H_0-z)^{-1} = -(H-z)^{-1}V(H_0 - z)^{-1}$. Noticing that $V(H_0-z)^{-1}$ is a compact operator (see H. Isozaki [7, p.27, Theorem 4.8]), we get that $\varphi(H) - \varphi(H_0)$ is a compact operator. This fact, together with Lemma 4.2, implies

$$\lim_{t\to\infty} \|\varphi(H)e^{-itH}u - \varphi(H_0)e^{-itH}u\| = 0. \tag{4.11}$$

Since $(1-\chi(x))\varphi(H_0)$ is a compact operator (check the integral kernel similarly in Lemma 4.5), and $e^{-itH}u$ converges weakly to 0 as $t \to \infty$ (Lemma 4.2), we get

$$\lim_{t\to\infty} \|(1-\chi(x))\varphi(H_0)e^{-itH}u\| = 0. \tag{4.12}$$

Noting that $\chi(x)\varphi(H_0) = P_+ + P_-$, we get

$$\lim_{t\to\infty} \|\varphi(H_0)e^{-itH}u - (P_+ + P_-)e^{-itH}u\| = 0. \tag{4.13}$$

Collecting (4.11), (4.12), (4.13), and Lemma 4.5, we have

$$\lim_{t\to\infty} \|e^{-itH}\varphi(H)u - P_+(A)e^{-itH}u\|_{L^2} = 0.$$ □

Lemma 4.7. *Let $u \in \mathcal{H}_{ac}(H)$, $d > 0$. For every $\varepsilon > 0$, there exists $s > 0$ and $A > 0$, such that, $\sup_{t>d} \|e^{-itH}u_s - e^{-itH_0}P_+(A)e^{-isH}u\|_{L^2} < \varepsilon$, where $u_s = e^{-isH}\varphi(H)u$.*

Proof. By the definition of $p_+(x,\xi)$, we get

$$\left|\partial_x^\alpha \partial_\xi^\beta p_+(x,\xi)\right| \le C_{\alpha\beta}\langle x\rangle^{-|\alpha|}\langle\xi\rangle^{-m-|\beta|},$$

for all $m > 0$, where $C_{\alpha\beta}$ is a positive constant. Since $\partial_x^\alpha \chi(x/A) = A^{-|\alpha|}(\partial_x^\alpha\chi)(x/A)$, we have

$$\left|\partial_x^\alpha \partial_\xi^\beta p_+(x,\xi)\right| \le C_{\alpha\beta}A^{-1}\langle\xi\rangle^{-m-|\beta|},$$

for all $|\alpha| \geq 1$. Then we get the symbol $q(x, \xi; A)$ of $P_+(A)^* - P_+(A)$ satisfying

$$\left|\partial_x^\alpha \partial_\xi^\beta q(x, \xi; A)\right| \leq C_{\alpha\beta} A^{-1} \langle \xi \rangle^{-m-|\beta|}.$$

Then

$$\| P_+(A)^* - P_+(A)\| \leq \frac{C}{A},$$

where $C > 0$ is a constant. This fact together with (4.5) yields $\sup_{t>d} \|(e^{-itH} - e^{-itH_0})P_+(A)\| \to 0$, as $A \to \infty$. From Lemma 4.6, we get that there exists $A > 0$, $s > 0$ such that, $\sup_{t>d} \|e^{-itH} u_s - e^{-itH_0} P_+(A)e^{-isH} u\|_{L^2} < \varepsilon$. Then we get the lemma. $\square$

Proof of Theorem 4.1. From Lemma 4.1, we get that there exists the limits

$$W_\pm = \lim_{t\to\pm\infty} e^{itH} e^{-itH_0}.$$

Then we just need to prove that

$$u \perp \mathcal{R}(W_+) \Rightarrow u = 0$$

for all $u \in \mathcal{H}_{ac}(H)$. (The case W_- is similar.)

Let $0 < a < c < d < b$, $\varphi(\lambda) \in C_0^\infty((a, b))$ satisfy

$$\varphi(\lambda) = 1 \quad (c < \lambda < d).$$

Let $u_s = e^{-isH}\varphi(H)u$. It follows from Lemma 4.7, that

$$\begin{aligned}\|u_s\|^2 &= (e^{-itH} u_s, e^{-itH} u_s) = (e^{-itH_0} P_+(A)e^{-isH} u, e^{-itH} u_s) + O(\varepsilon)\\ &\to (\varphi(H)e^{isH} W_+ e^{-itH_0} P_+(A)e^{-isH} u, u) + O(\varepsilon)\end{aligned}$$

as $t \to \infty$. Since

$$\begin{aligned}&(\varphi(H)e^{isH} W_+ e^{-itH_0} P_+(A)e^{-isH} u, u)\\ &= \int_{-\infty}^{\infty} \varphi(\lambda)e^{is\lambda}\, d(E_H(\lambda)W_+ e^{-itH_0} P_+(A)e^{-isH} u, u)\\ &= \int_{-\infty}^{\infty} \varphi(\lambda)e^{is\lambda}\, d(W_+(E_{H_0}(\lambda)e^{-itH_0} P_+(A)e^{-isH} u, u)\\ &= (\varphi(H_0)e^{isH_0} e^{-itH_0} P_+(A)e^{-isH} u, W_+^* u)\\ &= (W_+\varphi(H_0)e^{isH_0} e^{-itH_0} P_+(A)e^{-isH} u, u),\end{aligned}$$

we get $\|u_s\|^2 = (W_+\varphi(H_0)e^{isH_0} e^{-itH_0} P_+(A)e^{-isH} u, u) + O(\varepsilon)$. Applying that $u \perp \mathcal{R}(W_+)$, we get $\|u_s\| = O(\varepsilon)$. So $\varphi(H)u = 0$. Since $\varphi(\lambda)$ is an arbitrary $C_0^\infty((0, \infty))$ function, we get $u = 0$. $\square$

5. Eigenfunction expansions

In this section, we assum that the dimension n is an odd integer, $n \geq 3$, and $\sigma > (n+1)/2$. We consider the completeness of the generalized eigenfunction in this section. The main idea is the same as the idea in H. Kitada [10] and S.T. Kuroda [13], besides, in this section, we use the method in T. Ikebe [6, Section 11]. It is known that

$$\sigma_e(H) = \sigma_e(H_0) = [0, \infty).$$

We need to remark that $\sigma_p(H) \cap (0, \infty)$ is a discrete set. This fact was first proved by B. Simon [19, Theorem 2.1]. Moreover, B. Simon [19, Theorem 2.1] proved that each eigenvalue in the set $\sigma_p(H) \cap (0, \infty)$ has finite multiplicity.

The main theorem is

Theorem 5.1. *Assume the dimension n ($n \geq 3$) is an odd integer, $\sigma > (n+1)/2$, $s > n/2$ and $[a, b] \subset (0, \infty) \setminus \sigma_p(H)$. For $u \in L^{2,s}(\mathbb{R}^n)$, let $\mathcal{F}_\pm$ be defined by*

$$\mathcal{F}_\pm u(k) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x)\overline{\varphi^\pm(x, k)}\, dx. \tag{5.1}$$

For an arbitrary $L^{2,s}(\mathbb{R}^n)$-function $f(x)$,

$$E_H([a, b]) f(x) = (2\pi)^{-n/2} \int_{a \leq |k| \leq b} \mathcal{F}_\pm f(k)\varphi^\pm(x, k)\, dk,$$

where E_H is the spectral measure on H, and $\varphi^\pm(x, k)$ are defined in Theorem 1.3.

Lemma 5.1. *Let $[a, b] \subset (0, \infty) \setminus \sigma_p(H)$. Then $(W_\pm\varphi_0(\,\cdot\,, k), g) = (\varphi^\pm(\,\cdot\,, k), g)$ for all $g \in C_0^\infty(\mathbb{R}^n)$ and $k \in [a, b]$, where $\varphi_0(x, k) = e^{ix\cdot k}$, and $W_\pm$ is the same as in* Theorem 4.1.

Proof. Noticing that

$$e^{itH}e^{-itH_0} = I + i\int_0^t e^{i\tau H}Ve^{-i\tau H_0}\, d\tau,$$

and letting $t \to \pm\infty$, we get

$$(W_\pm\varphi_0(\,\cdot\,, k), g) = (\varphi_0(\,\cdot\,, k), g) + i\int_0^{\pm\infty} (e^{i\tau H}Ve^{-i\tau H_0}\varphi_0(\,\cdot\,, k), g)\, d\tau.$$

Putting $f = e^{ix\cdot k}$, we have

$$i\int_0^{\pm\infty} (e^{i\tau H}Ve^{-i\tau H_0} f, g)\, d\tau = i\lim_{\varepsilon\downarrow 0}\int_0^{\pm\infty} e^{\mp\varepsilon\tau}(e^{i\tau H}Ve^{-i\tau H_0} f, g)\, d\tau$$

$$= i \lim_{\varepsilon \downarrow 0} \int_0^{\pm\infty} e^{\mp\varepsilon\tau}(f, e^{i\tau H_0} V e^{-i\tau H} g)\, d\tau = i \lim_{\varepsilon \downarrow 0} \int_0^{\pm\infty} e^{\mp\varepsilon\tau}(f, \mathcal{F}^{-1} e^{i\tau|k|} \mathcal{F} V e^{-i\tau H} g)\, d\tau$$

Since $g \in C_0^\infty(\mathbb{R}^n)$, $k \in [a, b]$, and $\varphi_0(x, k)$ is bounded for $(x, k) \in \mathbb{R}^n \times \{k \mid a \leq k \leq b\}$, we can interchange the τ-, x-, and k-integrations. Then we get

$$\begin{aligned} i \int_0^{\pm\infty} (e^{i\tau H} V e^{-i\tau H_0} f, g)\, d\tau &= i \lim_{\varepsilon \downarrow 0} \int_0^{\pm\infty} e^{\mp\varepsilon\tau}(\hat{f}, \mathcal{F} V e^{-i\tau(H - |k|)} g)\, d\tau \\ &= i \lim_{\varepsilon \downarrow 0} \int_0^{\pm\infty} (f, V e^{-i\tau(H - (|k| \pm i\varepsilon))} g)\, d\tau = (f, V R^{\pm}(|k|) g) = (R^{\mp}(|k|) V f, g). \end{aligned}$$

So, by the definition of $\varphi^{\pm}(x, k)$, and $k \in [a, b]$, we get $(W_{\pm}\varphi_0(\,\cdot\,, k), g) = (\varphi^{\pm}(\,\cdot\,, k), g)$. $\square$

Lemma 5.2. *Let* $[a, b] \subset (0, \infty) \setminus \sigma_p(H)$, $\operatorname{supp} \hat{g}(k) \subset \{k \mid a \leq |k| \leq b\}$ *and* $f(x) \in C_0^\infty(\mathbb{R}^n)$. *Then*

$$(\mathcal{F}_{\pm} f, \hat{g}) = (\mathcal{F} W_{\pm}^* f, \hat{g}),$$

where $\mathcal{F}_{\pm}$ *are defined by* (5.1).

Proof. By the definition of $\mathcal{F}^{-1}$, we get

$$(\mathcal{F} W_{\pm}^* f, \hat{g}) = (f, W \mathcal{F}^{-1} \hat{g}) = \left(f, W_{\pm} \int \varphi_0(\,\cdot\,, k) \hat{g}(k)\, dk \right).$$

Since $f \in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} \hat{g}(k) \subset \{k \mid a \leq |k| \leq b\}$, and $\varphi_0(x, k)$ is bounded for $(x, k) \in \mathbb{R}^n \times \{k \mid a \leq k \leq b\}$, we can interchange the x-, and k-integrations. Then, we have

$$(\mathcal{F} W_{\pm}^* f, \hat{g}) = \int (f, W_{\pm}\varphi_0(\,\cdot\,, k)) \hat{g}(k)\, dk.$$

Noticing $\operatorname{supp} \hat{g}(k) \subset \{k \mid a \leq |k| \leq b\}$ and using Lemma 5.1, we obtain Lemma 5.2. $\square$

Finally, we start to prove our main Theorem 5.1.

Proof of Theorem 5.1. It follows from Theorem 3.1 and Theorem 4.1 that the wave operators $W_{\pm}$ are complete, and the eigenfunctions $\varphi^{\pm}(x, k)$ are bounded for $(x, k) \in \mathbb{R}^n \times \{k \mid a \leq |k| \leq b\}$. Then, noticing that $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, together with Lemma 5.2, and using the idea of S.T. Kuroda [13, p.160], we can obtain Theorem 5.1. $\square$

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References

[1] S. Agmon: *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), 151–218.

[2] M. Ben-Artzi and J. Nemirovsky: *Remarks on relativistic Schrödinger operators and their extensions*, Ann. Inst. H. Poincaré Phys. Théor. **67** (1997), 29–39.

[3] V. Enss: *Asymptotic completeness for quantum mechanical potential scattering* I, *Short range potentials*, Comm. Math. Phys. **61** (1978), 285–291.

[4] A. Erdélyi: Tables of Integral Transforms, 1, McGraw-Hill (1952).

[5] T. Ikebe and Y. Saitō: *Limiting absorption method and absolute continuity for the Schrödinger operator*, J. Math. Kyoto Univ. **12** (1972), 513–542.

[6] T. Ikebe: *Eigenfunction expansions associated with the Schröedinger operators and their applications to scattering theory*, Arch. Rational Mech. Anal. **5** (1960), 1–34.

[7] H. Isozaki, Many-Body Schrödinger Equation, Springer Tokyo, 2004, in Japanese.

[8] H. Kitada, *Quantum Mechanics*, `http://kims.ms.u-tokyo.ac.jp/quantum.html`, (2003).

[9] H. Kitada: *Fundamental solutions and eigenfunction expansions for Schrödinger operators* I: *Fundamental solutions*, Math. Z. **198** (1988), 181–190.

[10] A. Jensen and H. Kitada: *Fundamental solutions and eigenfunction expansions for Schrödinger operators* II: *Eigenfunction expansions*, Math. Z. **199** (1988), 1–13.

[11] H. Kitada and H. Kumano-go: *A family of Fourier integral operators and the fundamental solution for a Schrödinger equation*, Osaka J. Math. **18** (1981), 291–360.

[12] H. Kumano-go: Pseudo-Differential Operators, MIT Press, 1981.

[13] S.T. Kuroda: Spectral Theory II, Iwanami Shoten, Tokyo, 1979, in Japanese.

[14] T. Kato and S.T. Kuroda: *Theory of simple scattering and eigenfunction expansions*; in Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., 1968), Springer, New York, 1970, 99–131.

[15] S.T. Kuroda: *Scattering theory for differential operators* I, J. Math. Soc. Japan **25** (1973), 75–104.

[16] S.T. Kuroda: *Scattering theory for differential operators* II, J. Math. Soc. Japan **25** (1973), 222–234.

[17] M. Nagase and T. Umeda: *On the essential selfadjointness of pseudo-differential operators*, Proc. Japan Acad. Ser. A Math. Sci. **64** (1988), 94–97.

[18] M. Reed and B. Simon: Methods of Modern Mathematical Physics IV, Analysis of Operators, Academic Press, New York, 1978.

[19] B. Simon: *Phase space analysis of simple scattering systems*: *extensions of some work of Enss*, Duke Math. J. **46** (1979), 119–168.

[20] E.M. Stein: Singular Integrals and Differentiability Properties of Functions, Princeton Univ. Press, Princeton, N.J., 1970.

[21] R.S. Strichartz: A Guide to Distribution Theory and Fourier Transforms, World Sci. Publishing, River Edge, NJ, 2003.

[22] T. Umeda: *Radiation conditions and resolvent estimates for relativistic Schrödinger operators*, Ann. Inst. H. Poincaré Phys. Théor. **63** (1995), 277–296.

[23] T. Umeda: *The action of* $\sqrt{-\Delta}$ *on weighted Sobolev spaces*, Lett. Math. Phys. **54** (2000), 301–313.

[24] T. Umeda, *Generalized eigenfunctions of ralativistic Schrödinger operators* I, Department of Mathematics Himeji Institute of Technology, Japan, 2001.

[25] T. Umeda, *Generalized eigenfunctions of ralativistic Schrödinger operators* II, private communication.

[26] T. Umeda: *Eigenfunction expansions associated with relativistic Schrödinger operators*; in Partial Differential Equations and Spectral Theory (Clausthal, 2000), Oper. Theory Adv. Appl. **126**, Birkhäuser, Basel, 2001, 315–319.

[27] D. Wei, *Completeness of the generalized eigenfunctions for relativistic Schrödinger operators* II, in preparation.

Department of Information Physics and Computing
Graduate School of Information Science and Technology
University of Tokyo
Hongou 7–3–1, bunkyo-ku, Tokyo 113–8656
Japan
e-mail: dabi@alab.t.u-tokyo.ac.jp