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ASYMPTOTIC STABILITY OF A STATIONARY SOLUTION TO A HYDRODYNAMIC MODEL OF SEMICONDUCTORS

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Abstract

We study the existence and the asymptotic stability of a stationary solution to the initial boundary value problem for a one-dimensional hydrodynamic model of semiconductors. This problem is considered, in the previous researches [2] and [11], under the assumption that a doping profile is flat, which makes the stationary solution also flat. However, this assumption is too narrow to cover the doping profile in actual diode devices. Thus, the main purpose of the present paper is to prove the asymptotic stability of the stationary solution without this assumption on the doping profile. Firstly, we prove the existence of the stationary solution. Secondly, the stability is shown by an elementary energy method, where the equation for an energy form plays an essential role.

1. Introduction

The present paper is concerned with the existence and the asymptotic stability of a stationary solution to the initial boundary value problem for a one-dimensional hydrodynamic model of semiconductors. The motion of electrons in semiconductors is governed by the system of equations

$$\rho_t + (\rho u)_x = 0, \tag{1.1a}$$

$$(\rho u)_t + (\rho u^2 + p(\rho))_x = \rho\phi_x - \rho u, \tag{1.1b}$$

$$\phi_{xx} = \rho - D. \tag{1.1c}$$

We study the asymptotic behavior of a solution to this system over bounded domain $\Omega := (0, 1)$. Here, the unknown functions ρ, u and ϕ stand for the electron density, the electron velocity and the electrostatic potential, respectively. Thus, the product $j := \rho u$ means the current density. The pressure p is assumed to be a function of the electron density ρ given by

$$p = p(\rho) = K\rho^{\gamma}, \tag{1.2}$$

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where the constants K and γ are supposed to satisfy $K > 0$ and $\gamma \geq 1$. The case $\gamma = 1$ is important from the physical point of view. The doping profile $D \in \mathcal{B}^0(\overline{\Omega})$ is a function of the spatial variable $x \in \overline{\Omega} := [0, 1]$ and satisfies

$$\inf_{x\in\overline{\Omega}} D(x) > 0. \tag{1.3}$$

The initial and the boundary data are prescribed as

$$(\rho, u)(0, x) = (\rho_0, u_0)(x), \tag{1.4}$$

$$\rho(t, 0) = \rho_l > 0, \quad \rho(t, 1) = \rho_r > 0, \tag{1.5}$$

$$\phi(t, 0) = 0, \quad \phi(t, 1) = \phi_r > 0, \tag{1.6}$$

where ρ_l, ρ_r and ϕ_r are constants. In addition, the compatibility conditions on $\rho(t, x)$ with orders 0 and 1 are supposed to hold at $(t, x) = (0, 0)$ and $(t, x) = (0, 1)$. Namely,

$$\rho(0, 0) = \rho_l, \quad \rho(0, 1) = \rho_r, \quad (\rho u)_x(0, 0) = 0, \quad (\rho u)_x(0, 1) = 0. \tag{1.7}$$

This initial boundary value problem is considered in the region where the subsonic condition (1.8a) and positivity of the density (1.8b) hold

$$\inf_{x\in\Omega}(p'(\rho) - u^2) > 0, \tag{1.8a}$$

$$\inf_{x\in\Omega} \rho > 0. \tag{1.8b}$$

Thus, we need to suppose that the initial data (1.4) satisfy these conditions

$$\inf_{x\in\Omega}(p'(\rho_0(x)) - u_0^2(x)) > 0, \quad \inf_{x\in\Omega} \rho_0(x) > 0. \tag{1.9}$$

We construct the solution in the neighborhood of the initial data (1.9) as the conditions (1.8) hold. Notice that the subsonic condition is equivalent to that one characteristic speed of the hyperbolic equations (1.1a) and (1.1b) is negative and another is positive, that is,

$$\lambda_1 := u - \sqrt{p'(\rho)} < 0, \quad \lambda_2 := u + \sqrt{p'(\rho)} > 0. \tag{1.10}$$

Hence the subsonic condition implies that two boundary conditions (1.5), (1.6) are necessary and sufficient for the wellposedness of this initial boundary value problem.

The initial boundary value problem (1.1) and (1.4) for (ρ, j, ϕ) is rewritten as

$$\rho_t + j_x = 0, \tag{1.11a}$$

$$j_t + \left(p'(\rho) - \frac{j^2}{\rho^2}\right)\rho_x + 2\frac{j}{\rho} j_x = \rho\phi_x - j, \tag{1.11b}$$

$$\phi_{xx} = \rho - D \tag{1.11c}$$

with the initial data $(\rho_0, j_0) := (\rho_0, \rho_0 u_0)$, which is derived from (1.4). In Section 2, we discuss the existence of the solution to (1.1) satisfying the conditions (1.8). Apparently, (1.1) is equivalent to (1.11), if the density ρ is positive. Thus once we prove the existence of a solution to the initial boundary value problem (1.11), (1.4), (1.5) and (1.6) for (ρ, j, ϕ) with $\rho > 0$, the existence of the solution to the problem (1.1), (1.4), (1.5) and (1.6) immediately follows. Integrating (1.11c) and using the boundary condition (1.6), we obtain an explicit formula of the electrostatic potential

$$\begin{aligned} \phi(t,x) &= \Phi[\rho](t,x) \\ &:= \int_0^x \int_0^y (\rho - D)(t,z)\,dzdy + \left(\phi_r - \int_0^1 \int_0^y (\rho - D)(t,z)\,dzdy\right)x. \end{aligned} \tag{1.12}$$

The main purpose of the present paper is to show the asymptotic stability of a stationary solution, which is a solution to (1.1) independent of a time variable t, satisfying the same boundary conditions (1.5) and (1.6). Hence, the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ verifies the system of equations

$$(\tilde{\rho}\tilde{u})_x = 0, \tag{1.13a}$$

$$(\tilde{\rho}\tilde{u}^2 + p(\tilde{\rho}))_x = \tilde{\rho}\tilde{\phi}_x - \tilde{\rho}\tilde{u}, \tag{1.13b}$$

$$\tilde{\phi}_{xx} = \tilde{\rho} - D \tag{1.13c}$$

and the boundary condition

$$\tilde{\rho}(0) = \rho_l > 0, \quad \tilde{\rho}(1) = \rho_r > 0, \tag{1.14}$$

$$\tilde{\phi}(0) = 0, \quad \tilde{\phi}(1) = \phi_r > 0. \tag{1.15}$$

The equation (1.13a) means the product $\tilde{j} := \tilde{\rho}\tilde{u}$ is constant. Substituting $\tilde{j} = \tilde{\rho}\tilde{u}$ in (1.13b) and dividing (1.13b) by $\tilde{\rho}$, we have the system equations for $(\tilde{j}, \tilde{\rho}, \tilde{\phi})$

$$\tilde{j}_x = 0, \tag{1.16a}$$

$$\frac{\partial F}{\partial \rho}(\tilde{\rho}, \tilde{j})\tilde{\rho}_x = \tilde{\phi}_x - \frac{\tilde{j}}{\tilde{\rho}}, \tag{1.16b}$$

$$\tilde{\phi}_{xx} = \tilde{\rho} - D, \tag{1.16c}$$

where

$$F(\rho, j) := \frac{j^2}{2\rho^2} + h(\rho), \quad h(\xi) := \int_1^\xi \frac{p'(\zeta)}{\zeta}\,d\zeta. \tag{1.17}$$

Differentiating (1.16b) in x yields that

$$\left(\frac{\partial F}{\partial \rho}(\tilde{\rho}, \tilde{j})\tilde{\rho}_x\right)_x - \frac{\tilde{j}}{\tilde{\rho}^2}\tilde{\rho}_x - \tilde{\rho} = -D. \tag{1.18}$$

Integrating (1.16b) over the domain Ω, we have the current-voltage relationship

$$\phi_r = F(\rho_r, \tilde{j}) - F(\rho_l, \tilde{j}) + \tilde{j}\int_0^1 \frac{1}{\tilde{\rho}}\, dx. \tag{1.19}$$

Moreover, owing to the equation (1.16c) and the boundary condition (1.15), $\tilde{\phi}$ is given by the formula

$$\tilde{\phi}(x) = \int_0^x\int_0^y (\tilde{\rho} - D)(z)\, dzdy + \left(\phi_r - \int_0^1\int_0^y (\tilde{\rho} - D)(z)\, dzdy\right)x, \tag{1.20}$$

which corresponds to (1.12) for the non-stationary problem.

In showing the existence and the asymptotic stability of the stationary solution, the strength of the boundary data, which is defined by

$$\delta := |\rho_r - \rho_l| + |\phi_r|, \tag{1.21}$$

plays a crucial role. The existence of the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ is summarized in the next lemma.

Lemma 1.1. *Let the doping profile and the boundary data satisfy conditions* (1.3), (1.5) *and* (1.6). *For an arbitrary* ρ_l, *there exists a positive constant* δ_1 *such that if* $\delta \leq \delta_1$, *then the stationary problem* (1.13), (1.14) *and* (1.15) *has a unique solution* $(\tilde{\rho}, \tilde{u}, \tilde{\phi})(x)$ *satisfying the conditions* (1.8) *in the space* $\mathcal{B}^2(\overline{\Omega})$.

Proof. This lemma follows from Lemmas 2.1 and 2.3. □

In order to discuss the asymptotic stability of the stationary solution constructed in Lemma 1.1, we employ the function space

$$\mathfrak{X}_i^j([0, T]) := \bigcap_{k=0}^{i} C^k([0, T]; H^{j+i-k}(\Omega)) \quad \text{for} \quad i, j = 0, 1, 2,$$
$$\mathfrak{X}_i([0, T]) := \mathfrak{X}_i^0([0, T]) \quad \text{for} \quad i = 0, 1, 2,$$

in which the norms are denoted as in Notation below. The main theorem, the stability of the stationary solution, is summarized in the next theorem.

Theorem 1.2. *Let $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$ be the stationary solution of* (1.13), (1.14) *and* (1.15). *Suppose that the initial data $(\rho_0, u_0) \in H^2(\Omega)$ and the boundary data ρ_l, ρ_r and ϕ_r satisfy* (1.5), (1.6), (1.7) *and* (1.9). *Then there exists a positive constant δ_2 such that if $\delta + \|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_2 \leq \delta_2$, the initial boundary value problem* (1.1), (1.4), (1.5) *and* (1.6) *has a unique solution $(\rho, u, \phi)(t, x) \in \mathfrak{X}_2([0, \infty))$. Moreover, the solution $(\rho, u, \phi)(t, x)$ verifies the additional regularity $\phi - \tilde{\phi} \in \mathfrak{X}_2^2([0, \infty))$ and the decay estimate*

$$\|(\rho - \tilde{\rho}, u - \tilde{u})(t)\|_2 + \|(\phi - \tilde{\phi})(t)\|_4 \leq C\|(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_2 e^{-\alpha t}, \tag{1.22}$$

where C and α are positive constants independent of a time variable t.

Related results. The hydrodynamic model of semiconductors was introduced by Bløtekjær [1]. Recently, not only engineers but also mathematicians interested in this model. From the mathematical point of view, the text book [13] is the good reference for the derivation of the hydrodynamic model of semiconductors. It is important to study the initial boundary problem over bounded domain with the Dirichlet boundary condition since semiconductor devices are minute.

Degond and Markowich [2] investigated the stationary solution to the one-dimensional hydrodynamic model of semiconductors with the Dirichlet boundary condition. They proved the existence of the stationary solution, satisfying the subsonic condition (1.8a). We reconsider the existence of the stationary solution in the present paper since the research in [2] shows the existence for a given current density $\tilde{j}$, although physical interest is to investigate the amount of the current density $\tilde{j}$ for a given boundary voltage ϕ_r. Li, Markowich and Mei [11] studied the asymptotic stability of the stationary solution. However, they assumed that the doping profile is flat, that is, $|D(x) - \rho_l| \ll 1$. This assumption is too narrow to cover physical problems since the typical example of the doping profile does not satisfy this assumption (see [4]). For instance, the doping profiles of $n^+ - n - n^+$ diodes have two steep slops. Matsumura and Murakami [14] started to study the physically meaningful doping profile. Precisely, they proved the asymptotic stability of the stationary solution without flatness assumption on the doping profile. However, they studied this problem with the periodic boundary condition, which makes it the full space problem over $\mathbb{R}$. Consequently, our main concern goes to the problem to show the asymptotic stability of the stationary solution under the Dirichlet boundary condition without the flatness assumption on the doping profile.

Other kinds of hyperbolic-elliptic coupled systems, rather than (1.1), arise as models for radiating or self-gravitational fluid flow, (see [3, 7, 8, 9, 10, 12] for example). Especially, the model for the self-gravitational flow, the other Euler-Poisson equation, is studied in [3, 10, 12]. The stability of traveling waves is considered in [9] for radiating gas dynamics. In researches [7, 8], general systems of hyperbolic-elliptic coupled equations are considered. We have to mention that we borrow several ideas from these papers [7, 8] although they do not cover the semiconductor model (1.1). Hence, it is an important open problem to generalize the results in [7, 8] to the system including (1.1).

After completing the present paper, we have learned that the almost same theorem as Theorem 1.2 *concerning the stability of the stationary solution had been proved independently by Y. Guo and W. Strauss in* [5]. *However, we think that the present paper is still worth of publication since the estimates are derived by different methods and the paper* [5] *does not discuss on the existence and the regularity of the solution. Furthermore, it follows the research* [2] *for the existence of the stationary solution and thus it does not state the result in terms of the electrostatic potential. As we have addressed above, such a consideration is important for the researches, especially in physics and technology.*

Outline of the paper. The remaining part of the present paper is organized as follows. In Section 2, we begin detailed discussions with the proof of the existence and the uniqueness of the stationary solution. The existence is proved in Subsection 2.1 by the Schauder fixed-point theorem. The uniqueness follows from the maximum principle. In Subsection 2.2, we obtain the elliptic estimate and then we establish the unique existence of the time local solution by using an iteration method for solving the non-linear hyperbolic equations. Here we omit the discussion on the solvability of the linearized hyperbolic problem in Subsection 2.2. and postpone it until Appendix. Section 3 is devoted to showing the asymptotic stability of the stationary solution. First, we introduce the energy form to obtain the basic estimate. Next, we derive the system of the equations for the perturbation from the stationary solution. Then an elementary energy method yields the higher order estimates. Therefore, combining the existence of the time local solution and the a priori estimate in the H^2-Sobolev space, we complete the proof of the existence of the the time global solution. Finally, by using the uniform estimates previously obtained, we show the exponential convergence of the solution, for the non-stationary problem, to the corresponding stationary solution in Subsection 3.4.

Notation. For a nonnegative integer $l \geq 0$, $H^l(\Omega)$ denotes the l-th order Sobolev space in the L^2 sense, equipped with the norm $\|\cdot\|_l$. We note $H^0 = L^2$ and $\|\cdot\| := \|\cdot\|_0$. $C^k([0,T]; H^l(\Omega))$ denotes the space of the k-times continuously differentiable functions on the interval $[0, T]$ with values in $H^l(\Omega)$. For a nonnegative integer $k \geq 0$, $\mathcal{B}^k(\overline{\Omega})$ denotes the space of the functions whose derivatives up to k-th order are continuous and bounded over $\overline{\Omega}$, equipped with the norm

$$|f|_k := \sum_{i=0}^{k} \sup_{x \in \overline{\Omega}} |\partial_x^i f(x)|.$$

2. Preliminary observation

2.1. Unique existence of stationary solution. This subsection is devoted to the discussion on the existence and the uniqueness of the stationary solution. Firstly, we show the existence of the stationary solution by applying the Schauder fixed-point theorem. Secondly, we obtain the estimates of the stationary solution, as it is necessary in showing the uniqueness of the stationary solution. Finally, the uniqueness of the stationary solution is proved by the maximum principle.

Apparently, (1.13) is equivalent to (1.16) if density $\tilde{\rho}$ is positive. Hence once we show the existence and the uniqueness of the solution to the problem (1.16), (1.14) and (1.15) with $\tilde{\rho} > 0$, the existence and the uniqueness of the solution to the problem (1.13), (1.14) and (1.15) immediately follow. We use the following constants to discuss the properties of the stationary solution.

$$\overline{C}_m := \min\left\{\rho_l,\, \rho_r,\, \inf_{x\in(0,1)} D(x)\right\}, \quad \overline{C}_M := \max\left\{\rho_l,\, \rho_r,\, \sup_{x\in(0,1)} D(x)\right\},$$
$$\overline{C}_b := \phi_r - \{h(\rho_r) - h(\rho_l)\}.$$

The existence of the stationary solution is stated in the next lemma. The main idea of this proof is essentially same as in [11].

Lemma 2.1. *Let the doping profile and the boundary data satisfy conditions* (1.3), (1.5) *and* (1.6). *Moreover, suppose that the following inequalities hold*:

$$\sqrt{\gamma K \overline{C}_m^{\gamma+1}} > 2\overline{C}_b\left\{\overline{C}_M^{-1} + \sqrt{\overline{C}_M^{-2} + 2\overline{C}_b(\rho_r^{-2} - \rho_l^{-2})}\right\}^{-1}, \tag{2.1a}$$

$$\overline{C}_M^{-2} + 2\overline{C}_b(\rho_r^{-2} - \rho_l^{-2}) \geq 0 \quad \textit{if} \quad \rho_l < \rho_r. \tag{2.1b}$$

Then the stationary problem (1.16), (1.14) *and* (1.15) *has a solution* $(\tilde{\rho}, \tilde{j}, \tilde{\phi}) \in \mathcal{B}^2(\overline{\Omega})$ *satisfying the condition* (1.8). *Furthermore, it holds that* $\tilde{j} \lesseqqgtr 0$ *if and only if* $\overline{C}_b \lesseqqgtr 0$.

Proof. Firstly, we define the mapping $T : q \mapsto Q$ over $W := \{f \in \mathcal{B}^1(\overline{\Omega}); \overline{C}_m \leq f \leq \overline{C}_M\}$ by solving the linear problem

$$\left(\frac{\partial F}{\partial \rho}(q,\, J_q) Q_x\right)_x - \frac{J_q}{q^2} Q_x - Q = -D, \quad x \in \Omega, \tag{2.2a}$$

$$Q(0) = \rho_l, \quad Q(1) = \rho_r \tag{2.2b}$$

with the constant J_q defined by solving the current-voltage relationship (1.19) with $(q,\, J_q)$ in place of $(\tilde{\rho},\, \tilde{j})$. Namely, it is given by

$$J_q := 2\overline{C}_b\left\{\int_0^1 q^{-1}\, dx + \sqrt{\left(\int_0^1 q^{-1}\, dx\right)^2 + 2\overline{C}_b(\rho_r^{-2} - \rho_l^{-2})}\right\}^{-1} \tag{2.3}$$

due to the assumption (2.1b) for the case $\rho_l < \rho_r$. Now, we show that the mapping T is well-defined. Estimating (2.3) by using the assumption (2.1), we see that there exists a certain constant $\bar{c}$ such that

$$\frac{\partial F}{\partial \rho}(q, J_q) \geq \bar{c} > 0. \tag{2.4}$$

The estimate (2.4) implies the pair (q, J_q) satisfies the subsonic condition (1.8a). We have from (2.1a) that

$$|J_q|, \quad \sup_{x \in \bar{\Omega}} \left| \frac{\partial F}{\partial \rho}(q, J_q) \right| \leq C, \tag{2.5}$$

where C is a certain positive constant independent of q. The above estimates (2.4) and (2.5) mean that the equation (2.2a) is elliptic. Hence, by applying the standard theory for the linear elliptic equations, we see that the problem (2.2) has a unique solution $Q \in \mathcal{B}^2(\overline{\Omega})$. Moreover, we have the estimates $\overline{C}_m \leq Q \leq \overline{C}_M$ by the maximal principle for the elliptic equation (2.2a). Thus, we have seen that the mapping T is well-defined.

Then we show the estimate by the standard energy method

$$\| Q_x \| \leq \overline{C}_1, \tag{2.6}$$

where $\overline{C}_1$ is a certain constant depending on $\bar{c}$, ρ_l, ρ_r and D, but independent of q. In fact, from (2.2a) we have the equation

$$\left\{ \frac{\partial F}{\partial \rho}(\chi + A)_x \right\}_x - \frac{J_q}{q^2}(\chi + A)_x - (\chi + A) = -D, \tag{2.7}$$
$$A(x) := \rho_l(1 - x) + \rho_r x, \quad \chi := Q - A.$$

Multiply (2.7) by χ, integrate the resultant equality over (0, 1) and then estimate the resultant integration. These procedures yield the desired estimate (2.6). For the details of the derivation of (2.6), see [11].

Letting T_1 be the restriction of T on $W_1 := \{ f \in W; \| f_x \| \leq \overline{C}_1 \}$, we see from (2.6) that the restriction T_1 is a mapping of W_1 into itself. Namely $T_1 \colon W_1 \to W_1$. Moreover, the straight forward computation shows that the mapping T_1 is continuous. If $q \in W_1$, then we have the estimate

$$\| Q_{xx} \| \leq \overline{C}_2, \tag{2.8}$$

where $\overline{C}_2$ is a certain constant depending on $\bar{c}$, ρ_l, ρ_r and D, but independent of q. In fact, multiplying (2.7) by χ_{xx} and integrating the result over (0, 1), we have the estimate (2.8) (see [11] for the details).

The image of the mapping T_1, i.e., $T_1(W_1)$, is contained in the set $W_2 := \{f \in \mathcal{B}^2(\overline{\Omega}) \cap W_1; \|f_{xx}\| \leq \overline{C}_2\}$, which is a compact convex subset in $\mathcal{B}^1(\overline{\Omega})$. Moreover, let T_2 be a restriction of T_1 on W_2. Then, T_2 is a continuous mapping of W_2 into itself. Consequently, we see that there exists a fixed-point $\tilde{\rho} = T_2(\tilde{\rho}) \in W_2$ by the Schauder fixed-point theorem (see Theorem 11.1 in [6] for example). Apparently, the function $\tilde{\rho}$ is a solution to the scalar equation (1.18) with the boundary data (1.14).

We construct the solution to (1.16), (1.14) and (1.15) from the $\tilde{\rho}$ as follows. Define a constant $\tilde{j} := J_{\tilde{\rho}}$ by (2.3) and then define a function $\tilde{\phi}$ by the formula (1.20), i.e., $\tilde{\phi} := \Phi[\tilde{\rho}]$. Finally, it is a straight forward computation to confirm that $(\tilde{\rho}, \tilde{j}, \tilde{\phi}) \in \mathcal{B}^2(\overline{\Omega})$ is a desired solution to the stationary problem (1.16), (1.14) and (1.15). Furthermore, we see that $\tilde{j} \lesseqgtr 0$ holds if and only if $\overline{C}_b \lesseqgtr 0$ due to (2.3). Thus, the proof is completed. □

The above lemma ensures the existence of the stationary solution. In order to show its uniqueness, we need an additional assumption (see Lemma 2.3). We obtain several estimates for the stationary solution in the next lemma before discussing the uniqueness.

Lemma 2.2. *Let $(\tilde{\rho}, \tilde{j}, \tilde{\phi})$ be a stationary solution in $\mathcal{B}^2(\overline{\Omega})$ to the problem* (1.16), (1.14) *and* (1.15) *satisfying the condition* (1.8). *Then the solution $(\tilde{\rho}, \tilde{j}, \tilde{\phi})$ verifies the estimates*.

$$\overline{C}_m \leq \tilde{\rho} \leq \overline{C}_M, \tag{2.9}$$

$$|\tilde{\phi}|_2 \leq \overline{C}_M + \phi_r, \tag{2.10}$$

$$|\tilde{j}| \leq J_M := \overline{C}_M\big(K\gamma\overline{C}_M^{\gamma+1}|\rho_r^{-2} - \rho_l^{-2}| + |\overline{C}_b|\big), \tag{2.11}$$

$$|\tilde{\rho}_x|_0 \leq \frac{\overline{C}_M^2\{\overline{C}_M(\overline{C}_M + \phi_r) + J_M\}}{K\gamma\overline{C}_m^{\gamma+1} - J_M^2}, \tag{2.12}$$

$$|\tilde{\rho}_{xx}|_0 \leq \frac{\big(K\gamma(\gamma-1)\overline{C}_M^{\gamma} + 2J_M^2\overline{C}_m^{-1}\big)|\tilde{\rho}_x|_0^2 + \overline{C}_M^{-2}(|\tilde{\rho}|_0 + \overline{C}_M)(\overline{C}_M + \phi_r)}{K\gamma\overline{C}_m^{\gamma+1} - J_M^2}. \tag{2.13}$$

Moreover, for an arbitrary ρ_l, there exists a positive constant δ_1 such that if $\delta \leq \delta_1$, then the stationary solution satisfies the estimates in the Hölder space:

$$|(\tilde{\rho}, \tilde{\phi})|_2 \leq C, \tag{2.14}$$

$$|\tilde{u}|_2 = \left|\frac{\tilde{j}}{\tilde{\rho}}\right|_2 \leq C\delta, \tag{2.15}$$

where C is a positive constant independent of ρ_r and ϕ_r.

Proof. Applying the maximum principle to the elliptic equation (1.18) yields the estimate (2.9), since the stationary solution satisfies the subsonic condition (1.8a). Note that $\tilde{\phi}$ is given by the formula (1.20) or equivalently

$$\tilde{\phi}(x) = \int_x^1 \int_y^1 (\tilde{\rho} - D)(z)\,dzdy + \phi_r x - \left(\int_0^1 \int_y^1 (\tilde{\rho} - D)(z)\,dzdy \right)(1 - x). \tag{2.16}$$

By estimating the formula (1.20) for $x \in [0, 1/2]$ and the formul (2.16) for $x \in (1/2, 1]$, we obtain the estimate (2.10) due to the estimate (2.9) and the assumption (1.3). Owing to the subsonic condition (1.8a) and the estimate (2.9), the inequality $\overline{C}_M^2 p'(\overline{C}_M) - \tilde{j}^2 > \tilde{\rho}^2 p'(\tilde{\rho}) - \tilde{j}^2 > 0$ holds. Namely $\tilde{j}^2 < K\gamma \overline{C}_M^{\gamma+1}$, which yields the estimate (2.11) with the aid of the current-voltage relationship (1.19). The estimates (2.12) and (2.13) immediately follow from the equation (1.16b) and the estimates (2.9), (2.10) and (2.11). The estimate (2.14) apparently follows from (2.9), (2.10), (2.12) and (2.13). The straight forward computation with (2.11), (2.12) and (2.13) also yields (2.15). □

Even though Lemma 2.1 shows the existence of a stationary solution, the stronger assumption than in Lemma 2.1 is necessary for its uniqueness.

Lemma 2.3. *Suppose that the doping profile and the boundary data* (1.14) *and* (1.15) *satisfy* (1.3), (1.5) *and* (1.6) *as well as*

$$\gamma K \overline{C}_m^{\gamma+1} > J_M^2 + 2\overline{C}_M(\overline{C}_M + \phi_r)J_M. \tag{2.17}$$

If the solution $(\tilde{\rho}, \tilde{j}, \tilde{\phi})$ *to the stationary problem* (1.16), (1.14) *and* (1.15) *exists in* $\mathcal{B}^2(\overline{\Omega})$ *and satisfies* (1.8), *then the solution is unique.*

Proof. Let $(\tilde{\rho}_1, \tilde{j}_1, \tilde{\phi}_1)$ and $(\tilde{\rho}_2, \tilde{j}_2, \tilde{\phi}_2)$ be solutions to the stationary problem (1.16), (1.14) and (1.15). We can assume $\tilde{j}_1 \leq \tilde{j}_2$ without loss of generality. Since $(\tilde{\phi}_1 - \tilde{\phi}_2)(0) = (\tilde{\phi}_1 - \tilde{\phi}_2)(1) = 0$, the mean value theorem shows that $(\tilde{\phi}_1 - \tilde{\phi}_2)_x(x_1) = 0$ holds for a certain $x_1 \in [0, 1]$. Thus, we may assume without loss of generality that $(\tilde{\phi}_1 - \tilde{\phi}_2)_x$ attains the nonnegative maximum at a certain $x_0 \in [0, 1]$.

We show that the maximum of $(\tilde{\phi}_1 - \tilde{\phi}_2)_x$ is zero by the contradiction. Firstly, suppose that $(\tilde{\phi}_1 - \tilde{\phi}_2)_x(x_0) > 0$ with $0 < x_0 < 1$. Then it holds that $(\tilde{\phi}_1 - \tilde{\phi}_2)_x(x_0) > 0$, $(\tilde{\rho}_1 - \tilde{\rho}_2)(x_0) = (\tilde{\phi}_1 - \tilde{\phi}_2)_{xx}(x_0) = 0$ and $(\tilde{\rho}_1 - \tilde{\rho}_2)_x(x_0) = (\tilde{\phi}_1 - \tilde{\phi}_2)_{xx}(x_0) \leq 0$. Substitute $(\tilde{\rho}_1, \tilde{j}_1, \tilde{\phi}_1)$ and $(\tilde{\rho}_2, \tilde{j}_2, \tilde{\phi}_2)$ in (1.13b) and then take the difference of these two resultant qualities to see that the following inequality holds at x_0,

$$(\tilde{j}_1 - \tilde{j}_2)\left(1 - (\tilde{j}_1 + \tilde{j}_2)\frac{\tilde{\rho}_{1x}}{\tilde{\rho}_1^2}\right) + \left(p'(\tilde{\rho}_1) - \frac{\tilde{j}^2}{\tilde{\rho}_1^2}\right)(\tilde{\rho}_1 - \tilde{\rho}_2)_x = \tilde{\rho}_1(\tilde{\phi}_1 - \tilde{\phi}_2)_x. \tag{2.18}$$

Combining the condition (2.17) with the estimates (2.11) and (2.10) yields the inequality $\tilde{\rho}_1^2 p'(\tilde{\rho}_1) > \tilde{\rho}_1 \phi_{1x}(\tilde{j}_1 + \tilde{j}_2) - \tilde{j}_1 \tilde{j}_2$, which shows $1 - (\tilde{j}_1 + \tilde{j}_2)\tilde{\rho}_{1x}/\tilde{\rho}_1^2 > 0$ with the aid of the condition (1.8) and the equation (1.16b). Hence, the left hand side of the equation (2.18) is non-positive. On the other hand, the right hand side of the equation (2.18) is positive. This is a contradiction.

Hence, the remained possibility is that $(\tilde{\phi}_1 - \tilde{\phi}_2)_x$ attains the positive maximum at the boundary $x = 0$ or 1. We treat the former case that it attains at $x = 0$ only since the latter case is handled similarly. If $(\tilde{\rho}_1 - \tilde{\rho}_2)_x(0) \le 0$, the similar observation as above with $(\tilde{\rho}_1 - \tilde{\rho}_2)(0) = 0$ yields (2.18), which is contradiction. If $(\tilde{\rho}_1 - \tilde{\rho}_2)_x(0) > 0$, we see from $(\tilde{\rho}_1 - \tilde{\rho}_2)(0) = 0$ that there exists a positive constant δ_1 such that if $0 < x < \delta_1$, then $(\tilde{\rho}_1 - \tilde{\rho}_2)(x) > 0$ holds. Thus $(\tilde{\phi}_1 - \tilde{\phi}_2)_{xx}(x) = (\tilde{\rho}_1 - \tilde{\rho}_2)(x) > 0$ holds for $0 < x < \delta_1$, and then $(\tilde{\phi}_1 - \tilde{\phi}_2)_x(0) < (\tilde{\phi}_1 - \tilde{\phi}_2)_x(x)$ for $0 < x < \delta_1$, which also contradicts to the assumption that $(\tilde{\phi}_1 - \tilde{\phi}_2)_x$ attains the positive maximum at the boundary $x = 0$. So, the maximum of $(\tilde{\phi}_1 - \tilde{\phi}_2)_x$ must be zero.

Thus, we have shown $(\tilde{\phi}_1 - \tilde{\phi}_2)_x \le 0$. Since $(\tilde{\phi}_1 - \tilde{\phi}_2)(0) = (\tilde{\phi}_1 - \tilde{\phi}_2)(1) = 0$, $\tilde{\phi}_1 \equiv \tilde{\phi}_2$. Owing to the equation (1.16c), we see $\tilde{\rho}_1 \equiv \tilde{\rho}_2$. Since (2.18) holds for an arbitrary $x_0 \in (0, 1)$, we have $\tilde{j}_1 \equiv \tilde{j}_2$. The proof is completed. □

Consequently, Lemma 1.1 holds apparently from Lemmas 2.1 and 2.3 since the smallness of δ implies all assumptions in Lammas 2.1 and 2.3 hold.

2.2. Time local solution to non-stationary problem. This subsection is devoted to the discussion on the unique existence of the solution locally in time to the initial boundary value problem. The existence of the time local solution is proved by the similar method as in [7] and [8] with using the standard iteration method.

Lemma 2.4. *Suppose the initial data* $(\rho_0, u_0) \in H^2(\Omega)$ *and the boundary data* ρ_l, ρ_r *and* ϕ_r *satisfy* (1.9), (1.5), (1.6) *and* (1.7). *Then there exists a constant* $T_1 > 0$ *such that the initial boundary value problem* (1.1), (1.4), (1.5) *and* (1.6) *has a unique solution* $(\rho, u, \phi)(t, x) \in \mathfrak{X}_2([0, T_1])$ *satisfying the condition* (1.8).

In order to define the successive approximation sequence for solving the problem (1.11), (1.4), (1.5) and (1.6), we study the linearized system for the unknown $(\hat{\rho}, \hat{j})$:

$$\hat{\rho}_t + \hat{j}_x = 0, \tag{2.19a}$$

$$\hat{j}_t + \left(p'(\rho) - \frac{j^2}{\rho^2}\right)\hat{\rho}_x + 2\frac{j}{\rho}\hat{j}_x = \rho\phi_x - j, \tag{2.19b}$$

with the initial data (1.4) and the boundary data (1.5), where the function ϕ in (2.19b) is defined by (1.12), i.e. $\phi = \Phi(\rho)$. The functions (ρ, j) in the coefficients in (2.19) are supposed satisfy

$$(\rho, j) \in \mathfrak{X}_2([0, T]), \quad (\rho, j)(0, x) = (\rho_0, j_0), \tag{2.20}$$

$$\rho(t, x) \ge m, \quad \left(p'(\rho) - \frac{j^2}{\rho^2}\right)(t, x) \ge k \quad \text{for} \quad (t, x) \in [0, T] \times \Omega, \tag{2.21}$$

$$\|(\rho, j)(t)\|_2 + \|(\rho_t, j_t)(t)\|_1 + \|(\rho_{tt}, j_{tt})(t)\| \le M \quad \text{for} \quad t \in [0, T], \tag{2.22}$$

where T, m, k and M are positive numbers. We denote by $X_2(T; m, k, M)$ the set of functions (ρ, j) satisfying (2.20), (2.21) and (2.22). Hereafter we abbreviate $X_2(T; m, k, M)$ by $X_2(\,\cdot\,)$ without confusion. Due to (1.12),

$$\phi \in C^2([0, T]; H^2(\Omega)), \quad \|\partial_t^i \phi(t)\|_2 \le M \quad \text{for} \quad i = 0, 1, 2, \quad t \in [0, T].$$

Then the next lemma shows that for suitably chosen constants T, m, k and M, the set $X_2(\,\cdot\,)$ is invariant under the mapping $(\rho, j) \to (\hat{\rho}, \hat{j})$ defined by solving the problem (2.19), (1.12), (1.4) and (1.5). The solvability of this linear problem is discussed in Appendix. Then we have the next lemma. Since it is proved by the similar method as in [7, 8], we omit the details.

Lemma 2.5. *Suppose that the initial data* $(\rho_0, j_0) \in H^2(\Omega)$ *and the boundary data* ρ_l *and* ρ_r *satisfy* (1.9) *and* (1.5). *In addition, assume the compatibility conditions* (1.7) *hold. Then there exist positive constants* T, m, k *and* M *with the following property*: *If* $(\rho, j) \in X_2(\,\cdot\,)$, *then the problem* (2.19), (1.12), (1.4) *and* (1.6) *admits a unique solution* $(\hat{\rho}, \hat{j})(x, t)$ *in the same set* $X_2(\,\cdot\,)$.

Using above lemma, we can prove Lemma 2.4.

Proof of Lemma 2.4. We define the successive approximation sequence $\{(\rho^n, j^n)\}_{n=0}^{\infty}$ by $(\rho^0, j^0) = (\rho_0, j_0)$ and

$$\rho_t^{n+1} + j_x^{n+1} = 0, \tag{2.23a}$$

$$j_t^{n+1} + \left(p'(\rho^n) - \left(\frac{j^n}{\rho^n}\right)^2\right)\rho_x^{n+1} + 2\frac{j^n}{\rho^n} j_x^{n+1} = \rho^n \phi_x^n - j^n, \tag{2.23b}$$

$$\phi^n = \Phi[\rho^n] \tag{2.23c}$$

with the initial and the boundary conditions

$$(\rho^{n+1}, j^{n+1})(0, x) = (\rho_0, j_0)(x), \tag{2.24}$$

$$\rho^{n+1}(t, 0) = \rho_l, \quad \rho^{n+1}(t, 1) = \rho_r \tag{2.25}$$

for $n = 0, 1, \ldots,$ where Φ in (2.23c) is defined by (1.12). By virtue of Lemma 2.5, the sequence $\{(\rho^n, j^n)\}$ is well defined and satisfies $(\rho^n, j^n) \in X_2(\,\cdot\,)$. Lemma 2.5 also implies that the estimate

$$\|(\rho^n, j^n)(t)\|_2 + \|(\rho_t^n, j_t^n)(t)\|_1 + \|(\rho_{tt}^n, j_{tt}^n)(t)\| \le M$$

holds for $t \in [0, T]$. Moreover, applying the standard energy estimate for the linear symmetric hyperbolic system satisfied by the difference $(\rho^{n+1} - \rho^n, j^{n+1} - j^n)$, we see that $\{(\rho^n, j^n)\}$ is the Cauchy sequence in $\mathfrak{X}_1([0, T])$. Consequently, there exists a function $(\rho, j) \in \mathfrak{X}_1([0, T])$ such that $(\rho^n, j^n) \to (\rho, j)$ strongly in $\mathfrak{X}_1([0, T])$ as $n \to \infty$. Moreover, it holds $(\rho, j) \in \mathfrak{X}_2([0, T])$ by the standard theory for the hyperbolic equations (see [15] for example). For the function ρ thus obtained, define $\phi := \Phi[\rho]$ as (1.12). It is easy to see that (ρ, j, ϕ) is the desired solution to the problem (1.11), (1.4), (1.5) and (1.6) as well as satisfies (1.8). Thus the proof of Lemma 2.4 is completed. □

3. A priori estimate

3.1. Preliminary computation. In order to prove the stability of the stationary solution in Theorem 1.2, we regard the solution (ρ, u, ϕ) as a perturbation from the stationary solution $(\tilde{\rho}, \tilde{u}, \tilde{\phi})$. Thus, we introduce new unknown functions as

$$\psi(t, x) := \rho(t, x) - \tilde{\rho}(x), \quad \eta(t, x) := u(t, x) - \tilde{u}(x), \quad \omega(t, x) := \phi(t, x) - \tilde{\phi}(x).$$

Multiplying (1.1b) by $1/\rho$ and using the equation (1.1a), we have

$$u_t + uu_x + (h(\rho))_x = \phi_x - u. \tag{3.1}$$

Similarly, we have from (1.13b) that

$$\tilde{u}\tilde{u}_x + (h(\tilde{\rho}))_x = \tilde{\phi}_x - \tilde{u}. \tag{3.2}$$

Subtracting (1.13a) from (1.1a), (3.2) from (3.1) and (1.13c) from (1.1c), respectively, we obtain the equations for the perturbation (ψ, η, ω) as

$$\psi_t + \{(\tilde{\rho} + \psi)(\tilde{u} + \eta) - \tilde{\rho}\tilde{u}\}_x = 0, \tag{3.3a}$$

$$\eta_t + \frac{1}{2}\{(\tilde{u} + \eta)^2 - \tilde{u}^2\}_x + \{h(\tilde{\rho} + \psi) - h(\tilde{\rho})\}_x - \omega_x + \eta = 0, \tag{3.3b}$$

$$\omega_{xx} = \psi. \tag{3.3c}$$

The initial and the boundary conditions to the system (3.3) are derived from (1.4), (1.5), (1.6), (1.14) and (1.15) as

$$\psi(x, 0) = \psi_0(x) := \rho_0(x) - \tilde{\rho}(x), \quad q\eta(x, 0) = \eta_0(x) := u_0(x) - \tilde{u}(x), \tag{3.4}$$

$$\psi(t, 0) = \psi(t, 1) = 0, \tag{3.5}$$

$$\omega(t, 0) = \omega(t, 1) = 0. \tag{3.6}$$

Since $(\tilde{\rho}, \tilde{u}, \tilde{\phi}) \in \mathfrak{X}_2([0, T])$ and ω satisfies (3.3c), the local existence of the solution (ψ, η, ω) to the initial boundary value problem (3.3), (3.4), (3.5) and (3.6) follows from Lemmas 1.1 and 2.4.

Corollary 3.1. *Suppose that the initial data* (ψ_0, η_0) *belongs to* $H^2(\Omega)$ *and* $(\tilde{\rho} + \psi_0, \tilde{u} + \eta_0)$ *satisfies* (1.9). *Then there exists a constant* $T_2 > 0$, *such that the initial boundary value problem* (3.3), (3.4), (3.5) *and* (3.6) *has a unique local solution* $(\psi, \eta, \omega) \in \mathfrak{X}_2([0, T_2]) \times \mathfrak{X}_2([0, T_2]) \times \mathfrak{X}_2^2([0, T_2])$ *with the property that* $(\tilde{\rho} + \psi, \tilde{u} + \eta)$ *satisfies* (1.8).

Owing to Corollary 3.1, it suffices to derive an a priori estimate (3.7) in order to show the existence of the solution globally in time.

Proposition 3.2. *Let* $(\psi, \eta, \omega)(t, x) \in \mathfrak{X}_2([0, T]) \times \mathfrak{X}_2([0, T]) \times \mathfrak{X}_2^2([0, T])$ *be a solution to* (3.3), (3.4), (3.5) *and* (3.6). *Then there exists a positive constant* ϵ_0 *such that if* $N(T) + \delta \leq \epsilon_0$, *then the following estimate holds for* $t \in [0, T]$.

$$\|(\psi, \eta)(t)\|_2^2 + \|\omega(t)\|_4^2 + \int_0^t \|(\psi, \eta)(\tau)\|_2^2 + \|\omega(\tau)\|_4^2 \, d\tau \leq C\|(\psi, \eta)(0)\|_2^2, \tag{3.7}$$

where C *is a positive constant independent of* T.

The remainder of the present paper is devoted to showing the uniform estimate (3.7). To this purpose, it is convenient to use notations

$$N(t) := \sup_{0 \leq \tau \leq t} \|(\psi, \eta)(\tau)\|_2, \quad M^2(t) := \int_0^t \|\psi_x(\tau)\|^2 + \|\eta_x(\tau)\|^2 \, d\tau.$$

3.2. Basic estimate. This subsection is devoted to the derivation of the basic estimate. First, we define an energy E as

$$E := \frac{1}{2}\rho u^2 + \int_1^\rho h(\xi) \, d\xi + \frac{1}{2}(\phi_x)^2. \tag{3.8}$$

Using the equality (3.1), we see that the energy E satisfies the equation

$$E_t + \rho u^2 = -\frac{1}{2}\{\rho u\}_x u^2 - \rho u^2 u_x - \{h(\rho)\rho u\}_x + \{\rho u \phi\}_x + \{\phi_{xt}\phi\}_x. \tag{3.9}$$

In order to show the basic estimate, we define the energy form $\mathcal{E}$ as

$$\mathcal{E} := \frac{1}{2}\rho(u - \tilde{u})^2 + \Psi(\rho, \tilde{\rho}) + \frac{1}{2}\{(\phi - \tilde{\phi})_x\}^2, \quad \Psi(\rho, \tilde{\rho}) := \int_{\tilde{\rho}}^\rho h(\xi) - h(\tilde{\rho}) \, d\xi. \tag{3.10}$$

Notice that $\mathcal{E}$ is equivalent to $|(\psi, \eta, \omega_x)|^2$ if $|(\psi, \eta, \omega_x)| < c$, since $\Psi(\rho, \tilde{\rho})$ is equivalent to $|\psi|^2$. Namely, there exist positive constants c_1 and C_1 such that

$$c_1|(\psi, \eta, \omega_x)|^2 \leq \mathcal{E} \leq C_1|(\psi, \eta, \omega_x)|^2 \tag{3.11}$$

if $|(\psi, \eta, \omega_x)| \le c$. Multiply the equation (3.3b) by $\rho u - \tilde{\rho}\tilde{u}$. Apply the integration by parts with respect to t and x to the first and the second terms of the left hand side, respectively. Moreover, after integrating the third term of the left hand by parts with respect to x, substitute the equation (3.3b) in the resultant. Rewrite the fourth term by similar method with using (3.3c). These computations yield an equation for the energy form $\mathcal{E}$:

$$\mathcal{E}_t + \tilde{\rho}\eta^2 = R_{1x} + R_2, \tag{3.12a}$$

$$R_1 := \omega\omega_{xt} + \omega(\rho\phi - \tilde{\rho}\tilde{\phi}) - \{h(\rho) - h(\tilde{\rho})\}(\rho u - \tilde{\rho}\tilde{u}) + \{h(\rho) - h(\tilde{\rho})\}\psi\tilde{u}, \tag{3.12b}$$

$$\begin{aligned} R_2 := &-\left\{\frac{1}{2}(u^2 - \tilde{u}^2)(\rho u - \tilde{\rho}\tilde{u})\right\}_x - \psi\eta u + (\rho u - \tilde{\rho}\tilde{u})_x\eta\tilde{u} \\ &+ \left\{\frac{1}{2}(u^2 - \tilde{u}^2)_x - \omega_x + \eta\right\}\psi\tilde{u} - \{h(\rho) - h(\tilde{\rho})\}(\psi\tilde{u})_x. \end{aligned} \tag{3.12c}$$

Applying the Sobolev inequality on R_2 with the estimates (2.14) and (2.15), we have the following estimate:

$$|R_2| \le C(N(T) + \delta)|(\psi, \psi_x, \eta, \eta_x, \omega_x)|^2. \tag{3.13}$$

We show Lemma 3.3 to drive the basic estimate.

Lemma 3.3. *Suppose the same assumptions as in* Proposition 3.2 *hold. Then the following estimates hold for* $t \in [0, T]$.

$$\|\partial_t^i \omega(t)\|_2^2 \le C\|\partial_t^i \psi(t)\|^2 \quad \textit{for} \quad i = 0, 1, 2, \tag{3.14}$$

$$|\partial_t^i \omega(t)|_1^2 \le C\|\partial_t^i \psi(t)\|^2 \quad \textit{for} \quad i = 0, 1, 2, \tag{3.15}$$

$$\|\omega_{xt}(t)\|^2 \le C(N(T) + \delta)\|\psi(t)\|^2 + C\|\eta(t)\|^2, \tag{3.16}$$

where C is a positive constant independent of T.

Proof. The estimate (3.14) follows easily from (3.3c), (3.6) and the Poincaré inequality. Applying the Sobolev inequality on the estimate (3.14), we have the estimate (3.15). Substituting (3.3c) in (3.3a) yields the equality $\{\omega_{xt} + (\tilde{\rho} + \psi)(\tilde{u} + \eta) - \tilde{\rho}\tilde{u}\}_x = 0$. Thus, a function $k(t) := \omega_{xt} + (\tilde{\rho} + \psi)(\tilde{u} + \eta) - \tilde{\rho}\tilde{u}$ is independent of x. Hence, we obtain the following inequality from the boundary condition $\omega_t(t, 0) = \omega_t(t, 1) = 0$.

$$\int_0^1 \{(\tilde{\rho} + \psi)(\tilde{u} + \eta) - \tilde{\rho}\tilde{u}\}^2 - (\omega_{xt})^2\, dx = k^2(t) \ge 0. \tag{3.17}$$

The estimate (3.16) follows easily from the inequality (3.17), the estimates (2.14) and (2.15) and the assumption $N(T) + \delta \le \epsilon_0$. □

Next, we prove Lemma 3.4, which gives the basic estimate.

Lemma 3.4. *Suppose the same assumptions as in* Proposition 3.2 *hold. Then there exists a positive constant ϵ_0 such that if $N(T)+\delta \le \epsilon_0$, then the following estimate holds for $t \in [0, T]$.*

$$\|(\psi,\eta,\omega_x)(t)\|^2+\int_0^t \|(\psi,\eta,\omega_x)(\tau)\|^2 d\tau \le C\|(\psi,\eta,\omega_x)(0)\|^2+C(N(T)+\delta)M^2(t), \tag{3.18}$$

where C is a positive constant independent of T.

Proof. First, integrating (3.12a) over $[0,t]\times\Omega$ and substituting the estimate (3.13) to handle the integration of R_2, we have

$$\int_0^1 \mathcal{E}(t,x)\,dx + \int_0^t\int_0^1 \tilde{\rho}\eta^2\,dxd\tau = \int_0^1 \mathcal{E}(0,x)\,dx + \int_0^t\int_0^1 R_2\,dxd\tau \tag{3.19a}$$

$$\le \int_0^1 \mathcal{E}(0,x)\,dx + C(N(T)+\delta)\left(M^2(t) + \int_0^t\int_0^1 \psi^2+\eta^2+(\omega_x)^2\,dxd\tau\right) \tag{3.19b}$$

since $\int_0^1 R_{1x}\,dx = 0$ owing to the boundary conditions (3.5) and (3.6). Multiplying (3.3b) by ω_x, integrating the resultant equality over $[0,t]\times\Omega$, applying the integration by parts, and then using the equation (3.3c) and the boundary condition (3.5), we obtain that

$$\begin{aligned} &\int_0^1 \{-\eta\omega_x\}(t,x)\,dx + \int_0^t\int_0^1 (h(\tilde{\rho}+\psi)-h(\tilde{\rho}))\psi+(\omega_x)^2\,dxd\tau \\ &= \int_0^1 \{-\eta\omega_x\}(0,x)\,dx + \int_0^t\int_0^1 \eta\omega_x - \eta\omega_{xt} + \frac{1}{2}(2\tilde{u}\eta+\eta^2)_x\omega_x\,dxd\tau \end{aligned} \tag{3.20a}$$

$$\begin{aligned} &\le \int_0^1 \{\eta^2+(\omega_x)^2\}(0,x)\,dx + \int_0^t\int_0^1 C\eta^2+\frac{1}{2}(\omega_x)^2\,dxd\tau \\ &\quad + C(N(T)+\delta)\left(M^2(t)+\int_0^t\int_0^1 \psi^2+(\omega_x)^2\,dxd\tau\right). \end{aligned} \tag{3.20b}$$

In deriving the above inequality, we have also used the Schwarz and the Sobolev inequalities as well as the estimates (2.14), (2.15), (3.15) and (3.16). Multiply (3.20) by α, where α is a positive constant to determined, and then add the resultant inequality to (3.19). Then use the inequality $|-\eta\omega_x| \le \eta^2+(\omega_x)^2$ and take α and $N(T)+\delta$ are sufficiently small. These procedures yield the desired estimate (3.18). □

3.3. Higher order estimates. This subsection is devoted to the derivation of the higher order estimates. It is necessary to justify these computations by the discussion using the mollifier with respect to time variable t since the regularity of the solution

(ψ, η) constructed in Corollary 3.1 is not enough. However we omit this discussion as it is a well known argument. In the following computations, we differentiate the equations with respect to t to make use of the equalities $(\partial_t^i \psi, \partial_t^i \omega)(t, 0) = (\partial_t^i \psi, \partial_t^i \omega)(t, 1) = 0$ for $i = 0, 1, 2$. Thus, it is convenient to use a notation

$$A_i^2(t) := \sum_{j=0}^{i} \|(\partial_t^j \psi, \partial_t^j \eta)(t)\|^2 \quad \text{for} \quad i = 1, 2.$$

Differentiating (3.3b) with respect to t, we have the following equation

$$\partial_t^i \eta_t + (\tilde{u} + \eta)\partial_t^i \eta_x + \{h'(\tilde{\rho} + \psi)\partial_t^i \psi\}_x - \partial_t^i \omega_x + \partial_t^i \eta = F_i \quad \text{for} \quad i = 1, 2, \tag{3.21a}$$

$$F_1 := -(\tilde{u} + \eta)_x \eta_t, \quad F_2 := -(\tilde{u} + \eta)_x \eta_{tt} - 2\eta_t \eta_{xt} - \{h''(\tilde{\rho} + \psi)(\psi_t)^2\}_x. \tag{3.21b}$$

The absolute values of F_1 and F_2 are estimated as

$$|F_1| \leq C(N(T) + \delta)|\eta_t|, \quad |F_2| \leq C(N(T) + \delta)(|\eta_{tt}| + |\eta_{tx}| + |\psi_{tx}|), \tag{3.22}$$

where C is a positive constant independent of T. In deriving (3.22), we have also used the estimates (2.14) and (2.15) and the following inequality,

$$|\psi_t(t)|_0 + |\eta_t(t)|_0 \leq CN(T), \tag{3.23}$$

where C is a positive constant independent of T. In fact, we see that $(\psi, \eta) \in \mathfrak{X}_2([0, T])$ satisfies (3.23) by applying the Sobolev inequality on the equations (3.3a) and (3.3b) with using the estimates (2.14) and (2.15). Next, differentiating (3.3a) with respect to t, we have

$$\begin{aligned} \{(\tilde{\rho} + \psi)\partial_t^i \eta\}_x &= -\partial_t^i \psi_t - (\tilde{u} + \eta)\partial_t^i \psi_x + G_i \\ &= -\partial_t^i \omega_{xxt} - (\tilde{u} + \eta)\partial_t^i \psi_x + G_i \quad \text{for} \quad i = 0, 1, 2, \end{aligned} \tag{3.24a}$$

$$G_0 := -\tilde{u}_x \psi + \eta\psi_x, \quad G_1 := -(\tilde{u} + \eta)_x \psi_t, \quad G_2 := -(\tilde{u} + \eta)_x \psi_{tt} - 2(\psi_t \eta_t)_x. \tag{3.24b}$$

The estimates (2.14), (2.15) and (3.23) give that

$$|G_0| \leq C(N(T) + \delta)(|\eta| + |\psi|), \quad |G_1| \leq C(N(T) + \delta)|\psi_t|,$$
$$|G_2| \leq C(N(T) + \delta)(|\psi_{tt}| + |\eta_{tx}| + |\psi_{tx}|),$$

where C is a positive constant independent of T.

Lemma 3.5. *Suppose the same assumptions as in* Proposition 3.2 *hold. Then there exists a positive constant* ϵ_0 *such that if* $N(T) + \delta \leq \epsilon_0$, *then the following esti-*

mate holds for $t \in [0, T]$ *and* $i = 1, 2$

$$\begin{aligned}
&\|(\partial_t^i \psi, \partial_t^i \eta_t, \partial_t^i \omega_x)(t)\|^2 + \int_0^t \|(\partial_t^i \psi, \partial_t^i \eta, \partial_t^i \omega_x)(\tau)\|^2 \, d\tau \\
&\le C\left(A_i^2(0) + A_{i-1}^2(t) + \int_0^t A_{i-1}^2(\tau)\, d\tau \right),
\end{aligned} \tag{3.25}$$

where C *is a positive constant independent of* T.

Proof. The estimate

$$\|\partial_t^{i-1}\psi_x(t)\|^2 + \|\partial_t^{i-1}\eta_x(t)\|^2 + \|\partial_t^i \omega(t)\|_2^2 \le C A_i^2(t) \quad \text{for} \quad i = 1, 2 \tag{3.26}$$

holds from the smallness of $N(T) + \delta$ and the equations (3.3), (3.21a) and (3.24a) for $i = 1$. In deriving (3.26), we have also used the estimates (2.14) and (2.15). Multiply (3.21a) by $(\tilde{\rho} + \psi)\partial_t^{i-1}\eta$ for $i = 1, 2$ and integrate the resultant equality over Ω to obtain that

(3.27)
$$\begin{aligned}
&\int_0^1 \{\partial_t^i \eta_t + (\tilde{u} + \eta)\partial_t^i \eta_x + \partial_t^i \eta\}(\tilde{\rho} + \psi)\partial_t^{i-1}\eta \, dx + \int_0^1 \{h'(\tilde{\rho} + \psi)\partial_t^i \psi - \partial_t^i \omega\}_x (\tilde{\rho} + \psi)\partial_t^{i-1}\eta \, dx \\
&= \int_0^1 F_i (\tilde{\rho} + \psi)\partial_t^{i-1}\eta \, dx.
\end{aligned}$$

We rewrite the first term on the left hand side of (3.27) by applying the integration by parts with respect to t as

(3.28)
(the first term)
$$\begin{aligned}
&= \frac{d}{dt}\int_0^1 (\tilde{\rho} + \psi)\partial_t^i \eta \partial_t^{i-1}\eta \, dx - \int_0^1 \{(\tilde{\rho} + \psi)\partial_t^{i-1}\eta\}_t \partial_t^i \eta \, dx \\
&\quad + \frac{d}{dt}\int_0^1 (\tilde{\rho} + \psi)(\tilde{u} + \eta)\partial_t^{i-1}\eta \partial_t^{i-1}\eta_x \, dx - \int_0^1 \{(\tilde{\rho} + \psi)(\tilde{u} + \eta)\partial_t^{i-1}\eta\}_t \partial_t^{i-1}\eta_x \, dx \\
&\quad + \frac{d}{dt}\int_0^1 \frac{1}{2}(\tilde{\rho} + \psi)(\partial_t^{i-1}\eta)^2 \, dx - \int_0^1 \frac{1}{2}\psi_t (\partial_t^{i-1}\eta)^2 \, dx.
\end{aligned}$$

Using the boundary conditions $\partial_t^i \psi(t, 0) = \partial_t^i \psi(t, 1) = \partial_t^i \omega(t, 0) = \partial_t^i \omega(t, 1) = 0$ and applying the integration by parts with respect to x, we rewrite the second term on the left hand side of (3.27) as

$$
\begin{aligned}
&\text{(the second term)}\\
&= -\int_0^1 \{(\tilde\rho+\psi)\partial_t^{i-1}\eta\}_x\{h'(\tilde\rho+\psi)\partial_t^i\psi - \partial_t^i\omega\}\,dx\\
&= \int_0^1 h'(\tilde\rho+\psi)(\partial_t^i\psi)^2 + h'(\tilde\rho+\psi)(\tilde u+\eta)\partial_t^i\psi\,\partial_t^{i-1}\psi_x - h'(\tilde\rho+\psi)G_{i-1}\partial_t^i\psi\,dx\\
&\quad + \int_0^1 (\partial_t^i\omega_x)^2 - (\tilde u+\eta)\partial_t^i\omega\,\partial_t^{i-1}\psi_x + G_{i-1}\partial_t^i\omega\,dx,
\end{aligned}
\tag{3.29}
$$

where we have also used the equation (3.24a).

Substitute the equalities (3.28) and (3.29) in the equality (3.27) and then integrate the resultant equality over $(0, t)$. The result is

$$
\begin{aligned}
&I_1^{(i)}(t) + \int_0^t\int_0^1 h'(\tilde\rho+\psi)(\partial_t^i\psi)^2 + (\partial_t^i\omega_x)^2\,dxd\tau\\
&= \int_0^t\int_0^1 (\tilde\rho+\psi)(\partial_t^i\eta)^2\,dxd\tau + I_1^{(i)}(0) + \int_0^t J_1^{(i)}(\tau)\,d\tau,
\end{aligned}
\tag{3.30a}
$$

$$
I_1^{(i)}(t) := \int_0^1 (\tilde\rho+\psi)\left(\partial_t^i\eta\,\partial_t^{i-1}\eta + (\tilde u+\eta)\partial_t^{i-1}\eta\,\partial_t^{i-1}\eta_x + \frac{1}{2}(\partial_t^{i-1}\eta)^2\right)dx,
\tag{3.30b}
$$

$$
\begin{aligned}
J_1^{(i)}(t) := &\int_0^1 \eta_t(\tilde\rho+\psi)\partial_t^{i-1}\eta\,\partial_t^{i-1}\eta_x + \psi_t(\tilde u+\eta)\partial_t^{i-1}\eta\,\partial_t^{i-1}\eta_x + \psi_t\partial_t^i\eta\,\partial_t^{i-1}\eta\,dx\\
&+ \int_0^1 (\tilde u+\eta)(\tilde\rho+\psi)\partial_t^i\eta\,\partial_t^{i-1}\eta_x - h'(\tilde\rho+\psi)(\tilde u+\eta)\partial_t^i\psi\,\partial_t^{i-1}\psi_x\,dx\\
&+ \int_0^1 h'(\tilde\rho+\psi)G_{i-1}\partial_t^i\psi + (\tilde u+\eta)\partial_t^i\omega\,\partial_t^{i-1}\psi_x + G_{i-1}\partial_t^i\omega\,dx\\
&+ \int_0^1 \frac{1}{2}\psi_t(\partial_t^{i-1}\eta)^2 + (\tilde\rho+\psi)F_i\,\partial_t^{i-1}\eta\,dx.
\end{aligned}
\tag{3.30c}
$$

Applying the Schwarz and the Sobolev inequalities, we estimate the first term on the left-hand side of the equality (3.30a) as

$$
|I_1^{(i)}(t)| \le \epsilon\|(\partial_t^i\eta, \partial_t^i\psi)(t)\|^2 + C_\epsilon A_{i-1}^2(t),
\tag{3.31}
$$

where we have also used the estimates (2.14), (2.15) and (3.26) as well as the smallness assumption $N(T) + \delta \le \epsilon_0$. In (3.31), ϵ is an arbitrary positive constant and C_ϵ is a constant depending only on ϵ. Substitute $t = 0$ and $\epsilon = 1$ in the estimate (3.31) to obtain that

$$
|I_1^{(i)}(0)| \le CA_i^2(0).
\tag{3.32}
$$

Moreover, apply the Schwarz and the Sobolev inequalities to the first term of $J_1^{(i)}(t)$ with using the estimates (2.14), (2.15), (3.23) and (3.26) as well as the smallness assumption $N(T)+\delta \le \epsilon_0$ to obtain that

$$\left|\int_0^1 \eta_t(\tilde{\rho}+\psi)\partial_t^{i-1}\eta\partial_t^{i-1}\eta_x\,dx\right| \le C|\eta_t(t)|_0\,|(\tilde{\rho}+\psi)(t)|_0\,\|(\partial_t^{i-1}\eta,\,\partial_t^{i-1}\eta_x)(t)\|^2$$
$$\le C(N(T)+\delta)A_i^2(t)+CA_{i-1}^2(t).$$

The other terms in $J_1^{(i)}(t)$ are estimated by the similar method since $|(\tilde{u}+\eta)(t)|_1 \le C(N(T)+\delta)$. Consequently, we have

$$|J_1^{(i)}(t)| \le C(N(T)+\delta)A_i^2(t)+CA_{i-1}^2(t). \tag{3.33}$$

Substituting the estimates (3.31), (3.32) and (3.33) in the equation (3.30a) yields the inequality

(3.34)

$$-\epsilon\|(\partial_t^i\eta,\,\partial_t^i\psi)(t)\|^2+\int_0^t\!\!\int_0^1 h'(\tilde{\rho}+\psi)(\partial_t^i\psi)^2+(\partial_t^i\omega_x)^2\,dxd\tau-\int_0^t\!\!\int_0^1(\tilde{\rho}+\psi)(\partial_t^i\eta)^2\,dxd\tau$$
$$\le C_\epsilon A_{i-1}^2(t)+C\int_0^t A_{i-1}^2(\tau)\,d\tau+C\left(A_i^2(0)+(N(T)+\delta)\int_0^t A_i^2(\tau)\,d\tau\right).$$

Next, multiply (3.21a) by $(\tilde{\rho}+\psi)\partial_t^i\eta$ for $i=1,\,2$ and integrate the resultant equality over Ω to obtain that

$$\begin{aligned}&\int_0^1(\tilde{\rho}+\psi)\partial_t^i\eta\partial_t^i\eta_t\,dx+\int_0^1(\tilde{u}+\eta)(\tilde{\rho}+\psi)\partial_t^i\eta\partial_t^i\eta_x\,dx+\int_0^1(\tilde{\rho}+\psi)(\partial_t^i\eta)^2\,dx\\&+\int_0^1(\tilde{\rho}+\psi)\{h'(\tilde{\rho}+\psi)\partial_t^i\psi-\partial_t^i\omega\}_x\partial_t^i\eta\,dx\\&=\int_0^t\!\!\int_0^1(\tilde{\rho}+\psi)F_i\partial_t^i\eta\,dxd\tau.\end{aligned} \tag{3.35}$$

We rewrite the first term on the left-hand side of the equality (3.35) by applying the integration by parts with respect to t as

$$(\text{the first term})=\frac{d}{dt}\int_0^1\frac{1}{2}(\tilde{\rho}+\psi)(\partial_t^i\eta)^2\,dx-\int_0^1\frac{1}{2}\psi_t(\partial_t^i\eta)^2\,dx. \tag{3.36}$$

Use the boundary conditions $\partial_t^i\psi(t,0)=\partial_t^i\psi(t,1)=0$ and apply the integration by parts to rewrite the second term on the left-hand side of the equality (3.35) as

$$
\begin{aligned}
&(\text{the second term})\\
&= -\int_0^1 (\tilde{u}+\eta)\partial_t^i\eta\{\partial_t^i\psi_t + (\tilde{u}+\eta)\partial_t^i\psi_x + (\tilde{\rho}+\psi)_x\partial_t^i\eta - G_i\}\,dx\\
&= \frac{d}{dt}\int_0^1 -(\tilde{u}+\eta)\partial_t^i\eta\partial_t^i\psi\,dx + \int_0^1 \{(\tilde{u}+\eta)\partial_t^i\eta\}_t\partial_t^i\psi\,dx\\
&\quad + \int_0^1 \{(\tilde{u}+\eta)^2\partial_t^i\eta\}_x\partial_t^i\psi - (\tilde{u}+\eta)(\tilde{\rho}+\psi)_x(\partial_t^i\eta)^2 + (\tilde{u}+\eta)G_i\partial_t^i\eta\,dx\\
&= \frac{d}{dt}\int_0^1 -(\tilde{u}+\eta)\partial_t^i\eta\partial_t^i\psi\,dx + \int_0^1 \{((\tilde{u}+\eta)^2)_x + \eta_t\}\partial_t^i\eta\partial_t^i\psi\,dx\\
&\quad - \int_0^1 (\tilde{u}+\eta)(\tilde{\rho}+\psi)_x(\partial_t^i\eta)^2 - (\tilde{u}+\eta)G_i\partial_t^i\eta\,dx\\
&\quad + \int_0^1 \{\partial_t^i\eta_t + (\tilde{u}+\eta)\partial_t^i\eta_x\}(\tilde{u}+\eta)\partial_t^i\psi\,dx,
\end{aligned}
\tag{3.37}
$$

Note that we have used the equation (3.24a) too in deriving the first equality above. By using the boundary conditions $\partial_t^i\psi(t,0) = \partial_t^i\psi(t,1) = 0$ and applying the integration by parts with respect to x, we rewrite the last term on the right-hand side of (3.37) as

$$
\begin{aligned}
&\int_0^1 (\tilde{u}+\eta)\{\partial_t^i\eta_t + (\tilde{u}+\eta)\partial_t^i\eta_x\}\partial_t^i\psi\,dx\\
&= \int_0^1 (\tilde{u}+\eta)\{(-h'(\tilde{\rho}+\psi)\partial_t^i\psi)_x + \partial_t^i\omega_x - \partial_t^i\eta + F_i\}\partial_t^i\psi\,dx\\
&= \int_0^1 \frac{1}{2}\{h'(\tilde{\rho}+\psi)(\tilde{u}+\eta)\}_x(\partial_t^i\psi)^2 - h''(\tilde{\rho}+\psi)(\tilde{\rho}+\psi)_x(\tilde{u}+\eta)(\partial_t^i\psi)^2\,dx\\
&\quad + \int_0^1 (\tilde{u}+\eta)\{\partial_t^i\omega_x - \partial_t^i\eta + F_i\}\partial_t^i\psi\,dx.
\end{aligned}
\tag{3.38}
$$

where we have used the equation (3.21a) to obtain the first equality.

The last term on the left-hand side of the equality (3.35) is rewritten by the boundary conditions $\partial_t^i\psi(t,0) = \partial_t^i\psi(t,1) = \partial_t^i\omega(t,0) = \partial_t^i\omega(t,1) = 0$ and the integration by parts as

$$
\begin{aligned}
&(\text{the last term})\\
&= -\int_0^1 \{(\tilde\rho+\psi)\partial_t^i\eta\}_x\{h'(\tilde\rho+\psi)\partial_t^i\psi - \partial_t^i\omega\}\,dx\\
&= \int_0^1 h'(\tilde\rho+\psi)\partial_t^i\psi\,\partial_t^i\psi_t + h'(\tilde\rho+\psi)(\tilde u+\eta)\partial_t^i\psi_x\partial_t^i\psi - h'(\tilde\rho+\psi)G_i\partial_t^i\psi\,dx\\
&\quad + \int_0^1 -\partial_t^i\omega\partial_t^i\omega_{xxt} - (\tilde u+\eta)\partial_t^i\omega\partial_t^i\psi_x + G_i\partial_t^i\omega\,dx\\
&= \frac{d}{dt}\int_0^1 \frac12 h'(\tilde\rho+\psi)(\partial_t^i\psi)^2\,dx - \int_0^1 \frac12 h''(\tilde\rho+\psi)\psi_t(\partial_t^i\psi)^2\,dx\\
&\quad + \int_0^1 -\frac12\{h'(\tilde\rho+\psi)(\tilde u+\eta)\}_x(\partial_t^i\psi)^2 - h'(\tilde\rho+\psi)G_i\partial_t^i\psi\,dx\\
&\quad + \frac{d}{dt}\int_0^1 \frac12(\partial_t^i\omega_x)^2\,dx + \int_0^1 \{(\tilde u+\eta)\partial_t^i\omega\}_x\partial_t^i\psi + G_i\partial_t^i\omega\,dx,
\end{aligned}
\tag{3.39}
$$

where we have also used the equation (3.24a) again. Substitute the equalities (3.36), (3.37), (3.38) and (3.39) in the equality (3.35) and integrate the resultant equality over $(0,t)$ to obtain that

(3.40a)

$$
I_2^{(i)}(t) + \int_0^t\int_0^1 (\tilde\rho+\psi)(\partial_t^i\eta)^2\,dxd\tau = I_2^{(i)}(0) + \int_0^t J_2^{(i)}(\tau)\,d\tau,
$$

(3.40b)

$$
I_2^{(i)}(t) := \int_0^1 \frac12(\tilde\rho+\psi)(\partial_t^i\eta)^2 + \frac12 h'(\tilde\rho+\psi)(\partial_t^i\psi)^2 + \frac12(\partial_t^i\omega_x)^2 - (\tilde u+\eta)\partial_t^i\psi\,\partial_t^i\eta\,dx,
$$

(3.40c)

$$
\begin{aligned}
J_2^{(i)}(t) := &\int_0^1 \frac12\psi_t(\partial_t^i\eta)^2 + \frac12 h''(\tilde\rho+\psi)\psi_t(\partial_t^i\psi)^2\,dx\\
&+ \int_0^1 G_i h'(\tilde\rho+\psi)\partial_t^i\psi - \{(\tilde u+\eta)\partial_t^i\omega\}_x\partial_t^i\psi - G_i\partial_t^i\omega\,dx\\
&+ \int_0^1 (\tilde u+\eta)(\tilde\rho+\psi)_x(\partial_t^i\eta)^2 - (\tilde u+\eta)G_i\partial_t^i\eta - \eta_t\partial_t^i\eta\partial_t^i\psi + (\tilde\rho+\psi)F_i\partial_t^i\eta\,dx\\
&+ \int_0^1 \{h'(\tilde\rho+\psi)(\tilde\rho+\psi)_x\partial_t^i\psi - 2(\tilde u+\eta)_x\partial_t^i\eta\}(\tilde u+\eta)\partial_t^i\psi\,dx\\
&+ \int_0^1 (\tilde u+\eta)\{\partial_t^i\eta - \partial_t^i\omega_x - F_i\}\partial_t^i\psi\,dx.
\end{aligned}
$$

The 4th term in $I_2^{(i)}(t)$ is estimated, with the aid of the estimates (2.14) and (2.15) as

$$\left|\int_0^1 \{-(\tilde{u}+\eta)\partial_t^i\psi\,\partial_t^i\eta\}(t,x)\,dx\right| \le C(N(T)+\delta)\|(\partial_t^i\psi,\,\partial_t^i\eta)(t)\|^2. \tag{3.41}$$

Moreover $I_2^{(i)}(0)$ and $J_2^{(i)}(t)$ are estimated similarly as the estimation of $I_1^{(i)}(0)$ and $J_1^{(i)}(t)$. Thus the estimates

$$|I_2^{(i)}(0)| \le CA_i^2(0), \quad |J_2^{(i)}(t)| \le C\{(N(T)+\delta)A_i^2(t)+A_{i-1}^2(t)\} \tag{3.42}$$

hold. Finally substituting (3.41) and (3.42) in (3.40a) gives the inequality

(3.43)
$$\begin{aligned}
&\frac{1}{2}\int_0^1 \{(\tilde{\rho}+\psi)(\partial_t^i\eta)^2+h'(\tilde{\rho}+\psi)(\partial_t^i\psi)^2+(\partial_t^i\omega_x)^2\}(t,x)\,dx - C(N(T)+\delta)\|(\partial_t^i\psi,\,\partial_t^i\eta)(t)\|^2\\
&+\int_0^t\int_0^1 (\tilde{\rho}+\psi)(\partial_t^i\eta)^2\,dx d\tau\\
&\le C\int_0^t A_{i-1}^2(\tau)\,d\tau + C\left(A_i^2(0)+(N(T)+\delta)\int_0^t A_i^2(\tau)\,d\tau\right).
\end{aligned}$$

Multiplying (3.43) by 2, adding the resulting inequality to (3.34) and then letting both $N(T)+\delta$ and ϵ be small enough, we arrive at the desired estimate (3.25). □

Using the estimate (3.25) thus obtained, we complete the proof of Proposition 3.2.

Proof of Proposition 3.2. Using the smallness $N(T)+\delta$, the equations (3.3a) and (3.3b), we have the estimate

$$c\|(\psi,\,\eta)(t)\|_i^2 \le A_i^2(t) \le C\|(\psi,\,\eta)(t)\|_i^2 \quad \text{for} \quad i=1,\,2. \tag{3.44}$$

Moreover, the estimate

$$\|\omega(t)\|_4 \le C\|\psi(t)\|_2 \tag{3.45}$$

holds from (3.3c). Hence we obtain the a priori estimate (3.7) by combining (3.18) with (3.25) and using the smallness of $N(T)+\delta$ again. □

3.4. Decay estimate. Since the existence of the time global solution to the problem (1.1), (1.4), (1.5) and (1.6) is proved owing to the continuation argument on Corollary 3.1 and Proposition 3.2, it is sufficient to show decay estimate (1.22) in order to complete of the proof of Theorem 1.2.

Proof of Theorem 1.2. Multiply (3.20a) by β, (3.30a) with $i=1$ by β^2, (3.40a) with $i=1$ by $2\beta^2$, (3.30a) with $i=2$ by β^3, (3.40a) with $i=2$ by $2\beta^3$, where β

is a positive constant to the determined. Then sum up (3.19a) and these results to obtain that

$$\tilde{E}(t) + \int_0^t \tilde{F}(\tau)\,d\tau = \tilde{E}(0) \quad \text{for} \quad t \in [0,\infty), \tag{3.46a}$$

$$\tilde{E}(t) := \int_0^1 \mathcal{E} - \beta\{\eta\omega_x\}\,dx + \beta^2\big(I_1^{(1)}(t) + 2I_2^{(1)}(t)\big) + \beta^3\big(I_1^{(2)}(t) + 2I_2^{(2)}(t)\big), \tag{3.46b}$$

(3.46c)

$$\begin{aligned}\tilde{F}(t) := &\int_0^1 \tilde{\rho}\eta^2 + \beta\{(h(\xi) - h(\tilde{\rho}))\psi + (\omega_x)^2\}\,dx \\ &+ \sum_{i=1}^{2} \beta^{i+1} \int_0^1 (\tilde{\rho} + \psi)(\partial_t^i \eta)^2 + h'(\tilde{\rho} + \psi)(\partial_t^i \psi)^2 + (\partial_t^i \omega_x)^2\,dx \\ &- \int_0^1 R_2 + \beta\left(\eta\omega_x - \eta\omega_{xt} + \frac{1}{2}(2\tilde{u}\eta + \eta^2)_x\omega_x\right)dx - \sum_{i=1}^{2} \beta^{i+1}(J_1^{(i)} + 2J_2^{(i)})(t).\end{aligned}$$

Now we take β and $N(T) + \delta$ sufficiently small in this order so that $0 < N(T) + \delta \ll \beta^3 \ll \beta^2 \ll \beta \ll 1$. This procedure yields that both quantities $\tilde{E}(t)$ and $\tilde{F}(t)$ are equivalent to $\|(\psi, \eta, \psi_t, \eta_t, \psi_{tt}, \eta_{tt})(t)\|^2$. Hence $\tilde{E}(t)$ and $\tilde{F}(t)$ are also equivalent to $\|(\psi, \eta)(t)\|_2^2$ due to (3.44). These facts are confirmed by applying the Schwarz and the Sobolev inequalities as well as the estimates (2.14) (2.15), (3.16), (3.23) and (3.26).

Since $\tilde{E}(t)$ and $\tilde{F}(t)$ are equivalent, there exists a certain positive constant α such that $\alpha\tilde{E}(t) \le \tilde{F}(t)$. Then differentiate (3.46a) and substitute this inequality in the resultant inequality to obtain the ordinary differential inequality

$$\frac{d}{dt}\tilde{E}(t) + \alpha\tilde{E}(t) \le 0 \quad \text{for} \quad t \in [0,\infty). \tag{3.47}$$

As the quantity $\tilde{E}(t)$ is also equivalent to $\|(\psi, \eta)(t)\|_2^2$, solving (3.47) yields that

$$c\|(\psi, \eta)(t)\|_2^2 \le \tilde{E}(t) \le \tilde{E}(0)e^{-\alpha t} \le C\|(\psi, \eta)(0)\|_2^2 e^{-\alpha t}, \tag{3.48}$$

where c and C are positive constants independent of t. The inequality (3.48) and the elliptic estimate (3.45) yield the decay estimate (1.22). Consequently, the proof of Theorem 1.2 is completed. □

4. Appendix

In this section we discussed the solvability of the linearized problem (2.19), (1.12), (1.4) and (1.5). For this purpose, we firstly study the system of equations

$$A^0\begin{pmatrix} v \\ w \end{pmatrix}_t + A^1\begin{pmatrix} v \\ w \end{pmatrix}_x + B\begin{pmatrix} v \\ w \end{pmatrix} = F, \tag{4.1a}$$

$$(4.1b)\qquad A^0 = \begin{pmatrix} p'(\rho) - \dfrac{j^2}{\rho^2} & 0 \\ 0 & 1 \end{pmatrix}, \quad A^1 = \begin{pmatrix} 0 & -\left(p'(\rho) - \dfrac{j^2}{\rho^2}\right) \\ -\left(p'(\rho) - \dfrac{j^2}{\rho^2}\right) & \dfrac{2j}{\rho} \end{pmatrix},$$

$$(4.1c)\qquad B = \begin{pmatrix} 0 & 0 \\ -\left(p'(\rho) - \dfrac{j^2}{\rho^2}\right)_x & \left(\dfrac{2j}{\rho}\right)_x \end{pmatrix}, \quad F = \begin{pmatrix} 0 \\ (-\phi_x\rho + j)_x \end{pmatrix}$$

with the initial and the boundary data

$$(4.2)\qquad v(0, x) = (\rho_0)_x(x), \quad w(0, x) = -(j_0)_x(x),$$

$$(4.3)\qquad w(t, 0) = w(t, 1) = 0.$$

Here, notice that the matrices A^0 and A^1 are symmetric.

The above initial boundary problem is derived from (2.19) as follows. Differentiating (2.19b) with respect to x and using the equation (2.19a), we see that if $(\hat\rho, \hat j) \in \mathfrak{X}_2([0, T])$ is a solution to the initial boundary value problem (2.19), (1.12), (1.4) and (1.5), and then $(v, w) = (\hat\rho_x, \hat\rho_t) \in \mathfrak{X}_1([0, T])$ is a solution to the initial boundary value problem (4.1), (4.2) and (4.3). So, we consider the existence of solution (v, w) to (4.1), (4.2) and (4.3). After that, we construct the solution $(\hat\rho, \hat j)$ to (2.19), (1.12), (1.4) and (1.5) from the (v, w).

In order to apply Theorem-A1 in [16] to the symmetric liner problem (4.1), (4.2) and (4.3), we use approximation sequences $\{B_i\}_{i=0}^\infty \subset C^2([0, T]; H^2(\Omega))$ such that B_i converges to B strongly in $\mathfrak{X}_1([0, T])$ as i tends to infinity. Similarly take $\{F_i\}_{i=0}^\infty \subset C^1([0, T]; H^1(\Omega))$ such that $F_i \to F$ strongly in $C^1([0, T]; L^2(\Omega))$. In addition, we define a successive approximation sequence $\{(v^i, w^i)\}_{i=0}^\infty$ by solving

$$(4.4)\qquad A^0\begin{pmatrix} v^i \\ w^i \end{pmatrix}_t + A^1\begin{pmatrix} v^i \\ w^i \end{pmatrix}_x + B_i\begin{pmatrix} v^i \\ w^i \end{pmatrix} = F_i$$

with the initial data (4.2) and the boundary data (4.3). The solvability of this symmetric liner problem (4.4), (4.2), (4.3) is ensured by Theorem-A1 in [16]. For the system (4.1), the following estimate holds from the energy method.

$$\|(v^i, w^i)(t)\|_1 + \|(v^i_t, w^i_t)(t)\| \le C \quad \text{for} \quad t \in [0, T],$$

where C is a positive constant, independent of $i = 0, 1, \ldots$. Similarly, by applying the Energy method on the equations for $(v^i - v^j, w^i - w^j)$ together with the above estimate, we see that $\{(v^i, w^i)\}_0^\infty$ is the Cauchy sequence in $\mathfrak{X}_1([0, T])$. Hence, there exists a certain function $(v, w) \in \mathfrak{X}_1([0, T])$ such that $(v^i, w^i) \to (v, w)$ strongly in $\mathfrak{X}_1([0, T])$ as $i \to \infty$. Moreover, we see from the standard argument that (v, w) is a unique solution to the initial boundary value problem (4.1), (4.2), (4.3).

Next, we proceed to construct the solution $(\hat{\rho}, \hat{j})$ to the initial boundary value problem (2.19), (1.12), (1.4) and (1.5). For this purpose, define $(\hat{\rho}, \hat{j})$ by

$$\hat{\rho}(t, x) := \int_0^x v(t, x)\, dx + \rho_l, \tag{4.5a}$$

$$\hat{j}(t, 0) := \int_0^t \left\{ -\left(p'(\rho) - \frac{j^2}{\rho^2} \right) v + \frac{2j}{\rho} w + \phi_x \rho - j \right\}(t, 0)\, dt + j_0(0), \tag{4.5b}$$

$$\hat{j}(t, x) := \int_0^x -w(t, x)\, dx + \hat{j}(t, 0). \tag{4.5c}$$

We show that $(\hat{\rho}, \hat{j}) \in \mathfrak{X}_2([0, T])$ is a desired solution to the linearized problem (2.19), (1.12), (1.4) and (1.5). Apparently, the equalities $\hat{\rho}_x = v$ and $\hat{\rho}_t = \int_0^x v_t\, dx = \int_0^x w_x\, dx = w = -\hat{j}_x$ hold from (4.1), (4.5) and (4.3). In addition, differentiating (4.5c) with respect to t and using (4.1) and (4.5b), we have the equality

$$\begin{aligned}
\hat{j}_t(t, x) &= \int_0^x -w_t(t, x)\, dx + \hat{j}_t(t, 0) \\
&= \int_0^x \left\{ -\left(p'(\rho) - \frac{j^2}{\rho^2} \right) v + \frac{2j}{\rho} w + \phi_x \rho - j \right\}_x (t, x)\, dx \\
&\quad + \left\{ -\left(p'(\rho) - \frac{j^2}{\rho^2} \right) v + \frac{2j}{\rho} w + \phi_x \rho - j \right\}(t, 0) \\
&= \left\{ -\left(p'(\rho) - \frac{j^2}{\rho^2} \right) v + \frac{2j}{\rho} w + \phi_x \rho - j \right\}(t, x) \\
&= \left\{ -\left(p'(\rho) - \frac{j^2}{\rho^2} \right) \hat{\rho}_x - \frac{2j}{\rho} \hat{j}_x + \phi_x \rho - j \right\}(t, x),
\end{aligned}$$

where we have also used $w = -j_x$ and $v = \hat{\rho}_x$. Thus, $(\hat{\rho}, \hat{j})$ satisfies the equation (2.19). Next, we confirm that $(\hat{\rho}, \hat{j})$ satisfies initial condition (1.4). Actually, the equalities $\hat{\rho}(0, x) = \int_0^x \rho_{0x}(x)\, dx + \rho_l = \rho_0(x)$ and $\hat{j}(0, x) = \int_0^x j_{0x}\, dx + j_0(0) = j_0(x)$ holds from (4.5), (4.2) and the compatibility condition (1.7). Moreover, the boundary condition (1.5) holds, i.e., $\hat{\rho}(t, 0) = \rho_l$ and $\hat{\rho}(t, 1) = \rho_r$, due to (4.5a), $\hat{\rho}_t(t, 1) = w(t, 1) = 0$ and $\hat{\rho}(0, 1) = \rho_0(1) = \rho_r$. Consequently, $(\hat{\rho}, \hat{j})$ is the solution to the linearized problem (2.19), (1.12), (1.4) and (1.5).

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