Kamada, S.
Osaka J. Math.
44 (2007), 441–458

BRAID PRESENTATION OF VIRTUAL KNOTS AND WELDED KNOTS

SEIICHI KAMADA

(Received October 14, 2005, revised June 23, 2006)

Abstract

Virtual knot theory, introduced by L. Kauffman, is a generalization of classical knot theory. It naturally yields the notion of a virtual braid, which is closely related to the notion of a welded braid due to R. Fenn, R. Rimányi and C. Rourke. In this paper we prove that any virtual link or welded link can be described as the closure of a virtual braid or welded braid, respectively, which is unique up to certain basic moves. This is analogous to the Alexander and Markov theorems for classical braids and links.

1. Introduction

The theory of a virtual knot was introduced by L. Kauffman as a generalization of classical knot theory (cf. [14], [15]). It is related to quandles/biquandles and their homology groups (cf. [5], [6], [18]). It naturally yields the notion of a virtual braid, defined in §2 (cf. [14], [15], [16]). The virtual braid group contains the braid group in a natural way. This group is closely related to the welded braid group introduced by R. Fenn, R. Rimányi and C. Rourke [7]. In this paper we prove that any virtual link or welded link can be described as the closure of a virtual braid or welded braid, respectively, which is unique up to certain basic moves. This is analogous to the Alexander and Markov theorems for classical braids and links.

The Alexander theorem states that any link is described as the closure of a braid, and the Markov theorem states that such a braid presentation is unique up to conjugations and stabilizations (cf. [1], [19], [20], [22], [26], [27], [28], etc.). The Alexander theorem for virtual links (Proposition 3) and for welded links (Proposition 8) are easily obtained by observing a relationship between virtual links and Gauss code diagrams given in [10] and [14] . In his talk at the AMS Meeting, Washington D.C. in January 2000, Kauffman asked whether there is a result analogous to the Markov theorem for virtual links. The following theorem answers the question and ensures a relationship between virtual braids and virtual links.

Theorem 1. *Two virtual braid diagrams* (*or two virtual braids*, *respectively*) *have equivalent closures as virtual link if and only if they are related to each other by a*

2000 Mathematics Subject Classification. Primary 57M25; Secondary 20F36.

finite sequence of the following VM0-, VM1-, VM2- *and* VM3-*moves* (*or* VM1-, VM2- *and* VM3-*moves*, *resp*.):

- (VM0-move) *a virtual braid move*,
- (VM1-move) *a conjugation in the virtual braid group*,
- (VM2-move) *a right stabilization of positive*, *negative or virtual type*, *and its inverse operation*,
- (VM3-move) *a right/left virtual exchange move*.

The VM0-, VM1- and VM2-moves are analogous to the Markov moves for classical braids. The VM3-moves are analogous to exchange moves (cf. [2], [3]). It is remarkable that VM3-moves are not consequences of VM0-, VM1- and VM2-moves [12], whereas exchange moves for classical braids are consequences of Markov moves. We also note that left stabilizations of positive/negative type for virtual braids are not consequences of VM0-, VM1- and VM2-moves [12], whereas left stabilizations of positive/negative for classical braids are consequences of Markov moves.

For welded braids and links, we have an analogous result as follows:

Theorem 2. *Two welded braid diagrams* (*or welded braids*, *respectively*) *have equivalent closures as welded link if and only if they are related by a finite sequence of the following* WM0-, WM1- *and* WM2-*moves* (*or* WM1- *and* WM2-*moves*, *resp*.):

- (WM0-move) *a welded braid move*,
- (WM1-move) *a conjugation in the welded braid group*,
- (WM2-move) *a right stabilization of positive*, *negative or welded type*, *and its inverse operation*.

The original version [11] of this paper was archived in 2000, and was not published since virtual knot theory was not popular yet. However these days the author has been asked by a lot of researchers about the paper, and he decided to submit it for publication here. Note that this current paper is shorter that the original one [11] because Section 6 of [11], concerned with virtual braid invariants, was separated as [12] in order to be discussed in more general situation. It is also updated. Recently, L. Kauffman and S. Lambropoulou discovered an alternative approach to the Alexander and Markov theorems for virtual links using 'L-moves' [17].

2. Virtual braids and welded braids

Let m be a positive integer and Q_m a set of m interior points of the interval $[0, 1]$. We denote by E the 2-disk $[0, 1] \times [0, 1]$ and by $p_2 \colon E \to [0, 1]$ the second factor projection. A *virtual braid diagram of degree* m is an immersed 1-manifold $b = a_1 \cup \cdots \cup a_m$ in E such that

1. $\partial b = Q_m \times \{0, 1\} \subset E$,
2. for each $i \in \{1, \ldots, m\}$, $p_2|_{a_i} \colon a_i \to [0, 1]$ is a homeomorphism,

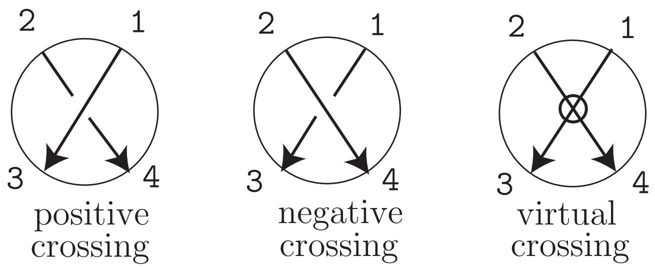

Fig. 1. Crossings

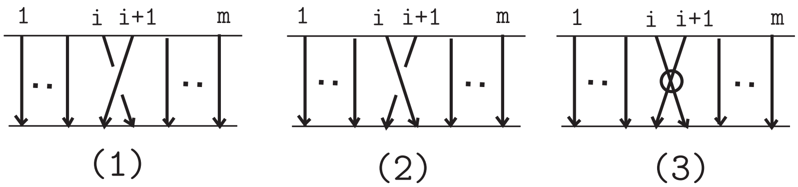

Fig. 2. Standard generators

3. the multiple point set $V(b)$ consists of transverse double points,
4. $p_2|_{V(b)}\colon V(b) \to [0,1]$ is injective,
5. each point of $V(b)$ is assigned information of *positive*, *negative* or *virtual crossing* as in Fig. 1. (The labels $1,\dots,4$ in the figure are used later. Ignore them at this moment.)

The arcs $a_1,\dots,a_m$ are assumed to be oriented from the top ($[0,1]\times\{1\}$) to the bottom ($[0,1]\times\{0\}$) of E. Two virtual braid diagrams are identified if one is transformed to the other continuously keeping the above conditions. The set of virtual braid diagrams of degree m, with the concatenation product, forms a monoid generated by $\sigma_i, \sigma_i^{-1}, \tau_i$ ($i = 1,\dots,m-1$) illustrated in Fig. 2. The identity element is $Q_m \times [0,1] \subset E$.

DEFINITION (cf. [14], [15], [16], [17]). The *virtual braid group* VB_m *of degree* m is the group obtained from the monoid of virtual braid diagrams of degree m by introducing the following relations:

$$\begin{array}{ll}
\text{(Trivial relations)} & \sigma_i\sigma_i^{-1} = \sigma_i^{-1}\sigma_i = 1 \\
\text{(Braid relations)} & \begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j|>1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \end{cases} \\
\text{(Permutation group relations)} & \begin{cases} \tau_i^2 = 1 \\ \tau_i\tau_j = \tau_j\tau_i, & |i-j|>1 \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} \end{cases} \\
\text{(Mixed relations)} & \begin{cases} \sigma_i\tau_j = \tau_j\sigma_i, & |i-j|>1 \\ \sigma_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\sigma_{i+1}. \end{cases}
\end{array}$$

A *virtual braid of degree* m is an element of VB_m.

DEFINITION ([7]). The *welded braid group* WB_m is the group that is obtained from VB_m by introducing additional relations $\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}$ $(i = 1, \dots, m-2)$. A *welded braid diagram* is a diagram representing an element of this group.

REMARK. There is a canonical epimorphism $VB_m \to WB_m$. Fenn, Rimányi and Rourke [7] proved that the welded braid group WB_m is isomorphic to the braid-permutation group BP_m. By an argument in [7], we see that the subgroup of VB_m generated by σ_i $(i = 1, \dots, m)$ is isomorphic to the braid group B_m and the subgroup generated by τ_i $(i = 1, \dots, m)$ is isomorphic to the symmetric group S_m.

3. Braid presentation of virtual links

A *virtual link diagram* is a closed oriented 1-manifold K immersed in $\mathbf{R}^2$ such that the multiple point set $V(K)$ consists of transverse double points each of which has information of positive, negative or virtual crossing as in Fig. 1. Positive and negative crossings are called *real crossings*. The set of real crossings will be denoted by $V_R(K)$. We assume that virtual link diagrams are the same if they are isotopic in $\mathbf{R}^2$. *Virtual Reidemeister moves* are the local moves illustrated in Fig. 3. (The moves indicated by (b) are consequences of the moves indicated by (a) and R2-moves or V2-moves.) Two virtual link diagrams are *equivalent* if they are related by a finite sequence of virtual Reidemeister moves. A *virtual link* or a *virtual link type* is the equivalence class of a virtual link diagram, [10], [14], [15].

The *closure* of a virtual braid diagram (or a virtual link) is defined in the standard way in knot theory (Fig. 4). The following proposition is well-known. We shall give a proof in §4.

Proposition 3. *Any virtual link can be described as the closure of a virtual braid.*

When virtual braid diagrams b_1 and b_2 represent the same virtual braid, we say that b_2 is obtained from b_1 by a *virtual braid move* or a VM0-*move*.

For virtual braid diagrams b_1 and b_2 of the same degree, we say that the virtual braid diagram b_1b_2 is obtained from b_2b_1 by a *conjugation* or a VM1-*move*.

For a virtual braid diagram b of degree m, we denote by $\iota_s^t(b)$ the virtual braid diagram of degree $m+s+t$ obtained from b by adding s trivial arcs to the left of b and t trivial arcs to the right. (This defines a monomorphism $\iota_s^t \colon VB_m \to VB_{m+s+t}$.)

For a virtual braid diagram b of degree m, a *right stabilization* of *positive*, *negative* or *virtual type* is the replacement of b by the virtual braid diagram $\iota_0^1(b)\sigma_m$, $\iota_0^1(b)\sigma_m^{-1}$ or $\iota_0^1(b)\tau_m$, respectively, of degree $m+1$. See Fig. 5. This operation and the inverse operation are called VM2-*moves*.

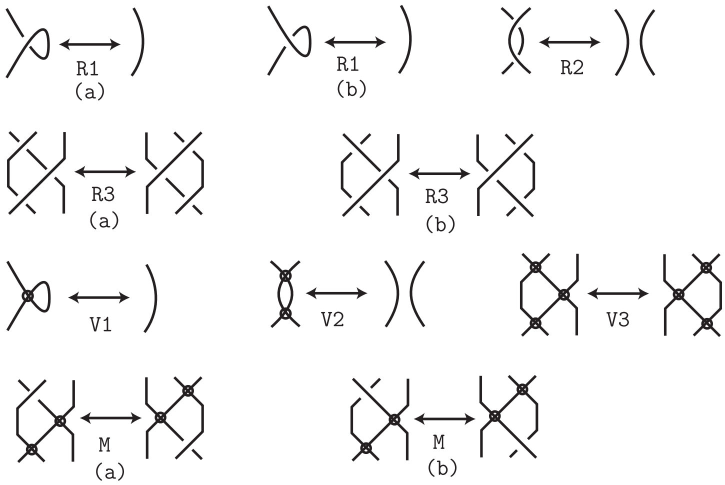

Fig. 3. Virtual Reidemeister moves

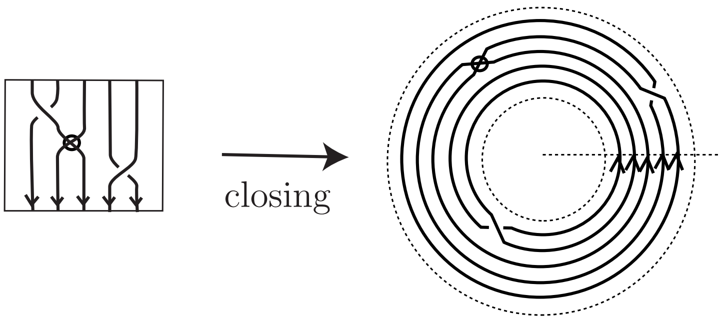

Fig. 4. Closure

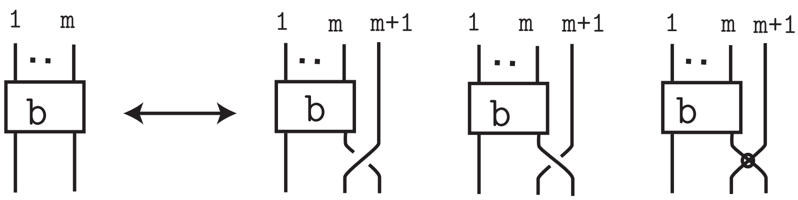

Fig. 5. Right stabilizations

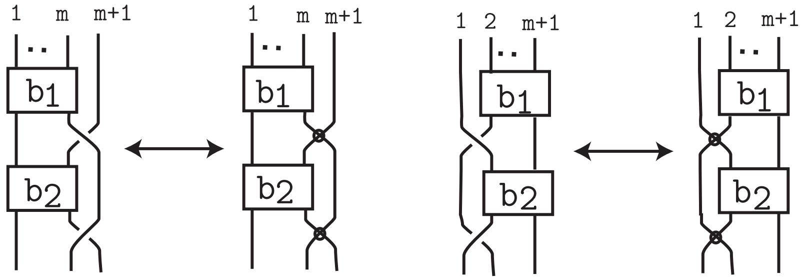

Fig. 6. Right/left virtual exchange moves

Similarly, a *left stabilization* is the replacement of b by $\iota_1^0(b)\sigma_1$, $\iota_1^0(b)\sigma_1^{-1}$ or $\iota_1^0(b)\tau_1$. (A left stabilization will be used in §6. Note that we do not call a left stabilization a VM2-move in this paper.)

A *right virtual exchange move* is the replacement

$$\iota_0^1(b_1)\sigma_m^{-1}\iota_0^1(b_2)\sigma_m \leftrightarrow \iota_0^1(b_1)\tau_m\iota_0^1(b_2)\tau_m$$

and a *left virtual exchange move* is a replacement

$$\iota_1^0(b_1)\sigma_1^{-1}\iota_1^0(b_2)\sigma_1 \leftrightarrow \iota_1^0(b_1)\tau_1\iota_1^0(b_2)\tau_1$$

where b_1 and b_2 are virtual braid diagrams of degree m. See Fig. 6. These moves are called VM3-*moves*.

4. Braiding process

For a virtual link diagram K, we denote by $S(K)\colon V_R(K) \to \{+1, -1\}$ the map assigning the real crossings their signs. For a real crossing $v \in V_R(K)$, let $N(v)$ be a regular neighborhood of v as in Fig. 1. We denote by $v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}$ the four points of $\partial N(v) \cap K$ ordered as in the figure. Put $W = W(K) = \mathrm{Cl}\big(\mathbf{R}^2 - \bigcup_{v\in V_R(K)} N(v)\big)$ and $V_R^\partial(K) = \{v^{(j)} \mid v \in V_R(K),\ j \in \{1,2,3,4\}\}$, where Cl means the closure. The restriction of K to W is denoted by $K|_W$, which is the union of some oriented arcs and loops immersed in W such that the multiple points are virtual crossings of K and that the boundary of the arcs is equal to the set $V_R^\partial(K)$.

Define a subset $G(K) \subset V_R^\partial(K) \times V_R^\partial(K)$ such that $(a, b) \in G(K)$ if and only if $K|_W$ has an oriented arc starting from a and terminating at b. We denote by $\mu(K)$ the number of components of K. For example, for a virtual link diagram illustrated in Fig. 7,

$V_R(K) = \{v_1, v_2, v_3\}$,

$S(K)\colon v_1 \mapsto +1, \quad v_2 \mapsto +1, \quad v_3 \mapsto -1,$

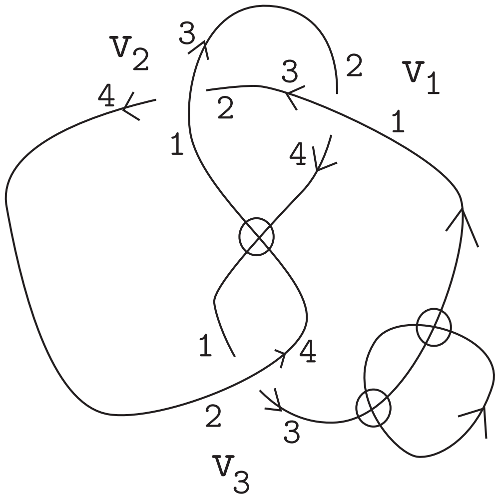

Fig. 7. A virtual link diagram

$G(K) = \{(v_3^{(3)}, v_1^{(1)}), (v_1^{(3)}, v_2^{(2)}), (v_2^{(4)}, v_3^{(2)}), (v_3^{(4)}, v_2^{(1)}), (v_2^{(3)}, v_1^{(2)}), (v_1^{(4)}, v_3^{(1)})\}$,

$\mu(K) = 2$.

The *Gauss data* of K is the quadruple $(V_R(K), S(K), G(K), \mu(K))$. We say that two virtual link diagrams K and K' have the *same Gauss data* if $\mu(K) = \mu(K')$ and if there is a bijection $g\colon V_R(K) \to V_R(K')$ such that g preserves the signs of the crossing points and that $(a, b) \in G(K)$ implies $(g(a), g(b)) \in G(K')$, where $g\colon V_R^\partial(K) \to V_R^\partial(K')$ is the bijection induced from $g\colon V_R(K) \to V_R(K')$. This condition is equivalent to the condition that K and K' have the same Gauss diagram in the sense of [10] or the same Gauss code in the sense of [14].

Let K be a virtual link diagram and let $W = W(K) = \mathrm{Cl}\big(\mathbf{R}^2 - \bigcup_{v \in V_R(K)} N(v)\big)$ be as before. Suppose that K' is a virtual link diagram with the same Gauss data as K. Then we can deform K' by an isotopy of $\mathbf{R}^2$ such that

1. K and K' are identical in $N(v)$ for every $v \in V_R(K)$,
2. K' has no real crossings in W, and
3. there is a one-to-one correspondence between the arcs/loops of $K|_W$ and those of $K'|_W$ satisfying a condition that each arc of $K|_W$ and the corresponding arc of $K'|_W$ have the same endpoints.

In this situation, we say that K' is obtained from K by *replacing* $K|_W$.

Lemma 4 ([10], [14]). *If two virtual link diagrams K and K' have the same Gauss data, then K is equivalent to K'. Moreover, we can transform K to K', up to isotopy of $\mathbf{R}^2$, by a finite sequence of* V1-, V2-, V3- *and* M-*moves*.

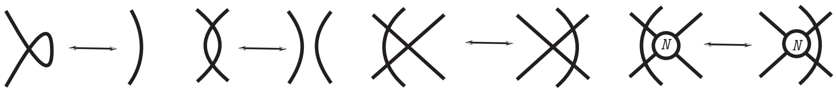

Fig. 8. Moves on immersed curves

Proof. Since K and K' have the same Gauss data, without loss of generality we may assume that K' is obtained from K by replacing $K|_W$. Let $a_1, \dots, a_s$ be the arcs/loops of $K|_W$, and let $a'_1, \dots, a'_s$ be the corresponding arcs/loops of $K'|_W$. We may assume that a'_1 intersects $a_2, \dots, a_s$ transversely. The arc or loop a_1 is homotopic to a'_1 in $\mathbf{R}^2$ (relative to the boundary of a_1 if a_1 is an arc). Taking the homotopy generically with respect to the arcs/loops $a_2, \dots, a_s$ and the 2-disks $N(v)$ ($v \in V_R(K)$), we see that the arc/loop a_1 is transformed to a'_1 by a finite sequence of moves as in Fig. 8 up to isotopy of $\mathbf{R}^2$, where N means $N(v)$ for $v \in V_R(K)$. Each move is a V1-, V2-, V3-, or M-move. Inductively, every a_i is transformed to a'_i by such moves. $\square$

Let O be the origin of $\mathbf{R}^2$. Identify $\mathbf{R}^2 - \{O\}$ with $\mathbf{R}_+ \times S^1$ by polar coordinates and let $\pi\colon \mathbf{R}^2 - \{O\} = \mathbf{R}_+ \times S^1 \to S^1$ be the projection, where $\mathbf{R}_+$ is the half-line consisting of positive numbers and we assume that S^1 is oriented counterclockwise. A *braided virtual link diagram* (of degree m) is a virtual link diagram K such that
(i) it is contained in $\mathbf{R}^2 - \{O\}$,
(ii) for the underlying immersion $k\colon \bigsqcup S^1 \to \mathbf{R}^2 - \{O\}$ of K, the composition $\pi \circ k\colon \bigsqcup S^1 \to S^1$ is an orientation preserving covering map of degree m (where $\bigsqcup S^1$ is the disjoint union of $\mu(K)$ circles), and
(iii) $\pi|_{V(K)}\colon V(K) \to S^1$ is injective.
A point θ of S^1 is called a *regular value* if $V(K) \cap \pi^{-1}(\theta) = \emptyset$. Cutting K along the half-line $\pi^{-1}(\theta)$ for a regular value θ, we obtain a virtual braid diagram whose closure is K. Such a virtual braid diagram is uniquely determined up to conjugation (VM1-move).

Proof of Proposition 3 (Braiding Process). Let K be a virtual link diagram and let $N_1, \dots, N_n$ be regular neighborhoods of the real crossings of K. By an isotopy of $\mathbf{R}^2$, we may assume that all N_i ($i = 1, \dots, n$) are in $\mathbf{R}^2 - \{O\}$, $\pi(N_i) \cap \pi(N_j) = \emptyset$ for $i \neq j$ and that the restriction of K to each N_i consists of two oriented arcs each of which is mapped into S^1 by π homeomorphically with respect to the orientation of S^1. Replace the remainder $K|_{W(K)}$ arbitrarily such that the result is a braided virtual link diagram. By Lemma 4, K is equivalent to this diagram. $\square$

5. Proof of Theorem 1

The terminologies 'virtual braid moves', 'right stabilizations' and 'right/left virtual exchange moves' defined in §3 are also used for braided virtual link diagrams. These moves and their inverse moves are also called VM0-, VM2- and VM3-moves,

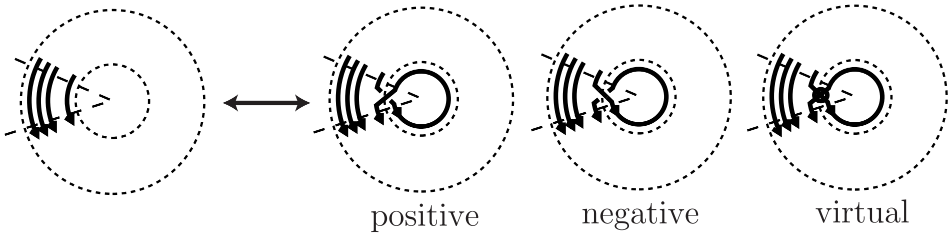

Fig. 9. Right stabilizations (VM2-moves)

respectively. For example, the moves illustrated in Fig. 9 are right stabilizations (VM2-moves) for braided virtual link diagrams. If two braided virtual link diagrams are related by a finite sequence of VM0- and VM2-moves, then we say that they are *virtually Markov equivalent in the strict sense*. If they are related by a finite sequence of VM0-, VM2- and VM3-moves, then we say that they are *virtually Markov equivalent*.

Lemma 5. *Let K and K' be braided virtual link diagrams* (*possibly of distinct degrees*) *such that K' is obtained from K by replacing $K|_{W(K)}$. Then K and K' are virtually Markov equivalent in the strict sense.*

Proof. Let $N_1, \ldots, N_n$ be regular neighborhoods of the real crossings of K (and hence of K') with $W = W(K) = \mathrm{Cl}\big(\mathbf{R}^2 - \bigcup_{i=1}^n N_i\big)$. Taking $N_1, \ldots, N_n$ to be smaller, without loss of generality we may assume that $\pi(N_i) \cap \pi(N_j) = \emptyset$ for $i \neq j$ and hence $\pi\big(\bigcup_{i=1}^n N_i\big) \neq S^1$. Let $a_1, \ldots, a_s$ be the arcs/loops of $K|_W$ and let $a'_1, \ldots, a'_s$ be the corresponding arcs/loops of $K'|_W$. Take a common regular value $\theta_0 \in S^1$ for K and K' such that θ_0 is not in $\pi\big(\bigcup_{i=1}^n N_i\big)$. If there exists an arc/loop a_i of $K|_W$ and the corresponding one a'_i of $K'|_W$ such that $\#(a_i \cap \pi^{-1}(\theta_0)) \neq \#(a'_i \cap \pi^{-1}(\theta_0))$, then move a small segment of a_i or a'_i toward the origin by a series of VM0-moves corresponding to $\tau_i^2 = 1$ and apply some VM2-moves of virtual type so that $\#(a_i \cap \pi^{-1}(\theta_0)) = \#(a'_i \cap \pi^{-1}(\theta_0))$. Thus we may assume that $\#(a_i \cap \pi^{-1}(\theta_0)) = \#(a'_i \cap \pi^{-1}(\theta_0))$ for $i = 1, \ldots, s$. Let k and k' be underlying immersions $\bigsqcup S^1 \to \mathbf{R}^2 - \{O\}$ of K and K' such that they are identical near the preimages of the real crossings. Let $I_1, \ldots, I_s$ be intervals or circles in $\bigsqcup S^1$ with $k(I_i) = a_i$ for $i = 1, \ldots, s$, and put $k_i = k|_{I_i}$. Let $k'_1, \ldots, k'_s$ be such immersions for K'. Note that $\pi \circ k_i \colon I_i \to S^1$ and $\pi \circ k'_i \colon I_i \to S^1$ are orientation preserving immersions and $\pi \circ k_i|_{\partial I_i} = \pi \circ k'_i|_{\partial I_i}$. Since a_i and a'_i have the same degree with respect to θ_0, there exists a homotopy $\{k_i^t \colon I_i \to \mathbf{R}^2 - \{O\}\}_{t\in[0,1]}$ in $\mathbf{R}^2 - \{O\}$ between $k_i = k_i^0$ and $k'_i = k_i^1$ relative to the boundary ∂I_i such that for each $t \in [0, 1]$, $\pi \circ k_i^t \colon I_i \to S^1$ is an immersion. Taking such a homotopy generically with respect to the other arcs/loops of $K|_W$ (and $K'|_W$) and the 2-disks $N_1, \ldots, N_n$, we have a finite sequence of VM0-moves transforming a_i to a'_i (recall the proof of Lemma 4). Applying this procedure inductively, we see that K is transformed to K' by VM0-moves. $\square$

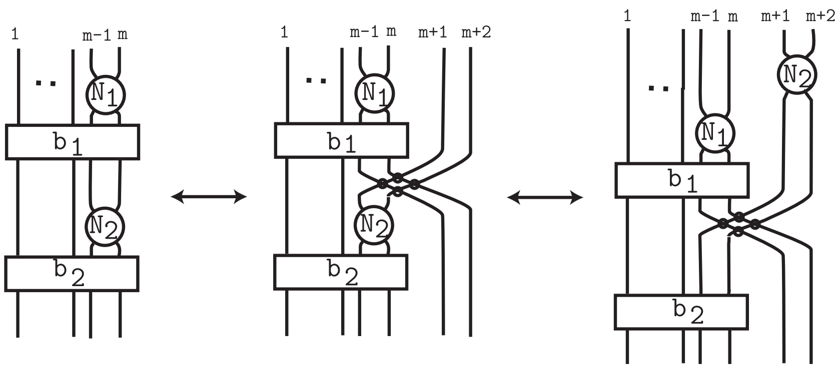

Fig. 10.

Lemma 6. *Two braided virtual link diagrams with the same Gauss data are virtually Markov equivalent in the strict sense.*

Proof. Let K and K' be braided virtual link diagrams with the same Gauss data. Let $N_1, \ldots, N_n$ be regular neighborhoods (as in Fig. 1) of the real crossings $v_1, \ldots, v_n$ of K, and $N'_1, \ldots, N'_n$ be regular neighborhoods of the corresponding real crossings $v'_1, \ldots, v'_n$ of K'.

CASE 1. Suppose that $\pi(N_1), \ldots, \pi(N_n)$ and $\pi(N'_1), \ldots, \pi(N'_n)$ appear in S^1 in the same (cyclic) order. By an isotopy of $\mathbf{R}^2$, deform K keeping the conditions of a braided virtual link diagram such that $N_i = N'_i$ $(i = 1, \ldots, n)$ and that the restrictions of K and K' to these disks are identical. By Lemma 5, K and K' are virtually Markov equivalent in the strict sense.

CASE 2. Suppose that $\pi(N_1), \ldots, \pi(N_n)$ and $\pi(N'_1), \ldots, \pi(N'_n)$ do not appear in S^1 in the same (cyclic) order. It is sufficient to consider a special case that $\pi(N_1), \ldots, \pi(N_n)$ and $\pi(N'_1), \ldots, \pi(N'_n)$ appear in S^1 in the same order except a pair, say $\pi(N_1)$ and $\pi(N_2)$. Applying VM0-moves, we may assume that K is the closure of a virtual braid diagram which looks like the left one of Fig. 10, where b_1 is a virtual braid diagram without real crossings and b_2 is a virtual braid diagram. The middle of the figure is obtained from the left by VM0- and VM2-moves. The right one is obtained from the middle by VM0-moves. By Case 1, the right one and K' are virtually Markov equivalent in the strict sense. Thus K and K' are virtually Markov equivalent in the strict sense. □

Since the braiding process (the proof of Proposition 3) does not change the Gauss data of a virtual link diagram, we have the following.

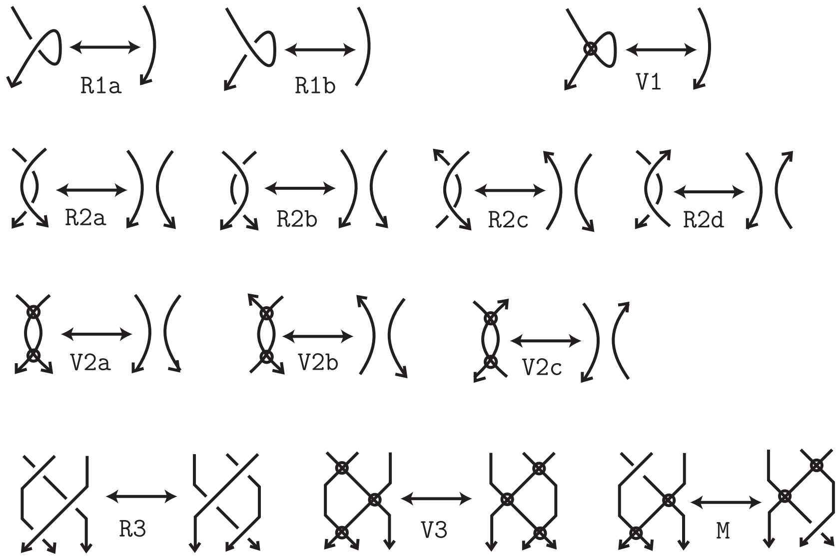

Fig. 11. Oriented virtual Reidemeister moves

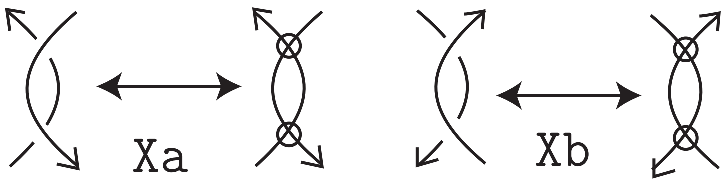

Fig. 12.

Corollary 7. *For a virtual link diagram K, a braided virtual link diagram obtained by the braiding process is uniquely determined up to virtual Markov equivalence in the strict sense.*

Proof of Theorem 1. The if part is obvious. We prove the only if part. Let K and K' be braided virtual link diagrams which represent the same virtual link. There is a finite sequence of virtual link diagrams from K to K' each step of which is one of the moves in Fig. 11 (cf. §7, Proposition 11). By use of V2-moves, an R2c-move and an R2d-move are obtained from an Xa-move and an Xb-move in Fig. 12, respectively. Therefore, there is a finite sequence of virtual link diagrams $K = K_0, K_1, \ldots, K_s = K'$ such that each K_i is obtained from K_{i-1} by an R1a-, R1b-, V1-, R2a-, R2b-, Xa-, Xb-, V2a-, V2b-, V2c-, R3-, V3- or M-move.

Apply the braiding process to each K_i and let $\widetilde{K}_i$ be a braided virtual link diagram with the same Gauss data as K_i. Note that $\widetilde{K}_i$ is uniquely determined up to virtual Markov equivalence in the strict sense (Lemma 6). We assume that $\widetilde{K}_0 = K_0 = K$ and

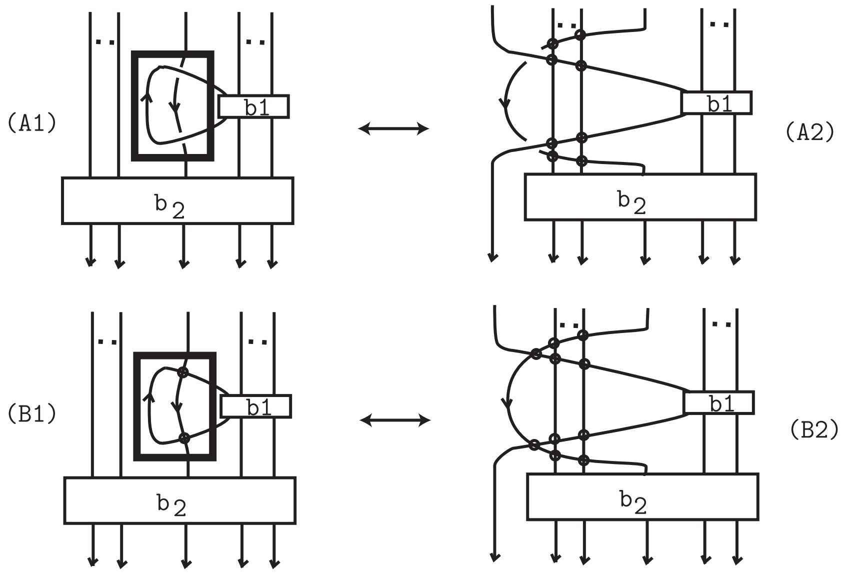

Fig. 13.

$\widetilde{K}_s = K_s = K'$. Then it is sufficient to prove that for each i ($i = 1, \ldots, s$), $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are virtually Markov equivalent.

If K_i is obtained from K_{i-1} by a V1-, V2a-, V2b-, V2c-, V3- or M-move, then K_i and K_{i-1} have the same Gauss data and so do $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$. By Lemma 6, $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are virtually Markov equivalent.

Suppose that K_i is obtained from K_{i-1} by an R1a-, R1b-, R2a-, R2b-, Xa-, Xb-, or R3-move. Let Δ be a 2-disk in $\mathbf{R}^2$ where the move is applied, and let Δ^c be the complement of Δ in $\mathbf{R}^2$ so that $K_i \cap \Delta^c = K_{i-1} \cap \Delta^c$.

If the move is not an Xb-move, then we can deform K_i and K_{i-1} by an isotopy of $\mathbf{R}^2$ such that $K_i \cap \Delta$ and $K_{i-1} \cap \Delta$ satisfy the conditions of a braided virtual link diagram. Apply the braiding process to the remainder $K_i \cap \Delta^c = K_{i-1} \cap \Delta^c$, and we have braided virtual link diagrams, say $\widetilde{K}'_i$ and $\widetilde{K}'_{i-1}$ such that $\widetilde{K}'_i \cap \Delta = K_i \cap \Delta$, $\widetilde{K}'_{i-1} \cap \Delta = K_{i-1} \cap \Delta$, and $\widetilde{K}'_i \cap \Delta^c = \widetilde{K}'_{i-1} \cap \Delta^c$. If the move is an R1a-, R1b-, or Xa-move, then Δ contains the origin O of $\mathbf{R}^2$ and $\widetilde{K}'_i$ and $\widetilde{K}'_{i-1}$ are related by a right stabilization of positive/negative type or a right virtual exchange move. If the move is an R2a-, R2b-, or R3-move, then Δ is disjoint from O and $\widetilde{K}'_i$ and $\widetilde{K}'_{i-1}$ are related by a VM0-move. Since $\widetilde{K}'_i$ has the same Gauss data as K_i, it is virtually Markov equivalent to $\widetilde{K}_i$ by Lemma 6. Similarly $\widetilde{K}'_{i-1}$ is virtually Markov equivalent to $\widetilde{K}_{i-1}$. Therefore $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are virtually Markov equivalent.

If the move is an Xb-move, then transform K_i and K_{i-1}, without changing their Gauss data, to the closures of the (virtual) tangles depicted as (A1) and (B1) in Fig. 13, say K'_i and K'_{i-1}, where b_1 and b_2 are virtual braid diagrams. (First deform $K_i \cap \Delta$

and $K_{i-1} \cap \Delta$ by isotopies of $\mathbf{R}^2$ such that they are locally as in the thick boxes of (A1) and (B1). Then apply the braiding process to the remainder.) Let $\widetilde{K}_i'$ and $\widetilde{K}_{i-1}'$ be the closures of the virtual braid diagrams depicted as (A2) and (B2) in the figure. Note that $\widetilde{K}_i'$ has the same Gauss data as K_i' and hence as K_i. Thus $\widetilde{K}_i'$ is virtually Markov equivalent to $\widetilde{K}_i$ (Lemma 6). Similarly $\widetilde{K}_{i-1}'$ is virtually Markov equivalent to $\widetilde{K}_{i-1}$. On the other hand, $\widetilde{K}_i'$ and $\widetilde{K}_{i-1}'$ are related by a left virtual exchange move. Therefore $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are virtually Markov equivalent. □

6. Welded links and their braid presentation

Throughout this section, a virtual link diagram is referred to as a *welded link diagram*. We call the local move illustrated in the left hand side of Fig. 14 a W-*move*. Two welded link diagrams are *equivalent as welded link* if they are related by a finite sequence of virtual Reidemeister moves and W-moves. The equivalence class is called a *welded link* or a *welded link type*. It is easily verified that the oriented W-move illustrated in the right of Fig. 14 is sufficient to realize all possible orientations for a W-move up to oriented moves in Fig. 11 (cf. the proof of Proposition 11, §7).

We refer to a virtual braid diagram as a *welded braid diagram*. Recall that the welded braid group WB_m is the quotient of VB_m by adding the relations $\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}$ $(i = 1, \dots, m-2)$ corresponding to W-moves.

Proposition 8. *Any welded link can be described as the closure of a welded braid.*

Proof. This is a direct consequence of Proposition 3. □

When two welded braid diagrams b and b' represent the same welded braid, we say that b' is obtained from b by a WM0-*move* or a *welded braid move*. A WM1-*move* or a WM2-*move* is a VM1-move or a VM2-move, respectively. A right/left stabilization of virtual type is referred to as a *right/left stabilization of welded type*.

Lemma 9. *A left stabilization of positive, negative or welded type is a consequence of* WM0-, WM1- *and* WM2-*moves*.

Proof. For the case of welded type, see the first row of Fig. 15.

For the case of positive type, see the second row. The step (6) → (7) is allowed in the welded braid group, whereas it is not allowed in the virtual braid group. The case of negative type is treated similarly. □

Lemma 10. *A right/left virtual exchange move is a consequence of* WM0-, WM1- *and* WM2-*moves*.

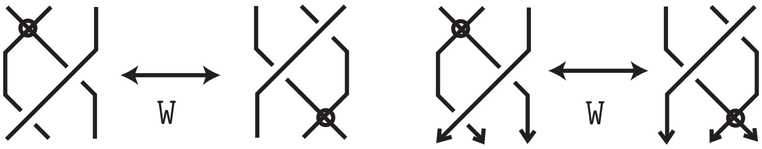

Fig. 14. W-move

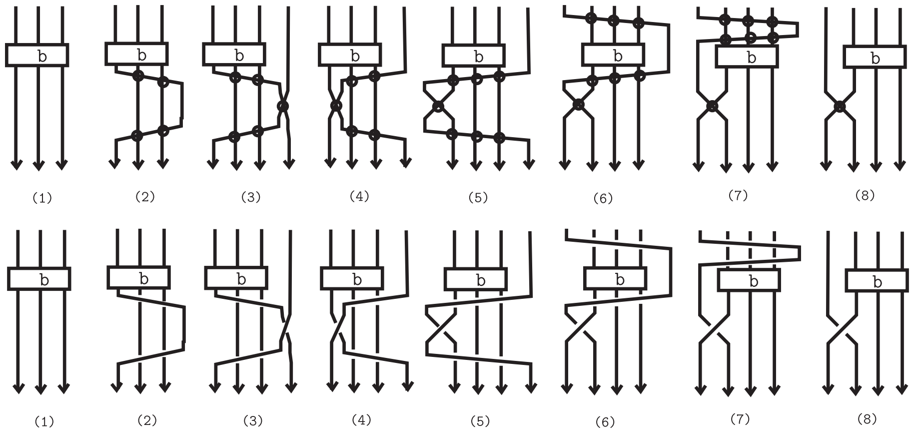

Fig. 15.

Proof. A right virtual exchange move is realized by WM0-, WM1- and WM2-moves as follows:

$$
\begin{aligned}
b_1\sigma_m^{-1}b_2\sigma_m &= b_1\sigma_m^{-1}\tau_m\tau_m b_2\sigma_m \in WB_{m+1}\\
&\leftrightarrow b_1\sigma_m^{-1}\tau_m\tau_{m+1}\tau_m b_2\sigma_m \in WB_{m+2} \quad (\text{WM1} + \text{WM2})\\
&= b_1\sigma_m^{-1}\tau_{m+1}\tau_m\tau_{m+1}b_2\sigma_m \in WB_{m+2}\\
&= b_1\tau_{m+1}\tau_m\sigma_{m+1}^{-1}\tau_{m+1}b_2\sigma_m \in WB_{m+2}\\
&= \tau_{m+1}b_1\tau_m b_2\sigma_{m+1}^{-1}\tau_{m+1}\sigma_m \in WB_{m+2}\\
&\leftrightarrow b_1\tau_m b_2\sigma_{m+1}^{-1}\tau_{m+1}\sigma_m\tau_{m+1} \in WB_{m+2} \quad (\text{WM1})\\
&= b_1\tau_m b_2\sigma_{m+1}^{-1}\tau_m\sigma_{m+1}\tau_m \in WB_{m+2}\\
&= b_1\tau_m b_2\sigma_m\tau_{m+1}\sigma_m^{-1}\tau_m \in WB_{m+2}\\
&\leftrightarrow b_1\tau_m b_2\sigma_m\sigma_m^{-1}\tau_m \in WB_{m+1} \quad (\text{WM1} + \text{WM2})\\
&= b_1\tau_m b_2\tau_m \in WB_{m+1},
\end{aligned}
$$

where $b_1, b_2 \in WB_m$ (and we also denote by b_i $(i = 1, 2)$ the natural images $\iota_0^1(b_i) \in WB_{m+1}$ and $\iota_0^2(b_i) \in WB_{m+2}$). Similarly, a left virtual exchange move is realized by

WM0-, WM1-moves and left stabilizations. By Lemma 9, we have the result. □

Here we also call a braided virtual link diagram a *braided welded link diagram*. Two braided welded link diagrams are *welded Markov equivalent* if they are related by a finite sequence of WM0- and WM2-moves. By Lemma 10, if two braided welded link diagrams are virtually Markov equivalent, then they are welded Markov equivalent.

Proof of Theorem 2. The if part is obvious. We prove the only if part. Let K and K' be braided welded link diagrams representing the same welded link. There is a finite sequence of welded link diagrams $K = K_0, K_1, \dots, K_s = K'$ such that each K_i is obtained from K_{i-1} by an R1a-, R1b-, V1-, R2a-, R2b-, Xa-, Xb-, V2a-, V2b-, V2c-, R3-, V3-, M- or W-move (in Figs. 11, 12 and 14). Apply the braiding process to each K_i and let $\widetilde{K}_i$ be a braided welded link diagram with the same Gauss data as K_i. By Lemmas 6 and 10, $\widetilde{K}_i$ is uniquely determined up to welded Markov equivalence. We assume that $\widetilde{K}_0 = K_0 = K$ and $\widetilde{K}_s = K_s = K'$. It is sufficient to prove that for each i ($i = 1, \dots, s$), $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are welded Markov equivalent. In the proof of Theorem 1, we have already seen that $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are welded Markov equivalent, except the case where K_i is obtained from K_{i-1} by a W-move. Suppose that K_i is obtained from K_{i-1} by a W-move. Let Δ be a 2-disk in $\mathbf{R}^2$ where the W-move is applied, and let Δ^c be the complement of Δ so that $K_i \cap \Delta^c = K_{i-1} \cap \Delta^c$. Deform K_i and K_{i-1} by an isotopy of $\mathbf{R}^2$ such that $K_i \cap \Delta$ and $K_{i-1} \cap \Delta$ satisfy the condition of a braided virtual (welded) link diagram. Apply the braiding process to the remainder $K_i \cap \Delta^c = K_{i-1} \cap \Delta^c$, and we have braided welded link diagrams, say $\widetilde{K}'_i$ and $\widetilde{K}'_{i-1}$ such that $\widetilde{K}'_i \cap \Delta = K_i \cap \Delta$, $\widetilde{K}'_{i-1} \cap \Delta = K_{i-1} \cap \Delta$, and $\widetilde{K}'_i \cap \Delta^c = \widetilde{K}'_{i-1} \cap \Delta^c$. Then $\widetilde{K}'_i$ and $\widetilde{K}'_{i-1}$ are related by a WM0-move corresponding to $\tau_k \sigma_{k+1} \sigma_k = \sigma_{k+1} \sigma_k \tau_{k+1}$. Since $\widetilde{K}'_i$ has the same Gauss data as K_i, it is welded Markov equivalent to $\widetilde{K}_i$. Similarly $\widetilde{K}'_{i-1}$ is welded Markov equivalent to $\widetilde{K}_{i-1}$. Therefore $\widetilde{K}_i$ and $\widetilde{K}_{i-1}$ are welded Markov equivalent. □

7. Remarks

The following proposition is folklore.

Proposition 11. *Two virtual link diagrams K and K' represent the same virtual link if and only if there is a finite sequence of virtual link diagrams from K to K' each step of which is one of the moves in* Fig. 11.

Proof. The if part is obvious by definition. The only if part is proved by showing that any move illustrated in Fig. 3 with the arcs oriented arbitrarily is a consequence of the moves in Fig. 11.

First we note that all possible orientations of arcs for an R2-move and V2-move in Fig. 3 are listed in Fig. 11.

For an R3-move (a) or (b) in Fig. 3, give orientations to the three arcs.

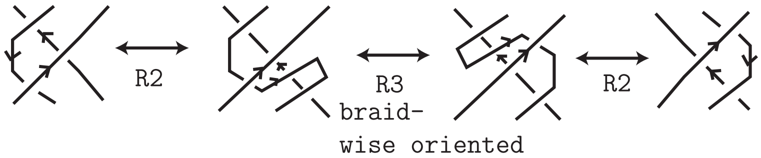

Fig. 16. Cyclically oriented R3-move

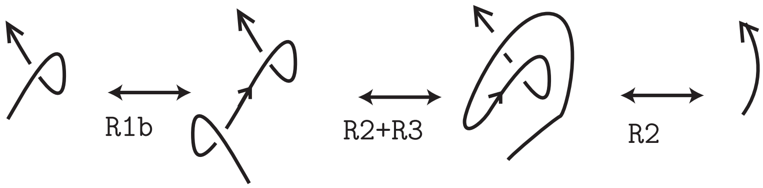

Fig. 17. Whitney trick

(1) If one can name the three crossings A, B and C such that the arcs are oriented from A to B, from B to C and from A to C, respectively, then we say that the arcs are oriented *braid-wise*. In this case, the oriented R3-move is expressed by replacement of braid words,

$$\sigma_i^{\epsilon_1}\sigma_j^{\epsilon_2}\sigma_i^{\epsilon_3} \leftrightarrow \sigma_j^{\epsilon_3}\sigma_i^{\epsilon_2}\sigma_j^{\epsilon_1},$$

where $\{i, j\} = \{1, 2\}$ and $\epsilon_1, \epsilon_2, \epsilon_3$ are ± 1 such that $\epsilon_1 = \epsilon_2$ or $\epsilon_2 = \epsilon_3$. However it is a consequence of a particular replacement with $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$ and some insertions and deletions of $\sigma_k^{\epsilon}\sigma_k^{-\epsilon}$ where $k = 1, 2$ and ϵ is ± 1. Thus, a braid-wise oriented R3-move is a consequence of an R3-move and some R2a-moves and R2b-moves in Fig. 11.

(2) If one can name the three crossings A, B and C such that the arcs are oriented from A to B, from B to C and from C to A, respectively, then we say that the arcs are oriented *cyclically*. A cyclically oriented R3-move is a consequence of a braid-wise oriented R3-move and some oriented R2-moves as in Fig. 16. Thus, it is a consequence of moves in Fig. 11.

For an R1-move, consider an orientation of the arc. If it is not in Fig. 11, then it is reduced to an R1-move in Fig. 11 by a sequence of oriented R2- and R3-moves as in Fig. 17; this process is sometimes called the Whitney trick. Since all oriented R2-moves and R3-moves are consequences of moves in Fig. 11, the oriented R1-move is so.

The other cases involving virtual crossings are shown similarly. □

REMARK. (1) J.S. Birman and R. Trapp introduced and studied the notion of a braided chord diagram [4]. It is different from our braided virtual link diagrams and braided welded link diagrams.

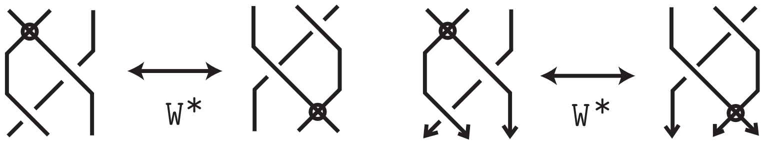

Fig. 18. W*-move

(2) D. Silver and S. Williams [25] proved that knot groups of virtual (or welded) links are isomorphic to knot groups of ribbon-wise knotted tori in the 4-sphere, and S. Satoh [24] showed a geometric relationship between them. From the point of view of [24], welded braids are related to the motion group of a trivial link in $\mathbf{R}^3$ (cf. [8], [9], [21]).
(3) When we use the move illustrated in Fig. 18, called a W*-*move*, instead of a W-move, we have another notion which is analogous to a welded link. Define the group WB_m^* to be the quotient of VB_m by the relations $\tau_i\sigma_{i+1}^{-1}\sigma_i^{-1} = \sigma_{i+1}^{-1}\sigma_i^{-1}\tau_{i+1}$ ($i = 1, \dots, m-2$), instead of $\tau_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\tau_{i+1}$. Then we have results analogous to those in this section. Note that one should not use both of W-moves and W*-moves simultaneously. If we use both moves, every virtual (or welded) knot diagram changes into the unknot (cf. [10], [13], [23]).

ACKNOWLEDGEMENTS. The author thanks J.S. Carter, O. Dasbach, R. Fenn, A. Ishii, T. Kadokami, N. Kamada, L. Kauffman, S. Lambropoulou, X.-S. Lin, L. Paris, S. Satoh, D. Silver and S. Williams for many stimulating conversations on virtual braids and links.

References

[1] J.S. Birman: Braids, Links, and Mapping Class Groups, Ann. of Math. Stud. **82**, Princeton Univ. Press, Princeton, NJ, 1974.
[2] J.S. Birman: *Studying links via closed braids*, Lecture Notes on the Ninth KAIST Mathematical Workshop **1** (1993), 1–67.
[3] J.S. Birman and W. Menasco: *Studying links via closed braids*. V. *The unlink*, Trans. Amer. Math. Soc. **329** (1992), 585–606.
[4] J.S. Birman and R. Trapp: *Braided chord diagrams*, J. Knot Theory Ramifications **7** (1998), 1–22.
[5] J.S. Carter, D. Jelsovsky, S. Kamada and M. Saito: *Quandle homology groups*, *their Betti numbers*, *and virtual knots*, J. Pure Appl. Algebra **157** (2001), 135–155.
[6] R. Fenn, M. Jordan-Santana, and L. Kauffman: *Biquandles and virtual links*, Topology Appl. **145** (2004), 157–175.
[7] R. Fenn, R. Rimányi, and C. Rourke: *The braid-permutation group*, Topology **36** (1997), 123–135.
[8] D.L. Goldsmith: *The theory of motion groups*, Michigan Math. J. **28** (1981), 3–17.
[9] D.L. Goldsmith: *Motion of links in the* 3-*sphere*, Math. Scand. **50** (1982), 167–205.
[10] M. Goussarov, M. Polyak and O. Viro: *Finite-type invariants of classical and virtual knots*, Topology **39** (2000), 1045–1068.

[11] S. Kamada: *Braid presentation of virtual knots and welded knots*, preprint (2000), arXiv: `math.GT/0008092`.

[12] S. Kamada: *Invariants of virtual braids and a remark on left stabilizations and virtual exchange moves*, Kobe J. Math. **21** (2004), 33–49.

[13] T. Kanenobu: *Forbidden moves unknot a virtual knot*, J. Knot Theory Ramifications **10** (2001), 89–96.

[14] L.H. Kauffman: *Virtual knot theory*, European J. Combin. **20** (1999), 663–690.

[15] L.H. Kauffman: *A survey of virtual knot theory*; in Knots in Hellas '98 (Delphi), (Greece, 1998), World Sci. Publ., River Edge, NJ, 2000, 143–202.

[16] L.H. Kauffman and S. Lambropoulou: *Virtual braids*, Fund. Math. **184** (2004), 159–186.

[17] L.H. Kauffman and S. Lambropoulou: *Virtual braids and the L-moves*, preprint (2005), arXiv: `math.GT/0507035 v2`.

[18] L.H. Kauffman and D. Radford: *Bi-oriented quantum algebras and a generalized Alexander polynomial for virtual links*; in Diagrammatic Morphisms and Applications (San Francisco, CA, 2000), Contemp. Math. **318**, Amer. Math. Soc., Providence, RI, 2003, 113–140.

[19] S. Lambropoulou and C.P. Rourke: *Markov's theorem in* 3*-manifolds*, Topology Appl. **78** (1997), 95–122.

[20] S. Lambropoulou and C.P. Rourke: *Algebraic Markov equivalence for links in* 3*-manifolds*, preprint (2004), arXiv: `math.GT/0405493`.

[21] Y. Marumoto, Y. Uchida, and T. Yasuda: *Motions of trivial links and its ribbon knots*, Michigan Math. J. **42** (1995), 463–477.

[22] H.R. Morton: *Threading knot diagrams*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 247–260.

[23] S. Nelson: *Unknotting virtual knots with Gauss diagram forbidden moves*, J. Knot Theory Ramifications **10** (2001), 931–935.

[24] S. Satoh: *Virtual knot presentation of ribbon torus-knots*, J. Knot Theory Ramifications **9** (2000) 531–542.

[25] D.S. Silver and S.G. Williams: *Virtual knot groups*; in Knots in Hellas '98 (Delphi), (Greece, 1998), World Sci. Publ., River Edge, NJ, 2000, 440–451.

[26] P. Traczyk: *A new proof of Markov's braid theorem*; in Knot Theory (Warsaw, 1995), Banach Center Publ., **42**, Polish Acad. Sci., Warsaw, 1998, 409–419.

[27] P. Vogel: *Representation of links by braids*: *a new algorithm*, Comment. Math. Helv. **65** (1990), 104–113.

[28] S. Yamada: *The minimal number of Seifert circles equals the braid index of a link*, Invent. Math. **89** (1987), 347–356.

Department of Mathematics
Hiroshima University
Hiroshima 739–8526
Japan