

## ASYMPTOTIC ARC-SINE LAWS FOR FINITE-DIMENSIONAL INTERACTING DIFFUSIONS

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### Abstract

We consider finite-dimensional interacting diffusions which are defined by adding a linear drift term to independent one dimensional diffusions. For these processes we prove that the distribution of the occupation time at the first quadrant converges to a generalized arc-sine law.

### 1. Introduction

Let  $S$  be a finite set, and let  $A = \{A_{ij}\}_{i \neq j \in S}$  be a matrix with non-negative elements. Let us consider the following stochastic differential equation (SDE):

$$(1.1) \quad dX_i(t) = \alpha(X_i(t)) dB_i(t) + \sum_{j \in S} A_{ij}(X_j(t) - X_i(t)) dt, \quad (i \in S),$$

where  $\{B_i(t)\}_{i \in S}$  is an independent system of one-dimensional standard Brownian motions.

Assume that  $\alpha: \mathbb{R} \rightarrow \mathbb{R}_+$  is a Borel measurable function satisfying the following conditions:

[A-1] For some positive constant  $C > 0$ ,

$$(1.2) \quad \alpha(x) \leq C(1 + |x|) \quad \text{for } x \in \mathbb{R}.$$

[A-2] For each compact set  $K$ , there exists a positive constant  $c_K$  such that  $\alpha(x) \geq c_K$  ( $x \in K$ ),

one can see by standard arguments to use the Girsanov theorem that for any initial distribution on  $\mathbb{R}^S$ , the SDE (1.1) has a unique weak solution, which defines a diffusion process  $(X(t), P_x)$  on  $\mathbb{R}^S$ . We call the diffusion process a *finite-dimensional interacting diffusion*.

In this paper we are concerned with limiting distribution as  $t \rightarrow \infty$  of the occupation time of  $X(t)$  at the first quadrant  $\mathbb{R}_+^S = [0, \infty)^S$  of  $\mathbb{R}^S$

$$(1.3) \quad \frac{1}{t} \int_0^t I_{\mathbb{R}_+^S}(X(s)) ds.$$

In non-interacting case where  $A = \{A_{ij}\}$  is absent, each coordinate process is a diffusion process  $(X(t), P_x)$  on  $\mathbb{R}$  governed by the following SDE:

$$(1.4) \quad dX(t) = \alpha(X(t)) dB(t).$$

For the one-dimensional diffusion process  $(X(t), P_x)$  governed by (1.4) Watanabe [5] proved that the distribution of

$$\frac{1}{t} \int_0^t I_{\mathbb{R}_+}(X(s)) ds$$

converges to a non-degenerate distribution as  $t \rightarrow \infty$  if and only if

$$m_+(x) = \int_0^x \alpha(u)^{-2} du, \quad m_-(x) = \int_{-x}^0 \alpha(u)^{-2} du \quad (x \geq 0)$$

satisfy the following condition; for some  $0 < p < 1$

$$(1.5) \quad m_{\pm}(x) = x^{1/p-1} K_{\pm}(x)$$

with slowly varying functions  $K_+(x)$  and  $K_-(x)$  at  $x = \infty$  and

$$(1.6) \quad \lim_{x \nearrow \infty} \frac{K_+(x)}{K_-(x)} = b \in (0, \infty).$$

Then it holds that

$$\frac{1}{t} \int_0^t I_{\mathbb{R}_+}(X(s)) ds \xrightarrow{(d)} Y_{p,q} \quad (t \rightarrow \infty),$$

where  $q$  is given by

$$q = \frac{b^p}{1 + b^p} \in (0, \infty),$$

and  $\xrightarrow{(d)}$  denotes convergence in distribution and  $Y_{p,q}$  is a  $[0, 1]$ -valued random variable with the Stieltjes transform given by

$$E \left[ \frac{1}{u + Y_{p,q}} \right] = \frac{q(u+1)^{p-1} + (1-q)u^{p-1}}{q(u+1)^p + (1-q)u^p}, \quad u > 0.$$

The family  $Y_{p,q}$ ,  $0 < p \leq 1$ ,  $0 < q < 1$ , was introduced by Lamperti [2], of which distribution is called a *generalized arc-sine law*. In particular, the distribution of  $Y_{1/2,1/2}$  is the arc-sine law, of which density function is given by

$$\frac{1}{\pi \sqrt{x(1-x)}}.$$

For general  $0 < p < 1$  and  $0 < q < 1$ ,  $Y_{p,q}$  has the density  $f_{p,q}(x)$  on  $[0, 1]$ ;

$$f_{p,q}(x) = \frac{\sin p\pi}{\pi} \frac{q(1-q)x^{p-1}(1-x)^{p-1}}{q^2(1-x)^{2p} + (1-q)^2x^{2p} + 2q(1-q)x^p(1-x)^p \cos p\pi}.$$

For the finite-dimensional interacting diffusion  $(X(t), P_x)$  governed by (1.1) we investigate the limiting distribution of (1.3) under the following condition:

[B-1]  $\alpha(x)$  is regularly varying both at  $x \rightarrow \infty$  and  $x \rightarrow -\infty$  with the common exponent  $-\infty < \gamma < 1/2$ , and

$$\lim_{x \rightarrow \infty} \frac{\alpha(-x)}{\alpha(x)} = c \in (0, \infty).$$

[B-2] An  $S \times S$ -matrix  $A = \{A_{ij}\}_{i,j \in S}$ , of which diagonal element is defined by

$$A_{ii} = - \sum_{j \in S, j \neq i} A_{ij} \quad (i \in S),$$

is irreducible.

We note that by [B-2]

$$Q_t = \exp tA$$

defines a transition matrix of an irreducible Markov process on  $S$ , so that there exists a probability vector  $m = \{m_i\}_{i \in S}$  with  $m_i > 0$  such that for some  $\delta > 0$

$$(1.7) \quad |Q_t(i, j) - m_j| \leq e^{-\delta t} \quad (i, j \in S).$$

The main result of this paper is the following.

**Theorem 1.1.** *Assume the conditions [B-1] and [B-2]. Then*

$$(1.8) \quad \frac{1}{t} \int_0^t \delta_{X(s)} ds \xrightarrow{(d)} Y_{p,q} \delta_{+\infty} + (1 - Y_{p,q}) \delta_{-\infty} \quad (t \rightarrow \infty),$$

where  $+\infty = \{x_i \equiv +\infty\}$ ,  $-\infty = \{x_i \equiv -\infty\}$ ,  $\delta_{X(s)}$ ,  $\delta_{+\infty}$  and  $\delta_{-\infty}$  stand for the one point mass at  $X(s)$ ,  $+\infty$  and  $-\infty$  respectively, and  $\xrightarrow{(d)}$  denotes the weak convergence as  $\mathcal{P}([-\infty, \infty]^S)$ -valued random variables, and here  $p, q$  are given by

$$p = \frac{1}{2(1-\gamma)}, \quad q = \frac{c^{2p}}{1+c^{2p}}.$$

From Theorem 1.1 it follows immediately that

**Corollary 1.2.** *Assume the same assumptions as in Theorem 1.1. Then*

$$(1.9) \quad \frac{1}{t} \int_0^t I_{\mathbb{R}^S}(X(s)) ds \xrightarrow{(d)} Y_{p,q} \quad (t \rightarrow \infty).$$

The result of Theorem 1.1 can be interpreted as follows. Since  $S$  is a finite set, the effect of the interaction  $A = \{A_{ij}\}$  is so strong that all component processes diverge to  $\infty$  or  $-\infty$  as  $t \rightarrow \infty$  simultaneously. Hence the phenomena would be quite similar to the one-dimensional case. Nevertheless the one-dimensional analysis as in Watanabe [5] cannot be applied, so, in the next section, we will investigate a scaling limit for the finite-dimensional interacting diffusion  $(X(t), P_x)$  on  $\mathbb{R}^S$ .

## 2. A scaling limit of $X(t)$

By the condition [B-1]  $\alpha(x)$  has the following form;

$$\alpha(x) = |x|^\gamma L(x) \quad (|x| > 0),$$

where  $L(x)$  is a slowly varying function both at  $\infty$  and  $-\infty$  satisfying that

$$\lim_{x \rightarrow \infty} \frac{L(-x)}{L(x)} = c \in (0, \infty).$$

Let

$$p = \frac{1}{2(1-\gamma)} \quad \text{and} \quad \theta_\lambda = \lambda L(\lambda^p)^{-2} \quad (\lambda > 0).$$

We introduce a rescaled process  $(X^\lambda(t), B^\lambda(t))$  by

$$X_i^\lambda(t) = \lambda^{-p} X_i^\lambda(\theta_\lambda t), \quad B_i^\lambda(t) = \theta_\lambda^{-1/2} B_i(\theta_\lambda t), \quad i \in S.$$

Note that  $\{B_i^\lambda(t)\}_{i \in S}$  are independent Brownians motion and the rescaled process  $(X^\lambda(t), B^\lambda(t))$  satisfies the following SDE;

$$dX_i^\lambda(t) = \alpha_\lambda(X_i^\lambda(t)) dB_i^\lambda(t) + \theta_\lambda \sum_{j \in S} A_{ij}(X_j^\lambda(t) - X_i^\lambda(t)) dt,$$

where

$$\alpha_\lambda(x) = \lambda^{-p} \theta_\lambda^{1/2} \alpha(\lambda^p x).$$

Moreover it holds that

$$\lim_{\lambda \rightarrow \infty} \alpha_\lambda(x) = \begin{cases} x^\gamma & (0 < x), \\ c|x|^\gamma & (0 > x). \end{cases}$$

In order to describe the limiting processes of the  $(X^\lambda(t))$  we introduce a class of skew Bessel processes on natural scale.

Let

$$\bar{\alpha}(x) = \begin{cases} \|m\|_2 x^\gamma & (0 \leq x), \\ \|m\|_2 c |x|^\gamma & (0 > x). \end{cases}$$

where  $\|m\|_2 = \sqrt{\sum_{i \in S} m_i^2}$ ,  $\bar{\alpha}(0) = \infty$  if  $\gamma < 0$ , and  $\bar{\alpha}(0) = \|m\|_2$  if  $\gamma = 0$ .

Let us consider the following one-dimensional SDE:

$$(2.1) \quad \begin{aligned} dZ(t) &= \bar{\alpha}(Z(t)) dB(t), \\ Z(0) &= x \in \mathbb{R}. \end{aligned}$$

If  $-\infty < \gamma \leq 0$ , the SDE (2.1) has a law unique solution, however, if  $0 < \gamma < 1/2$ , the law uniqueness for (2.1) fails. In this case, if we add the non-sticky condition to (2.1), i.e.

$$(2.2) \quad \int_0^t I_{\{0\}}(Z(s)) ds = 0 \quad (\forall t > 0), \quad P\text{-a.s.},$$

the law uniqueness holds. In fact, the solution can be constructed from a Brownian motion through the time change method. Thus we have a diffusion process  $(Z(t), P_x)$  on  $\mathbb{R}$ , which is called a skew Bessel process on natural scale.

**Theorem 2.1.** *Assume the conditions [B-1] and [B-2], and  $X(0)$  is a  $\mathbb{R}^S$ -valued random variable independent of  $B(t) = \{B_i(t)\}_{i \in S}$ . Then*

$$(2.3) \quad (X^\lambda(t) = \{X_i^\lambda(t)\}_{i \in S}) \xrightarrow{(\mathcal{L})} (X^\infty(t) = \{X_i^\infty(t)\}_{i \in S}) \quad (\lambda \rightarrow \infty),$$

where  $\xrightarrow{(\mathcal{L})}$  stands for the weak convergence of the probability laws on the path space induced by  $\{X^\lambda(t)\}$ . Moreover, all component processes of  $\{X_i^\infty(t)\}_{i \in S}$  coincide with each other and the common process is equivalent to a skew Bessel diffusion on natural scale  $(Z(t))$  governed by (2.1) with  $Z(0) = 0$  being imposed the non-sticky condition whenever  $0 < \gamma < 1/2$ ;

$$(2.4) \quad \int_0^t I_{\{0\}}(Z(s)) ds = 0 \quad (t > 0), \quad P\text{-a.s.}$$

From Theorem 2.1 it follows the following

**Corollary 2.2.** *Under the same assumption of in Theorem 2.1,*

$$(2.5) \quad X(t) \xrightarrow{(d)} q\delta_{+\infty} + (1 - q)\delta_{-\infty} \quad (t \rightarrow \infty).$$

**Proof of Theorem 1.1.** Theorem 1.1 follows immediately from Theorem 2.1. In fact, since

$$\int_0^t I(Z(s) = 0) ds = 0,$$

by Theorem 2.1 we can see that for every bounded continuous function  $f$  on  $[-\infty, \infty]$  it holds that

$$\begin{aligned} \frac{1}{\theta_\lambda} \int_0^{\theta_\lambda} f(X(s)) ds &= \int_0^1 f(\lambda^p X^\lambda(s)) ds \\ &\stackrel{(d)}{\Rightarrow} f(+\infty) \int_0^1 I(Z(s) > 0) ds + f(-\infty) \int_0^1 I(Z(s) < 0) ds \\ &= Y_{p,q} f(+\infty) + (1 - Y_{p,q}) f(-\infty), \end{aligned}$$

because of

$$\int_0^1 I(Z(s) > 0) ds \stackrel{(d)}{=} Y_{p,q}.$$

For the last relation see Watanabe [5]. □

### 3. Proof of Theorem 2.1

To avoid complication of arguments we prove Theorem 2.1 under the following condition [B-3] instead of [B-1], since the proof is essentially the same even under the condition [B-1].

[B-3] Let  $-\infty < \gamma < 1/2$ , and for some  $\alpha_+ > 0$  and  $\alpha_- > 0$

$$(3.1) \quad \lim_{x \rightarrow \infty} \frac{\alpha(x)}{x^\gamma} = \alpha_+, \quad \lim_{x \rightarrow -\infty} \frac{\alpha(x)}{|x|^\gamma} = \alpha_-.$$

In what follows we assume the conditions [A-1], [A-2], [B-2] and [B-3]. Let

$$(3.2) \quad p = \frac{1}{2(1-\gamma)},$$

and for  $\lambda > 0$  we set

$$(3.3) \quad \alpha_\lambda(x) = \lambda^{-p+1/2} \alpha(\lambda^p x),$$

and

$$(3.4) \quad \alpha_\infty(x) = \begin{cases} \alpha_+ x^\gamma & (0 \leq x), \\ \alpha_- |x|^\gamma & (x < 0). \end{cases}$$

where

$$\alpha_\infty(0) = \begin{cases} \infty & (\gamma < 0), \\ \alpha_+ & (\gamma = 0). \end{cases}$$

Moreover we set

$$(3.5) \quad \bar{\alpha}(x) = \|m\|_2 \alpha_\infty(x),$$

where  $\{m_i\}_{i \in S}$  is a probability vector in (1.7), and  $\|m\|_2 = \sqrt{\sum_{i \in S} m_i^2}$ .

For the diffusion process  $(X(t), P_x)$  governed by (1.1) we introduce a rescaled process  $X^\lambda(t)$  ( $\lambda > 0$ ) by

$$X_i^\lambda(t) = \lambda^{-p} X_i(\lambda t) \quad (i \in S),$$

which satisfies the following SDE:

$$(3.6) \quad dX_i^\lambda(t) = \alpha_\lambda(X_i^\lambda(t)) dB_i^\lambda(t) + \lambda \sum_{j \in S} A_{ij}(X_j^\lambda(t) - X_i^\lambda(t)) dt.$$

For the proof of Theorem 2.1 we may assume that the initial condition  $X(0)$  is non-random, i.e.

$$X(0) = \{x_i\}_{i \in S} \in \mathbb{R}^S.$$

We first prepare several moment estimates of the rescaled process  $X_i^\lambda(t)$ .

**Lemma 3.1.** *Let  $-\infty < \gamma < 1/2$ . For  $a \geq 2$  there exists a constant  $C = C(a, p) > 0$  such that*

$$(3.7) \quad \sum_{i \in S} m_i E[|X_i^\lambda(t)|^a] \leq C \left( \lambda^{-pn} + \lambda^{-pn} \sum_{i \in S} m_i |x_i|^a + t^{pa} \right) \quad (t \geq 0, \lambda > 0).$$

*Proof.* Using the Itô formula and taking expectations, we have

$$(3.8) \quad \begin{aligned} \frac{d}{dt} \sum_{i \in S} m_i E[|X_i(t)|^a] &= a \sum_{i \in S} \sum_{j \in S} m_i A_{ij} E[|X_i(t)|^{a-1} \operatorname{sgn}(X_i(t))(X_j(t) - X_i(t))] \\ &\quad + \frac{1}{2} a(a-1) \sum_{i \in S} m_i E[|X_i(t)|^{a-2} \alpha^2(X_i(t))]. \end{aligned}$$

Note that

$$(3.9) \quad \sum_{i \in S} \sum_{j \in S} m_i A_{ij} |x_i|^{a-1} \operatorname{sgn}(x_i)(x_j - x_i) \leq 0,$$

because, using  $\sum_{j \in S} A_{ij} = 0$ ,  $\sum_{i \in S} m_i A_{ij} = 0$  and a simple inequality

$$t^{a-1}s \leq \frac{a-1}{a}t^a + \frac{1}{a}s^a \quad (t > 0, s > 0),$$

we see

$$\begin{aligned} & \sum_{i \in S} \sum_{j \in S} m_i A_{ij} |x_i|^{a-1} \operatorname{sgn}(x_i)(x_j - x_i) \\ & \leq \sum_{i \in S} \sum_{j \in S} m_i A_{ij} (|x_i|^{a-1}|x_j| - |x_i|^a) \\ & \leq \frac{1}{a} \sum_{i \in S} \sum_{j \in S} m_i A_{ij} (|x_j|^a - |x_i|^a) \\ & = 0. \end{aligned}$$

Note that by the conditions [A-1], [A-2] and [B-3] there exists constants  $C_1 > 0$  and  $C_2 > 0$  satisfying

$$(3.10) \quad C_1(1 + |x|)^\gamma \leq \alpha(x) \leq C_2(1 + |x|)^\gamma, \quad (x \in \mathbb{R}),$$

so that there exists a constant  $C_3$  such that

$$(3.11) \quad \sum_{i \in S} m_i |x_i|^{a-2} \alpha^2(x_i) \leq C_3 \left( 1 + \sum_{i \in S} m_i |x_i|^a \right)^{1-1/ap},$$

Hence, by (3.9), (3.10) and (3.11)  $F(t) = \sum_{i \in S} m_i E[|X_i(t)|^{2a}]$  satisfies

$$\frac{d}{dt} F(t) \leq C_3(1 + F(t))^{1-1/ap}.$$

Thus we obtain, for some  $C_4 > 0$ ,

$$(3.12) \quad \sum_{i \in S} m_i E[|X_i(t)|^a] \leq C_4 \left( 1 + \sum_{i \in S} m_i |x_i|^a + t^{ap} \right).$$

(3.7) follows immediately from (3.12). □

Let

$$U_{i,j}(t) = X_i(t) - X_j(t) \quad (i \neq j \in S),$$

and for  $\lambda > 0$  let

$$U_{i,j}^\lambda(t) = X_i^\lambda(t) - X_j^\lambda(t) \quad (i \neq j \in S).$$

**Lemma 3.2.** (i) For any  $a \geq 2$  there exists a constant  $C > 0$  such that

$$(3.13) \quad E[|U_{i,j}^\lambda(t)|^a] \leq \begin{cases} C\lambda^{-a/2}(1+t^{ap\gamma}) & (0 \leq \gamma < 1/2), \\ C\lambda^{-ap} & (-\infty < \gamma < 0). \end{cases}$$

(ii) For each  $T > 0$  there exists a constant  $C_T > 0$  such that for every  $\lambda \geq 1$

$$(3.14) \quad E[|U_{i,j}^\lambda(t) - U_{i,j}^\lambda(s)|^6] \leq C_T\lambda^{-1}|t-s|^2 \quad (0 \leq s \leq t \leq T).$$

Proof. First, note that  $X(t)$  satisfies

$$(3.15) \quad X_i(t) = \sum_{k \in S} \int_s^t Q_{t-u}(i, k) \alpha(X_k(u)) dB_k(u) + \sum_{k \in S} Q_{t-s}(i, j) X_k(s) \quad (i \in S),$$

so that

$$\begin{aligned} U_{i,j}(t) - U_{i,j}(s) &= \sum_{k \in S} \int_s^t (Q_{t-u}(i, k) - Q_{t-u}(j, k)) \alpha(X_k(u)) dB_k(u) \\ &\quad + \sum_{k \neq i} Q_{t-s}(i, k) U_{i,k}(s) + \sum_{k \neq j} Q_{t-s}(j, k) U_{j,k}(s). \end{aligned}$$

Using this and the Burkholder inequality, we have

$$(3.16) \quad \begin{aligned} &E[|U_{i,j}(t) - U_{i,j}(s)|^a] \\ &\leq C_1 \sum_{k \in S} E \left[ \left( \int_s^t (Q_{t-u}(i, k) - Q_{t-u}(j, k))^2 \alpha^2(X_k(u)) du \right)^{a/2} \right] \\ &\quad + C_1 E \left[ \left( \sum_{k \in S} Q_{t-s}(i, k) U_{i,k}(s) \right)^a \right] \\ &\quad + C_1 E \left[ \left( \sum_{k \in S} Q_{t-s}(j, k) U_{j,k}(s) \right)^a \right]. \end{aligned}$$

When  $0 \leq \gamma < 1/2$ , using this with  $s = 0$ , (1.7) and Lemma 3.1 we have a constant  $C_2 > 0$  satisfying that

$$E[|U_{i,j}(t)|^a] \leq C_2(1+t^{a\gamma p}),$$

which yields (3.13). Using (1.7), (3.10), and Lemma 3.1, we see that the first term of the r.h.s. of (3.16) with  $a = 6$  is dominated by

$$C_3 \sum_{k \in S} E \left[ \left( \int_s^t e^{-2\delta(t-u)} \alpha^2(X_k(u)) du \right)^3 \right] \leq C_4((t-s) \wedge 1)^3(1+t^{6p\gamma}).$$

Furthermore, by (3.13) the last two terms of (3.16) are dominated by

$$C_5((t - s) \wedge 1)^6(1 + t^{6p\gamma}),$$

thus we have

$$(3.17) \quad E[|U_{i,j}(t) - U_{i,j}(s)|^6] \leq C_6((t - s) \wedge 1)^3(1 + t^{6p\gamma}).$$

From this it follows that

$$\begin{aligned} E[|U_{i,j}^\lambda(t) - U_{i,j}^\lambda(s)|^6] &\leq C_7(\lambda(t - s) \wedge 1)^3\lambda^{-6p}(1 + (\lambda t)^{6p\gamma}) \\ &\leq C_T\lambda^{-1}(t - s)^2, \end{aligned}$$

which concludes (3.14). In the case  $-\infty < \gamma < 0$ , since  $\alpha(x)$  is bounded,  $E[U_{i,j}(t)^6]$  is also bounded in  $t \geq 0$ . Hence it is easy to obtain (3.14).  $\square$

**Lemma 3.3.** *Suppose that  $X(t)$  is a continuous martingale with  $X(0) = 0$  defined on a complete probability space  $(\Omega, \mathcal{F}, P)$  with filtration  $\{\mathcal{F}_t\}$ , of which quadratic variation process satisfies*

$$\langle X \rangle(t) = \int_0^t \bar{\alpha}^2(X(s)) ds,$$

where  $\bar{\alpha}(x)$  is of (3.4). If  $0 < \gamma < 1$ , we further assume the non-sticky condition;

$$\int_0^t I_{(0)}(X(s)) ds = 0 \quad (t > 0) \quad P\text{-a.s.}$$

Then the probability law on the path space  $W = C([0, \infty), \mathbb{R})$  induced by  $(X(t))$  coincides with that of the skew Bessel process on natural scale  $Z(t)$  starting at 0 governed by the SDE (2.1) with (2.2).

*Proof.* Proof is to verify that  $X(t)$  satisfies the SDE (2.1) for some Brownian motion  $\bar{B}(t)$  using the time-change method, that is quite standard, so we omit it.  $\square$

**Proof of Theorem 2.1 in case 0  $\gamma < 1/2$ .** In this case the proof is rather standard, that is, first to verify the tightness of the probability laws  $P^\lambda$  on  $W$  induced by  $\{X^\lambda(t)\}$  and next to identify the limit of  $\{P^\lambda\}$  as  $\lambda \rightarrow \infty$ .

For the stationary probability vector  $\{m_i\}$  of  $Q_t$  we set

$$Y^\lambda(t) = \sum_{i \in S} m_i X_i^\lambda(t),$$

which satisfies the following equation;

$$(3.18) \quad dY^\lambda(t) = \sum_{i \in S} m_i \alpha_\lambda(X_i^\lambda(t)) dB_i^\lambda(t).$$

**Lemma 3.4.** *Let  $0 \leq \gamma < 1/2$ . For each  $T > 0$  there exists constant  $C_T > 0$  such that for every  $\lambda > 0$ ,*

$$(3.19) \quad E[|Y^\lambda(t) - Y^\lambda(s)|^4] \leq C_T(t - s)^2, \quad (0 \leq s, t \leq T).$$

Proof. It is immediate from (3.18) and Lemma 3.1. □

**Lemma 3.5.** *Let  $0 \leq \gamma < 1/2$ .*

$$(3.20) \quad \lim_{\varepsilon \rightarrow 0^+} \limsup_{\lambda \rightarrow \infty} \int_0^t P(|X_i^\lambda(s)| \leq \varepsilon) ds = 0. \quad (i \in S, t > 0).$$

Proof. For each  $\varepsilon > 0$  define a function  $\varphi_\varepsilon$  by

$$\begin{aligned} \varphi_\varepsilon''(x) &= |x|^{-2\gamma} I(|x| \leq \varepsilon), \\ \varphi_\varepsilon(x) &= \int_0^{|x|} \int_0^y \varphi_\varepsilon''(u) du dy. \end{aligned}$$

Applying Itô formula we obtain

$$(3.21) \quad E[\varphi_\varepsilon(Y^\lambda(t))] = \varphi_\varepsilon\left(\lambda^{-p} \sum_{i \in S} m_i x_i\right) + \sum_{i \in S} \int_0^t m_i^2 E[\alpha_\lambda^2(X_i^\lambda(s)) \varphi_\varepsilon''(Y^\lambda(s))] ds.$$

Since

$$|\varphi_\varepsilon(x)| \leq \frac{\varepsilon^{1-2\gamma}}{(1-2\gamma)} |x|,$$

using Lemma 3.1 we have

$$(3.22) \quad \lim_{\varepsilon \rightarrow 0^+} \limsup_{\lambda \rightarrow \infty} \sum_{i \in S} \int_0^t m_j^2 E[\alpha_\lambda^2(X_i^\lambda(s)) \varphi_\varepsilon''(Y^\lambda(s))] ds = 0.$$

Note that for some  $C_1 > 0$

$$\alpha_\lambda^2(x) \geq C_1(\lambda^{-p} + |x|)^\gamma \quad (x \in \mathbb{R}, \lambda > 0),$$

and for  $y = \sum_i m_i x_i$

$$\begin{aligned} \sum_{i \in S} m_i \alpha_\lambda^2(x_i) \varphi_\varepsilon''(y) &\geq C_1 \sum_{i \in S} m_i (\lambda^{-p} + |x_i|)^{2\gamma} |y|^{-2\gamma} I(|y| < \varepsilon) \\ &\geq C_2 (\lambda^{-p} + |y|)^{2\gamma} |y|^{-2\gamma} I(|y| < \varepsilon) \\ &\geq C_2 I(|y| < \varepsilon). \end{aligned}$$

Hence from this and (3.22) it follows that

$$(3.23) \quad \lim_{\varepsilon \rightarrow 0^+} \limsup_{\lambda \rightarrow \infty} \int_0^t P(|Y^\lambda(s)| \leq \varepsilon) du = 0.$$

Here we notice that

$$P(|X_i^\lambda(s)| \leq \varepsilon) \leq P(|Y^\lambda(s)| \leq 2\varepsilon) + P(|X_i^\lambda(s) - Y^\lambda(s)| > \varepsilon),$$

and that for each  $\varepsilon > 0$  the second term vanishes as  $\lambda \rightarrow \infty$ . Hence (3.20) follows from (3.23).  $\square$

Now we proceed to the proof of Theorem 2.1 in the case  $0 \leq \gamma < 1/2$ . Let  $P^\lambda$  be the probability measure on  $W = C([0, \infty), \mathbb{R}^S)$  induced by  $X^\lambda(t)$ . We use the notation  $E^{P^\lambda}$  for the expectation by  $P^\lambda$ . Then by Lemma 3.4 and Lemma 3.2  $\{P^\lambda\}$  is tight. Suppose that for some  $\{\lambda_n\}$  tending to  $\infty$ ,  $P^{\lambda_n}$  converges weakly to  $P^\infty$ . Let

$$\bar{w}(t) = \sum_{i \in S} m_i w_i(t).$$

Since by (3.18)  $\bar{w}(t)$  is a  $P^\lambda$ -martingale with quadratic variation process

$$(3.24) \quad \langle \bar{w} \rangle(t) = \sum_{i \in S} m_i^2 \int_0^t \alpha_\lambda^2(w_i(s)) ds \quad P^\lambda\text{-a.s.},$$

using Lemma 3.1 we see easily that  $\bar{w}(t)$  is a  $P^\infty$ -martingale with  $\bar{w}(0) = 0$ . Moreover, it follows from Lemma 3.2 that

$$(3.25) \quad P^\infty(w_i(t) = w_j(t) \ (\forall t \geq 0)) = 1.$$

(3.24) implies that for every  $0 \leq s < t$  and a  $\mathcal{F}_s$ -measurable and bounded continuous function  $\Phi_s(w)$  on  $W$

$$(3.26) \quad E^{P^\lambda} \left[ \left( \bar{w}^2(t) - \bar{w}^2(s) - \sum_{i \in S} m_i^2 \int_s^t \alpha_\lambda^2(w_i(u)) du \right) \Phi_s(w) \right] = 0.$$

We claim that

$$(3.27) \quad \lim_{\lambda \rightarrow \infty} E^{P^\lambda} \left[ \left( \int_s^t \alpha_\lambda^2(w_i(u)) du \right) \Phi_s(w) \right] = E^{P^\infty} \left[ \left( \int_s^t \bar{\alpha}^2(w_i(u)) du \right) \Phi_s(w) \right].$$

For  $\varepsilon > 0$  let  $\varphi_\varepsilon$  be a smooth function on  $\mathbb{R}$  satisfying

$$I_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]}(x) \leq \varphi_\varepsilon(x) \leq I_{\mathbb{R} \setminus [-\varepsilon/2, \varepsilon/2]}(x).$$

Since  $\alpha_\lambda(x)$  converges to  $\alpha_\infty(x)$  as  $\lambda \rightarrow \infty$  compact uniformly in  $\mathbb{R} \setminus \{0\}$  and

$$\alpha_\lambda(x) \leq C_3(1 + |x|^\gamma) \quad (x \in \mathbb{R}),$$

using Lemma 3.1 we see that for every  $\varepsilon > 0$

$$\begin{aligned} (3.28) \quad & \lim_{\lambda \rightarrow \infty} E^{P^\lambda} \left[ \left( \int_s^t \alpha_\lambda^2(w_i(u)) \varphi_\varepsilon(w_i(u)) du \right) \Phi_s(w) \right] \\ & = E^{P^\infty} \left[ \left( \int_s^t \alpha_\infty^2(w_i(u)) \varphi_\varepsilon(w_i(u)) du \right) \Phi_s(w) \right]. \end{aligned}$$

On the other hand by Lemma 3.5

$$\begin{aligned} & \lim_{\varepsilon \rightarrow +0} \limsup_{\lambda \rightarrow \infty} E^{P^\lambda} \left[ \left( \int_s^t \alpha_\lambda^2(w_i(u))(1 - \varphi_\varepsilon)(w_i(u)) du \right) \Phi_s(w) \right] \\ & \leq C_4 \lim_{\varepsilon \rightarrow +0} \limsup_{\lambda \rightarrow \infty} \int_0^t P(|X_i^\lambda(u)| \leq \varepsilon) du = 0. \end{aligned}$$

(3.27) follows from this and (3.28). Thus,  $\bar{w}(t)$  is a  $P^\infty$ -martingale with quadratic variation process

$$\langle w \rangle(t) = \sum_{i \in S} m_i^2 \int_0^t \alpha_\infty^2(w_i(u)) du = \int_0^t \bar{\alpha}^2(\bar{w}(u)) du.$$

Therefore by Lemma 3.3  $P^\infty$  coincides with the probability law of the skew Bessel process on natural scale, which completes the proof of Theorem 2.1 in the case  $0 \leq \gamma < 1/2$ . □

**Proof of Theorem 2.1 in case  $\gamma < 0$ .** In this case it seems hard to obtain the moment estimate for  $Y^\lambda(t)$  as in Lemma 3.4 due to difficulty of negative power moment estimates, so we consider a spatial transformation by an asymptotic scale function  $S(x)$ ;

$$S(x) = \begin{cases} x^{2(1-\gamma)} & (\gamma \geq 0), \\ |x|^{2(1-\gamma)} & (\gamma < 0). \end{cases}$$

**Lemma 3.6.** *Let  $-\infty < \gamma < 0$ . For each  $T > 0$  there exists a constant  $C_T > 0$  such that for every  $\lambda \geq 1$*

$$(3.29) \quad E[|S(Y^\lambda(t)) - S(Y^\lambda(s))|^4] \leq C_T |t - s|^2, \quad (0 \leq s, t \leq T).$$

*Proof.* Recall that  $Y^\lambda(t)$  satisfies

$$(3.30) \quad dY^\lambda(t) = \alpha_\lambda(Y^\lambda(t)) dV^\lambda(t),$$

where  $V^\lambda(t)$  is a continuous martingale with quadratic variation process

$$(3.31) \quad \langle V^\lambda \rangle(t) = \sum_{i \in S} m_i^2 \int_0^t \frac{\alpha_\lambda^2(X_i^\lambda(u))}{\alpha_\lambda^2(Y^\lambda(u))} du.$$

Applying Itô formula to  $S(x)$  together with Burkholder's inequality we see that

$$(3.32) \quad \begin{aligned} & E[|S(Y^\lambda(t)) - S(Y^\lambda(s))|^4] \\ & \leq C_1 E \left[ \left( \int_s^t |S'(Y^\lambda(u))|^2 \alpha_\lambda^2(Y^\lambda(u)) d\langle V^\lambda \rangle(u) \right)^2 \right] \\ & \quad + C_1 E \left[ \left( \int_s^t S''(Y^\lambda(u)) \alpha_\lambda^2(Y^\lambda(u)) d\langle V^\lambda \rangle(u) \right)^4 \right] \\ & \leq C_1 \int_s^t E[(S' \alpha_\lambda)^4(Y^\lambda(u))] du \int_s^t E[(\langle V^\lambda \rangle'(u))^4] du \\ & \quad + C_1 \|S'' \alpha_\lambda^2\|_\infty^4 (t-s)^3 \int_s^t E[(\langle V^\lambda \rangle'(u))^4] du, \end{aligned}$$

where

$$\langle V^\lambda \rangle'(u) = \sum_{i \in S} m_i^2 \frac{\alpha_\lambda^2(X_i^\lambda(u))}{\alpha_\lambda^2(Y^\lambda(u))},$$

and we notice that  $S'' \alpha_\lambda^2(x)$  is bounded in  $x \in \mathbb{R}$  and  $\lambda \geq 1$ . Note that

$$C_2 \lambda^{2p-1} (1 + \lambda^p |x|)^{2\gamma} \leq \alpha_\lambda^2(x) \leq C_3 \lambda^{2p-1} (1 + \lambda^p |x|)^{2\gamma},$$

then

$$\frac{\alpha_\lambda^2(x)}{\alpha_\lambda^2(y)} \leq C_4 \left( \frac{1 + \lambda^p |y|}{1 + \lambda^p |x|} \right)^{2|\gamma|} \leq C_4 (1 + \lambda^p |x - y|)^{2|\gamma|}.$$

Hence,

$$E[(\langle V^\lambda \rangle'(u))^4] \leq C_5 \left( 1 + \lambda^{8p|\gamma|} \sum_{j \neq k} E[|U_{j,k}^\lambda(u)|^{8|\gamma|}] \right),$$

which is bounded in  $u \geq 0$  by Lemma 3.2. Accordingly, it follows from this and (3.32) that

$$E[(S(Y^\lambda(t)) - S(Y^\lambda(s)))^4] \leq C_7 (|t - s|^2 + |t - s|^4),$$

which completes the proof of Lemma 3.6.  $\square$

**Lemma 3.7.** *Let  $-\infty < \gamma < 0$ . Then*

$$(3.33) \quad \lim_{\varepsilon \rightarrow 0} \limsup_{\lambda \rightarrow \infty} \int_0^t E[\alpha_\lambda^2(X_i^\lambda(s))I(|X_i^\lambda(s)| \leq \varepsilon)] ds = 0 \quad (i \in S).$$

*Proof.* In the proof of Lemma 3.5, replacing  $\varphi_\varepsilon(x)$  by  $\varphi_\varepsilon''(x) = I_{[-\varepsilon, \varepsilon]}(x)$  we have

$$(3.34) \quad \lim_{\varepsilon \rightarrow 0^+} \limsup_{\lambda \rightarrow \infty} \sum_{i \in S} \int_0^t m_i^2 E[\alpha_\lambda^2(X_i^\lambda(s))I_{[-\varepsilon, \varepsilon]}(Y^\lambda(s)) ds] = 0.$$

Noting that

$$I_{[-\varepsilon, \varepsilon]}(X_i^\lambda(s)) \leq I_{[-2\varepsilon, 2\varepsilon]}(Y^\lambda(s)) + \sum_{j \in S} I_{[-\varepsilon, \varepsilon]}(X_j^\lambda(s) - X_i^\lambda(s)),$$

and by Lemma 3.2 we can see

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} \int_0^t E[\alpha_\lambda^2(X_i(s))I_{[-\varepsilon, \varepsilon]}(X_j^\lambda(s) - X_i^\lambda(s))] ds \\ & \leq \lim_{\lambda \rightarrow \infty} \lambda^{-2p+1} \|\alpha\|_\infty^2 \int_0^t P(|U_{i,j}^\lambda(s)| > \varepsilon) = 0. \end{aligned}$$

Thus (3.33) follows from this and (3.34). □

Now we are in position to complete the proof of Theorem 2.1 in the case  $-\infty < \gamma < 0$ , but one can proceed the proof as in the case of  $0 \leq \gamma < 1/2$ , so we shall only sketch the proof. By virtue of Lemma 3.2 and Lemma 3.6, we may assume that  $P^{\lambda_n}$  converges weakly to  $P^\infty$  as  $n \rightarrow \infty$  for some  $\lambda_n \nearrow \infty$ . Then,  $\bar{w}(t)$  is  $P^\infty$ -martingale with  $\bar{w}(0) = 0$  and

$$w_i(t) = w_j(t) = \bar{w}(t) \quad P^\infty\text{-a.s.} \quad (i, j \in S),$$

in the same way as  $0 \leq \gamma < 1/2$ . Using the Lemma 3.7 instead of Lemma 3.5, we also have (3.27) which implies

$$E^{P^\infty} \left[ \left( \bar{w}^2(t) - \bar{w}^2(s) - \int_s^t \bar{\alpha}^2(\bar{w}(u)) du \right) \Phi_s(w) \right] = 0,$$

and then by Lemma 3.3, the probability law  $(\bar{w}(t), P^\infty)$  coincides with that of the desired skew Bessel process on natural scale. Therefore Theorem 2.1 has been proved completely. □

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