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A STRUCTURE THEOREM OF COMPACT COMPLEX PARALLELIZABLE PSEUDO-KÄHLER SOLVMANIFOLDS

Dedicated to Professor Yusuke Sakane on his 60th birthday

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Abstract

In this paper, we prove that the Mostow fibration of a compact complex parallelizable pseudo-Kähler solvmanifold is a complex torus bundle over a complex torus.

Introduction

A complex manifold X^n of complex dimension n is called *complex parallelizable* if there exist n holomorphic vector fields which are linearly independent at each point. Wang [13] proved that a compact complex parallelizable manifold is of the form G/Γ, where G is a complex Lie group and Γ is a discrete subgroup of G. Wang also proved that if a compact complex parallelizable manifold X admits a Kähler structure, then X is a complex torus. On the other hand, Matsushima [10] proved that a compact homogeneous Kähler manifold is biholomorphic to a product of a homogeneous rational manifold and a complex torus. By a homogeneous Kähler manifold we mean a Kähler manifold on which the group of holomorphic isometric transformations acts transitively. Borel-Remmert [2] generalized the result of Matsushima to compact Kähler manifolds on which the group of holomorphic transformations acts transitively. Dorfmeister-Guan [3] proved that a compact homogeneous pseudo-Kähler manifold is also biholomorphic to a product of a homogeneous rational manifold and a complex torus. As for compact pseudo-Kähler manifolds on which the group of holomorphic transformations acts transitively, there exist non-toral compact complex parallelizable pseudo-Kähler solvmanifolds. In particular, we see that a compact non-homogeneous pseudo-Kähler manifold is not biholomorphic to a product of a homogeneous rational manifold and a complex torus in general (cf. [17]). It is therefore important to study compact complex parallelizable pseudo-Kähler solvmanifolds. In this paper we prove the following structure theorem, which is our main theorem:

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Theorem 1.6 *Let* $X = G/\Gamma$ *be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of* X *is a complex torus bundle over a complex torus.*

We also investigate the Dolbeault cohomology groups of a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure.

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1. Proof of main theorem

In this section we prove our main theorem.

DEFINITION 1.1. Let G be a Lie group. A discrete subgroup Γ of G is called a *lattice* if G/Γ has a finite invariant measure.

If G is a solvable Lie group, then a discrete subgroup Γ of G is a lattice if and only if Γ is a discrete co-compact subgroup of G.

Let $\mathcal{O}_X = \mathcal{O}$ be the sheaf of holomorphic functions on a complex manifold X. We denote the Hodge number of X by $h^{p,q}(X)$, i.e., $h^{p,q}(X) = \dim H_{\bar\partial}^{p,q}(X)$. Let G be a connected complex Lie group, Γ a lattice of G, N the maximal connected normal nilpotent subgroup. Let $G = S \cdot R$ be a Levi decomposition, where S is a semi-simple part, and R is the radical. We denote derived Lie subgroups of G, N and R by G', N' and R' respectively. Winkelmann has proven

Theorem 1.2 ([14]). *Let* $G, \Gamma, N, S, R, G', N'$ *and* R' *be as above. Let* $A = [S, R] \cdot N'$. *Furthermore let* W *denote the maximal linear subspace of the Lie algebra* Lie$(R'A/A)$ *of* $R'A/A$ *such that* $\mathrm{Ad}(\gamma)|_W$, *where* Ad *is the adjoint representation of* G, *is a semisimple linear endomorphism with only real eigenvalues for each* $\gamma \in \Gamma$. *Then*

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/G' + \dim H^1(G/R\Gamma, \mathcal{O}) + \dim W.$$

DEFINITION 1.3. Let X be a complex manifold. A real $(1,1)$-form ω of X is called a *pseudo-Kähler structure* if ω is a non-degenerate closed form.

In the case of a compact complex parallelizable manifold, we have shown the following in the paper [17]:

Theorem 1.4. *Let $X^n = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure. Then*

$$h^{p,q}(X) \geq \binom{n}{p} \cdot \binom{n}{q}.$$

Corollary 1.5. *Let $(G/\Gamma, \omega)$ be an n-dimensional compact complex parallelizable pseudo-Kähler manifold such that Γ is a lattice of G. If $h^{0,1}(G/\Gamma) = \dim H^0(G/\Gamma, d\mathcal{O})$, then G/Γ is a complex torus.*

Proof. Let $\mathfrak{g}$ be the Lie algebra of G, I the complex structure of $\mathfrak{g}$, and $\mathfrak{g}^+ = \{X \in \mathfrak{g}^{\mathbb{C}} \mid IX = \sqrt{-1}X\}$. We identify $\mathfrak{g}^+$ with the set of all right invariant holomorphic vector fields of G. Let $H^q(\mathfrak{g}^+)$ be the qth Lie algebra cohomology group of $\mathfrak{g}^+$. Since $H^0(G/\Gamma, d\mathcal{O}) \cong H^1(\mathfrak{g}^+)$ and $h^{0,1}(G/\Gamma) \geq n$, we see that $\dim H^1(\mathfrak{g}^+) = n$. Hence $\mathfrak{g}^+$ is abelian. $\square$

Let G/Γ be a compact complex parallelizable solvmanifold, i.e., G is a simply connected complex solvable Lie group and Γ is a lattice of G. Mostow proved that $\Gamma_N = N \cap \Gamma$ is a lattice of the maximal normal nilpotent Lie subgroup N of G. A fibration $N/\Gamma_N \to G/\Gamma \to G/N\Gamma$ is called the *Mostow fibration* of G/Γ.

Theorem 1.6. *Let $X^n = G/\Gamma$ be a compact complex parallelizable solvmanifold which admits a pseudo-Kähler structure. Then the Mostow fibration of G/Γ is a complex torus bundle over a complex torus.*

Proof. We use the notation of Theorem 1.2. Since G is solvable, we see that $G = R$, $S = \{e\}$ and $A = N'$. By Theorems 1.2 and 1.4, we see

$$n \leq \dim H^{0,1}_{\bar\partial}(G/\Gamma) \leq \dim G/G' + \dim W \leq \dim G - \dim G' + \dim G'/N',$$

which implies $\dim N' = 0$. $\square$

By the proof of our main theorem, we have

Corollary 1.7. *If a compact complex parallelizable solvmanifold X^n admits a pseudo-Kähler structure, then $h^{0,1}(X) = n$.*

Corollary 1.8. *If a compact complex parallelizable solvmanifold G/Γ admits a pseudo-Kähler structure, then N must be abelian and in particular the Lie algebra $\mathfrak{g}$ must satisfy $\mathcal{D}^{(2)}\mathfrak{g} = 0$.*

Proof. Since $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$, we have our corollary. $\square$

REMARKS 1.9. (i) There exists a complex solvable Lie group G which has lattices Γ_1, Γ_2 such that G/Γ_1 has a pseudo-Kähler structure, while G/Γ_2 has no pseudo-Kähler structures (see [17]).
(ii) It is well known that a simply connected complex solvable Lie group G is biholomorphic to $\mathbb{C}^n$. Moreover if its Lie algebra $\mathfrak{g}$ has a Chevalley decomposition, then there exists a good system of coordinates $(z_1, \ldots, z_n)$ of G which satisfies the following:

(a) The Lie group G is isomorphic to $(\mathbb{C}^n, *)$ as a complex Lie group, where the multiplication $*$ of $\mathbb{C}^n$ is given by

$$\begin{aligned}&(z_1, \ldots, z_n) * (y_1, \ldots, y_n)\\ &\ = (z_1 + y_1, \ldots, z_r + y_r, F_{r+1r+1}(y)z_{r+1} + y_{r+1} + F_{r+1}(z, y), \ldots, F_{nn}(y)z_n + y_n + F_n(z, y))\end{aligned}$$

for $z = (z_1, \ldots, z_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{C}^n$, where $r = \dim \mathfrak{g} - \dim[\mathfrak{g}, \mathfrak{g}]$, $F_{ii}(z) = \exp\bigl(-\sum_{j=1}^{k} C^i_{ji} z_j\bigr)$, where $k = \dim G/N\Gamma$ and C^i_{ji} are constant, and $F_\lambda(z, y) = F_\lambda(z_1, \ldots, z_{\lambda-1}, y_1, \ldots, y_{\lambda-1})$ is a holomorphic function with respect to $(z_1, \ldots, z_{\lambda-1}, y_1, \ldots, y_{\lambda-1})$ for each λ.

(b) Let Γ be a lattice of G. Using the above system of coordinates $(z_1, \ldots, z_n)$ of G, we see that any element of $H^{0,1}_{\bar\partial}(G/\Gamma)$ has a representative of the following form:

$$\psi = \sum_{\lambda=1}^{k} c_\lambda \, d\bar z_\lambda + \sum_{\lambda=k+1}^{k+r(N/\Gamma_N)} f_\lambda(z_1, \ldots, z_k) \, d\bar z_\lambda,$$

where $r(N/\Gamma_N) = \dim H^{0,1}_{\bar\partial}(N/\Gamma_N)$, c_λ are constant and $f_\lambda(z_1, \ldots, z_k)$ are holomorphic in $z_1, \ldots, z_k$.

We say that a complex solvable Lie algebra $\mathfrak{g}$ has a *Chevalley decomposition* if $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ as a vector space, where $\mathfrak{a}$ is a commutative subalgebra and $\mathfrak{n}$ is the maximal nilpotent ideal. For further details see [11].

Using the above system of coordinates, we give another proof of our main theorem for the case where the Lie algebra $\mathfrak{g}$ of G has a Chevalley decomposition (see Section 3).

2. The structure of the sheaf $R^1\pi_*\mathcal{O}_{G/\Gamma}$

For a holomorphic map $f : X \to Y$ between complex spaces, there exists a Leray spectral sequence for the sheaf $\mathcal{O}$. The respective lower term sequence yields the following:

$$\begin{aligned}0 \to H^1(Y, R^0 f_*\mathcal{O}_X) &\to H^1(X, \mathcal{O}_X)\\ &\to H^0(Y, R^1 f_*\mathcal{O}_X) \to H^2(Y, R^0 f_*\mathcal{O}_X),\end{aligned}$$

where $R^q f_*\mathcal{O}_X$ is the higher direct image sheaf. If f is connected and proper, then $R^0 f_*\mathcal{O}_X = \mathcal{O}_Y$.

In this section we prove the following:

Proposition 2.1. *Let $X^n = G/\Gamma$ be a compact complex parallelizable pseudo-Kähler solvmanifold and $\pi: G/\Gamma \to G/N\Gamma$ the Mostow fibration. Then $R^1\pi_*\mathcal{O}_X$ is the sheaf of sections of a trivial holomorphic vector bundle.*

To prove this proposition we follow a part of the proof of Theorem 1.2 due to Winkelmann ([14]).

Let G be a connected complex Lie group and Γ a lattice of G. Let V be a complex vector space, $\rho: G \to GL(V)$ an antiholomorphic representation and V_1 the set of all $v \in V$ which are invariant under $\rho(G')$, where G' is the derived Lie subgroup of G. We denote by V_0 the subspace spanned by all vectors $v \in V_1$ such that v is an eigenvector with a real eigenvalue for every $\rho(\gamma)$ $(\gamma \in \Gamma)$.

Let E, E_0 be flat vector bundles over $X = G/\Gamma$ which are induced by $\rho|_\Gamma$ on V, V_0 respectively, i.e., $E = G \times V/\sim$, where $(g, v) \sim (g', v')$ if and only if $(g', v') = (g\gamma^{-1}, \rho(\gamma)v)$. Note that E, E_0 are holomorphic vector bundles.

Proposition 2.2. *The flat vector bundle E_0 is a holomorphically trivial vector bundle and $\Gamma(X, E) = \Gamma(X, E_0) \cong V_0$.*

Proof. See [14], Propositions 7.9.1 and 7.9.2. □

Let $\pi: X \to B$ be a holomorphic fiber bundle with an n-dimensional complex torus $T^n_{\mathbb{C}}$ as typical fiber. Let $V = \Omega^1(T^n_{\mathbb{C}})$ denote the vector space of holomorphic 1-forms on $T^n_{\mathbb{C}}$. Note that $T^n_{\mathbb{C}}$ is a compact Kähler manifold. Let $\mathfrak{U} = \{U_i\}$ be a trivializing open cover of B such that X is given by transition functions $\phi_{ij}: U_i \cap U_j \to \mathrm{Aut}(T^n_{\mathbb{C}})$, where $\mathrm{Aut}(T^n_{\mathbb{C}})$ is the automorphism group of $T^n_{\mathbb{C}}$. We denote by $\mathrm{Aut}^0(T^n_{\mathbb{C}})$ the identity component of $\mathrm{Aut}(T^n_{\mathbb{C}})$.

Lemma 2.3. *Under the above assumptions $R^1\pi_*\mathcal{O}_X$ is a locally free coherent sheaf of B isomorphic to the sheaf of sections of the flat vector bundle E given by transition functions $\varphi_{ij} = \overline{\zeta \circ \phi_{ij}}: U_i \cap U_j \to GL(V)$, where $1 \to \mathrm{Aut}^0(T^n_{\mathbb{C}}) \to \mathrm{Aut}(T^n_{\mathbb{C}}) \xrightarrow{\zeta} GL(V)$ is exact.*

Proof. See [14], CLAIM 8.4.5. □

We apply this lemma to a complex parallelizable manifold G/Γ. Let K be a normal abelian complex Lie subgroup of G and $\mathfrak{k}$ its Lie algebra. Assume that $K/K \cap \Gamma$ is compact. Denote the natural projection map $X = G/\Gamma \to B = G/K\Gamma$ by π.

Proposition 2.4. *The sheaf $R^1\pi_*\mathcal{O}_X$ is the sheaf of sections of the flat vector bundle E of rank* $\dim K$ *over B induced by the representation $\rho\colon \Gamma \to GL(\mathfrak{k}^*)$ given by $\gamma \mapsto \overline{\mathrm{Ad}^*(\gamma)}$.*

Proof. See [14], Proposition 8.4.6. □

Moreover, if G/K is abelian, we have

Proposition 2.5 ([14]). *If G/K is abelian, then*

$$\dim H^1(G/\Gamma, \mathcal{O}) \leq \dim G/K + \dim U,$$

where U denotes the maximal linear subspace of $\mathfrak{k}$ such that $\mathrm{Ad}(\gamma)|_U$ *is a semisimple linear endomorphism with only real eigenvalues for every γ in Γ.*

Proof. Let us consider the lower term of the Leray spectral sequence for $\pi\colon G/\Gamma \to G/K\Gamma$. Then we have the following:

$$0 \to H^1(G/K\Gamma, \mathcal{O}) \to H^1(G/\Gamma, \mathcal{O}) \to H^0(G/K\Gamma, R^1\pi_*\mathcal{O}).$$

Since $G/K\Gamma \cong (G/K)/(K\Gamma/K)$ and G/K is abelian, by Propositions 2.2 and 2.4, we see

$$\begin{aligned}\dim H^1(G/\Gamma, \mathcal{O}) &\leq \dim H^1(G/K\Gamma, \mathcal{O}) + \dim U\\ &= \dim G/K + \dim U.\end{aligned}$$

Hence we have our proposition. □

Proof of Proposition 2.1. In Section 1, we have seen that if a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then the maximal normal nilpotent Lie subgroup N of G is abelian and $h^{0,1} = n$. Thus let us consider the Mostow fibration $\pi\colon G/\Gamma \to G/N\Gamma$. Since G/N is abelian and G/Γ admits a pseudo-Kähler structure, we have $W = \mathfrak{n}$ by Proposition 2.5, where W is the maximal linear subspace of $\mathfrak{n}$ such that $\mathrm{Ad}(\gamma)|_W$ is a semisimple endomorphism with only real eigenvalues for every $\gamma \in \Gamma$. By Propositions 2.2 and 2.4, this means that the flat vector bundle E induced by the representation $\rho|_\Gamma\colon \Gamma \to GL(\mathfrak{n}^*)$ given by $\gamma \mapsto \overline{\mathrm{Ad}^*(\gamma)}$ is trivial as a holomorphic vector bundle. □

3. Dolbeault cohomology of compact complex parallelizable pseudo-Kähler solvmanifolds

In this section we consider the Dolbeault cohomology groups of compact complex parallelizable pseudo-Kähler solvmanifolds.

Let G be a complex Lie group and $\mathfrak{g}$ its Lie algebra. Let I denote the complex structure of $\mathfrak{g}$, and $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) denote the vector space of the $+\sqrt{-1}$ (resp. $-\sqrt{-1}$) eigenvectors of the complex structure I respectively. Then we have $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$. In this section, we identify $\mathfrak{g}^+$ with the set of all right invariant holomorphic vector fields of G. Recall that $H_{\bar\partial}^{p,q}(G/\Gamma) \cong H_{\bar\partial}^{0,q}(G/\Gamma) \otimes \bigwedge^p(\mathfrak{g}^+)^*$ for a compact complex parallelizable manifold G/Γ. Sakane [12] has proved that if G is a complex nilpotent Lie group, then $H_{\bar\partial}^{p,q}(G/\Gamma) \cong H^q(\mathfrak{g}^-) \otimes \bigwedge^p(\mathfrak{g}^+)^*$, where $H^q(\mathfrak{g}^-)$ is the qth Lie algebra cohomology group of $\mathfrak{g}^-$.

Let $F \to X \xrightarrow{\pi} B$ be a holomorphic fiber bundle such that X, B, F are connected and F is compact. Then $\bigcup_{b\in B} H_{\bar\partial}^{p,q}(F_b)$ is the total space of a differentiable vector bundle over B. This bundle is denoted by $\mathbf{H}^{p,q}(F)$ and $\mathbf{H}_{\bar\partial}(F)$ is the direct sum of $\mathbf{H}^{p,q}(F)$. If every connected component of the structure group of $\pi : X \to B$ acts trivially on $\mathbf{H}_{\bar\partial}(F)$, then the vector bundle is a holomorphic vector bundle. Thus if the fiber F is a compact Kähler manifold, then $\mathbf{H}_{\bar\partial}(F)$ is a holomorphic vector bundle.

Theorem 3.1 ([8]). *Let $\xi = (X, B, F, \pi)$ be a holomorphic fiber bundle, where X, B, F are connected and F is compact. Assume that every connected component of the structure group of ξ acts trivially on $\mathbf{H}_{\bar\partial}(F)$, i.e., $\mathbf{H}_{\bar\partial}(F)$ is a holomorphic vector bundle. Then there exists a spectral sequence (E_r, d_r), $(r \geq 0)$, with the following properties*:
(i) *E_r is 4-graded, by the fiber-degree, the base-degree and the type. Let ${}^{p,q}E_r^{s,t}$ be the subspace of elements of E_r of type (p, q), fiber-degree s and base-degree t. We have ${}^{p,q}E_r^{s,t} = 0$ if $p + q \neq s + t$ or if one of p, q, s, t is negative. The differential d_r maps ${}^{p,q}E_r^{s,t}$ into ${}^{p,q+1}E_r^{s+r,t-r+1}$.*
(ii) *If $p + q = s + t$, then we have*

$$
{}^{p,q}E_2^{s,t} \cong \sum_{i\geq 0} H_{\bar\partial}^{i,s-i}(B, \mathbf{H}^{p-i,q-s+i}(F)).
$$

(iii) *The spectral sequence converges to $H_{\bar\partial}(X) = \bigoplus_{p,q} H_{\bar\partial}^{p,q}(X)$.*

We put ${}^{p,q}E_r = \sum_{s,t\geq 0} {}^{p,q}E_r^{s,t}$. We call the above spectral sequence the *Borel's spectral sequence*.

REMARK 3.2. If F is a complex torus, then the vector bundle $\mathbf{H}_{\bar\partial}^{0,1}(F) \to B$ is isomorphic to the holomorphic vector bundle E considered in Section 2 (see Lemma 2.3).

By Theorem 1.4 and the Borel's spectral sequence, we have

Proposition 3.3. *Let $X = G/\Gamma$ be a compact complex parallelizable manifold which admits a pseudo-Kähler structure and $F \to X \xrightarrow{\pi} B$ a holomorphic fiber bun-*

dle such that F, B are complex tori. If $\mathbf{H}_{\bar\partial}(F) \to B$ is trivial as a holomorphic vector bundle, then

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$

Proof. By our assumption we see that the Borel's spectral sequence satisfies

$${}^{p,q}E_2^{s,t} \cong \sum_{i\geq 0} H_{\bar\partial}^{i,s-i}(B) \otimes H_{\bar\partial}^{p-i,q-s+i}(F).$$

Thus by the relation $\dim {}^{p,q}E_\infty \leq \dim {}^{p,q}E_2$, we see that $h^{p,q}(X) \leq \binom{n}{p} \cdot \binom{n}{q}$. Thus we have our proposition by Theorem 1.4. □

If $\mathbf{H}^{0,1}(T^n_{\mathbb{C}}) \to B$, where $T^n_{\mathbb{C}}$ is an n-dimensional complex torus, admits global holomorphic sections $\sigma_1, \ldots, \sigma_n$ which are linearly independent at each point, then $\mathbf{H}^{0,q}(T^n_{\mathbb{C}}) \to B$ is trivial as a holomorphic vector bundle. Indeed, consider $\sigma_J = \sigma_{j_1} \wedge \cdots \wedge \sigma_{j_q}$ (Note that $h^{0,q}(T^n_{\mathbb{C}}) = \binom{n}{q}$). Thus by Proposition 2.1 and Lemma 2.3 we have

Corollary 3.4. *If a compact complex parallelizable solvmanifold $X^n = G/\Gamma$ admits a pseudo-Kähler structure, then*

$$h^{p,q}(X) = \binom{n}{p} \cdot \binom{n}{q}.$$

Proof. By our assumption, we see that $\mathbf{H}_{\bar\partial}^{0,q}(N/\Gamma_N) \to G/N\Gamma$ is trivial as a holomorphic vector bundle. □

Let (X^n, ω) be a compact pseudo-Kähler manifold. We say that (X^n, ω) has the *hard Lefschetz property with respect to the Dolbeault cohomology* if for any $p+q \leq n$, the homomorphism

$$L^{n-p-q}\colon H_{\bar\partial}^{p,q}(X) \to H_{\bar\partial}^{n-q,n-p}(X), \quad L^{n-p-q}([\alpha]) = [\alpha \wedge \omega^{n-p-q}]$$

is an isomorphism.

Corollary 3.5. *Let $(G/\Gamma, \omega)$ be an n-dimensional compact complex parallelizable pseudo-Kähler solvmanifold. Then $(G/\Gamma, \omega)$ has the hard Lefschetz property with respect to the Dolbeault cohomology.*

Proof. Put $\bar\tau_i = i(X_i^+)\omega$, where $\{X_1^+, \ldots, X_n^+\}$ is a basis of $\mathfrak{g}^+$. We denote the dual basis of $\mathfrak{g}^+$ by $\{\omega_1^+, \ldots, \omega_n^+\}$. Then ω can be written as $\omega = \sum_{i=1}^n \bar\tau_i \wedge \omega_i^+$. In particular, we see that $\bar\tau_i$ are non-exact $\bar\partial$-closed. We also see that $\alpha = \sum_{JK} a_{JK} \bar\tau_J \wedge \omega_K^+$

is non-exact $\bar\partial$-closed, where $a_{JK} \in \mathbb{C}$, $\bar\tau_J = \bar\tau_{j_1} \wedge \cdots \wedge \bar\tau_{j_q}$ for $J = (j_1, \ldots, j_q)$ and $\omega^+_K = \omega^+_{k_1} \wedge \cdots \wedge \omega^+_{k_p}$ for $K = (k_1, \ldots, k_p)$. Thus by Corollary 3.4 for each Dolbeault cohomology class we can choose a representative of the form $\alpha = \sum_{JK} a_{JK} \bar\tau_J \wedge \omega^+_K$. Hence $(G/\Gamma, \omega)$ has the hard Lefschetz property. □

By the proof of Corollary 3.5, we see that if an n-dimensional compact complex parallelizable solvmanifold G/Γ admits a pseudo-Kähler structure, then $H_{\bar\partial}(G/\Gamma) = \bigoplus_{p,q} H^{p,q}_{\bar\partial}(G/\Gamma)$ is isomorphic to the cohomology *ring* $H_{\bar\partial}(T^n_{\mathbb{C}})$.

REMARK 3.6. Mathieu's theorem of the Dolbeault cohomology on a compact pseudo-Kähler manifold (X, ω) also holds (see [18], [19]), i.e., the following two assertions are equivalent: (a) every Dolbeault cohomology class contains a $\bar\partial$-harmonic representative. (b) (X, ω) has the hard Lefschetz property with respect to the Dolbeault cohomology. We define $\partial^*\colon \Omega^{p,q}(X) \to \Omega^{p-1,q}(X)$ by $\partial^* = (-1)^{p+q} * \bar\partial\, *$, where $\Omega^{p,q}(X)$ is the set of all differential (p,q)-forms on X. A form α is called a $\bar\partial$-*harmonic* form if it satisfies $\bar\partial\alpha = \partial^*\alpha = 0$, where $*\colon \Omega^{p,q}(X^n) \to \Omega^{n-q,n-p}(X^n)$ is defined as an analogy of the star operator for a compact Riemannian manifold. In the above case, for each Dolbeault cohomology class, we can choose a $\bar\partial$-harmonic representative of the form $\alpha = \sum_{JK} a_{JK} \bar\tau_J \wedge \omega^+_K$.

Let $(G/\Gamma, \omega)$ be an n-dimensional compact complex parallelizable pseudo-Kähler solvmanifold. We now give another proof of our main theorem for the case where the Lie algebra $\mathfrak{g}$ of G has a Chevalley decomposition. We use a system of coordinates of Remarks 1.9 and the notation of the proof of Corollary 3.5. Then the pseudo-Kähler structure ω on G/Γ can be written as follows:

$$\omega = \sum_{i=1}^{n} \bar\tau_i \wedge \omega^+_i.$$

Note that $\bar\tau_i$, ω^+_i are $\bar\partial$-closed. By Remarks 1.9, $\bar\tau_i$ are expressed by

$$\bar\tau_i = \psi_i + \bar\partial\gamma_i,$$

where $\psi_i = \sum_{\lambda=1}^{k} c^i_\lambda\, d\bar z_\lambda + \sum_{\lambda=k+1}^{k+r(F)} f^i_\lambda(z)\, d\bar z_\lambda$, $F = N/\Gamma_N$, c^i_λ are constant and f^i_λ are holomorphic. Hence we can write

$$\omega = \sum_{i=1}^{n} \psi_i \wedge \omega^+_i + \bar\partial\theta.$$

Put $\omega' = \sum_{i=1}^{n} \psi_i \wedge \omega^+_i$. Then a volume form ω^n is expressed by $\omega^n = \omega'^n + \bar\partial\Omega$, where $\Omega \in \Omega^{n,n-1}(X)$. Since $\int_{G/\Gamma} \omega^n = \int_{G/\Gamma} \omega'^n + \int_{G/\Gamma} \bar\partial\Omega = \int_{G/\Gamma} \omega'^n$, we see that $r(F) = \dim H^{0,1}_{\bar\partial}(F) = \dim F = n - k$. Since F is a compact complex parallelizable nilmanifold, we see $n-k=\dim H_{\bar\partial}^{0,1}(N/\Gamma_N)=\dim H^1(\mathfrak{n}^-)$ by Sakane's theorem. Thus $\mathfrak{n}^-$ is abelian, which implies that F is a complex torus.

4. Examples

EXAMPLE 4.1 ([11]). Define a multiplication $*$ of $\mathbb{C}^{n+2m}$ by

$$\begin{aligned}&(z_1,\dots,z_n,w_1,w_2,\dots,w_{2m-1},w_{2m})*\left(z_1',\dots,z_n',w_1',w_2',\dots,w_{2m-1}',w_{2m}'\right)\\&=\left(z_1+z_1',\dots,z_n+z_n',\dots,e^{-\sum_i a_i^k z_i}w_{2k-1}'+w_{2k-1},e^{\sum_i a_i^k z_i}w_{2k}'+w_{2k},\dots\right),\end{aligned}$$

where a_i^k are integers. The solvable Lie group $G=(\mathbb{C}^{n+2m},*)$ has a lattice Γ such that G/Γ has a pseudo-Kähler structure. Indeed, for a suitable lattice Γ of G,

$$\omega=\sqrt{-1}\sum_{k=1}^{n}dz_k\wedge d\bar z_k+\sum_{k=1}^{m}(dw_{2k-1}\wedge d\bar w_{2k}+d\bar w_{2k-1}\wedge dw_{2k})$$

is a pseudo-Kähler structure on G/Γ (for details, see [17]). By Corollary 3.4, we see $h^{p,q}(G/\Gamma)=\binom{n+2m}{p}\cdot\binom{n+2m}{q}$.

EXAMPLE 4.2 (cf. [4]). Let us consider the following solvable Lie group:

$$G=\left\{\begin{pmatrix}e^{z_1}&0&z_2e^{z_1}&0&0&0&w_1\\0&e^{-z_1}&0&z_2e^{-z_1}&0&0&w_2\\0&0&e^{z_1}&0&0&0&w_3\\0&0&0&e^{-z_1}&0&0&w_4\\0&0&0&0&1&0&z_2\\0&0&0&0&0&1&z_1\\0&0&0&0&0&0&1\end{pmatrix}\,\middle|\, z_1,z_2,w_1,w_2,w_3,w_4\in\mathbb{C}\right\}.$$

The Lie algebra $\mathfrak{g}$ of G is given by

$$\mathfrak{g}=\operatorname{span}_{\mathbb{C}}\{Z_1,Z_2,W_1,W_2,W_3,W_4\}$$

with

$$\begin{aligned}&[Z_1,W_{2k-1}]=W_{2k-1},\quad &&[Z_1,W_{2k}]=-W_{2k},\\&[Z_2,W_3]=W_1,&&[Z_2,W_4]=W_2\end{aligned}$$

for $k=1,2$. The solvable Lie group G admits a lattice Γ (see [16]). Since the maximal nilpotent ideal $\mathfrak{n}$ is not abelian, we see that for any lattice Γ, G/Γ has no pseudo-Kähler structures by Corollary 1.8.

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