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EXCEPTIONAL SURGERY AND BOUNDARY SLOPES

Dedicated to Professor Yukio Matsumoto on his sixtieth birthday

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Abstract

Let X be a norm curve in the $SL(2,\mathbb{C})$-character variety of a knot exterior M. Let $t = \|\beta\|/\|\alpha\|$ be the ratio of the Culler-Shalen norms of two distinct non-zero classes $\alpha, \beta \in H_1(\partial M, \mathbb{Z})$. We demonstrate that either X has exactly two associated strict boundary slopes $\pm t$, or else there are strict boundary slopes r_1 and r_2 with $|r_1| > t$ and $|r_2| < t$. As a consequence, we show that there are strict boundary slopes near cyclic, finite, and Seifert slopes. We also prove that the diameter of the set of strict boundary slopes can be bounded below using the Culler-Shalen norm of those slopes.

1. Introduction

For a knot in a closed (i.e., compact and without boundary), connected, orientable 3-manifold, let M denote the exterior of the knot. We fix a basis (μ, λ) of $H_1(\partial M, \mathbb{Z})$. The slope of $\gamma \in H_1(\partial M, \mathbb{Z})$ with respect to this basis will be denoted r_γ. That is, if $\gamma = a\mu + b\lambda$, $r_\gamma = a/b \in \mathbb{Q} \cup \{1/0\}$. Let $M(r)$ denote *Dehn surgery* on a knot along slope r. That is, $M(r)$ is the manifold obtained by attaching a solid torus V to M by a homeomorphism of $\partial V \to \partial M$ which sends a meridian curve of V to a simple closed curve in ∂M of the given slope r. If $\pi_1(M(r))$ is cyclic (respectively, finite), we call r a *cyclic* (resp., *finite*) *slope*. If $M(r)$ admits the structure of a Seifert fibred space, we call r a *Seifert slope*. An *essential surface* F in M is an incompressible and orientable surface properly embedded in M, no component of which is ∂-parallel and no 2-sphere component of which bounds a B^3. A connected essential surface F is called a *semi-fibre* if either F is a fibre of a fibration of M over S^1, or F is a common frontier of two 3-dimensional submanifolds of M, each of which is a twisted I-bundle with associated ∂I-bundle F. An essential surface is *strict* [11] if no component of F is a semi-fibre. If the set $\{\partial F\}$ is not empty, it consists of a collection of parallel, simple

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closed curves in ∂M. We call the slope of such a curve obtained from an essential surface F a *boundary slope* and we say that it is a *strict boundary slope* if we can choose F so that it is strict.

This paper deals with the connection between boundary slopes and cyclic, finite, and Seifert slopes. If M is hyperbolic, these last three types of slopes are examples of exceptional slopes, i.e., $M(r)$ is not hyperbolic. We also show that the diameter of the set of strict boundary slopes can be bounded below in terms of the norms of such slopes.

The set of characters of representations $\rho\colon \pi_1(M) \to SL(2, \mathbb{C})$ can be identified with the points of a complex affine algebraic set $X(M)$, which is called the *character variety* [8]. For $\gamma \in \pi_1(M)$ we define the regular function $I_\gamma\colon X(M) \to \mathbb{C}$ by $I_\gamma(\chi_\rho) =$ trace$(\rho(\gamma))$. By the Hurewicz isomorphism, a class $\gamma \in L = H_1(\partial M, \mathbb{Z})$ determines an element of $\pi_1(\partial M) \subset \pi_1(M)$ well defined up to conjugacy. A *norm curve* X_0 is a one-dimensional irreducible component of $X(M)$ on which no I_γ $(\gamma \in L \setminus \{0\})$ is constant. In this paper we will assume that $X(M)$ contains a norm curve. For example, it is known that this assumption holds if M is hyperbolic.

The terminology reflects the fact that we may associate to X_0 a norm $\|\cdot\|_0$ on $H_1(\partial M, \mathbb{R})$ called a *Culler-Shalen norm* in the following manner. Let $\tilde{X}_0$ be the smooth projective model of X_0, which is birationally equivalent to X_0. The birational map is regular at all but a finite number of points of $\tilde{X}_0$, which are called *ideal points* of $\tilde{X}_0$. The function $f_\gamma = I_\gamma^2 - 4$ is regular on X_0, and so can be pulled back to $\tilde{X}_0$. We will also denote the pull-back by f_γ. For $\gamma \in L$, the Culler-Shalen norm $\|\gamma\|_0$ is the degree of $f_\gamma\colon \tilde{X}_0 \to \mathbb{CP}^1$. The norm is extended to $H_1(\partial M, \mathbb{R})$ by linearity.

Fix a norm curve X_0 in the character variety $X(M)$ and denote by $\mathcal{I}$ the set of ideal points on $\tilde{X}_0$. Let s_0 denote the minimal norm of $\|\cdot\|_0$, i.e., $s_0 = \min\{\|\gamma\|_0; \gamma \in L, \gamma \neq 0\}$. If M is hyperbolic, let X_i denote a component of $X(M)$ which contains the character of a discrete, faithful representation. Note that X_i is a norm curve by [7, Proposition 1.1.1]. Let $\|\cdot\|_i$ denote the norm of X_i. We define the *canonical norm* $\|\cdot\|_M$ on $H_1(M, \mathbb{Z})$ to be the sum

$$\|\cdot\|_M = \|\cdot\|_1 + \|\cdot\|_2 + \cdots + \|\cdot\|_k$$

as in [5]. Let s_M denote the minimal norm of $\|\cdot\|_M$.

Note that $L = H_1(M, \mathbb{Z})$ is a lattice of $V = H_1(M, \mathbb{R})$, i.e., a $\mathbb{Z}$-submodule of V which is finitely generated and spans V as a vector space over $\mathbb{R}$. Let $\tilde{L}$ denote a sublattice of L. For an element $\gamma \in L$, let $\tilde{\gamma}$ denote a primitive element in $\tilde{L}$ such that $\tilde{\gamma} = q\gamma$ in L for some $q \in \mathbb{N}$. Let $s_{\tilde{\gamma}}$ denote the slope of $\tilde{\gamma}$ with respect to a basis (α, β) of $\tilde{L}$.

Now we state our main theorem.

Theorem 1. *Let M be a knot exterior and α and β be distinct non-zero elements in $L = H_1(\partial M, \mathbb{Z})$ which span a sublattice $\tilde{L}$. Suppose $X(M)$ contains a norm curve*

with the Culler-Shalen norm $\|\cdot\|_0$. *Then one of the following holds.*
(1) *There are two distinct strict boundary classes* γ *and* δ *such that* $|s_{\tilde{\gamma}}| < \|\beta\|_0/\|\alpha\|_0$ *and* $|s_{\tilde{\delta}}| > \|\beta\|_0/\|\alpha\|_0$. *In case* $\alpha = \mu$ *and* $\beta = a\mu + b\lambda$ *with* $b > 0$, *we have* $|r_\gamma - r_\beta| < \|\beta\|_0/b\|\alpha\|_0$ *and* $|r_\delta - r_\beta| > \|\beta\|_0/b\|\alpha\|_0$.
(2) *There are exactly two distinct strict boundary classes* γ *and* δ *associated to* $\mathcal{I}$. *Moreover they satisfy* $-s_{\tilde{\gamma}} = \|\beta\|_0/\|\alpha\|_0$ *and* $s_{\tilde{\delta}} = \|\beta\|_0/\|\alpha\|_0$. *In case* $\alpha = \mu$ *and* $\beta = a\mu + b\lambda$ *with* $b > 0$, $r_\beta - r_\gamma = \|\beta\|_0/b\|\alpha\|_0$ *and* $r_\delta - r_\beta = \|\beta\|_0/b\|\alpha\|_0$.

If M *is hyperbolic, the same statement also holds for the canonical norm* $\|\cdot\|_M$.

As a direct corollary we have the following.

Corollary 2. *Let* M *be a knot exterior and* α *and* β *be distinct non-zero elements in* $L = H_1(\partial M, \mathbb{Z})$ *which span a sublattice* $\tilde{L}$. *Suppose* $X(M)$ *contains a norm curve with the Culler-Shalen norm* $\|\cdot\|_0$. *Suppose* β *has a particular property and we have an upper bound* c *on the norm of such a class, i.e.,* $\|\beta\|_0 \leq c$. *Then there is a strict boundary class* γ *with* $|s_{\tilde{\gamma}}| \leq c/\|\alpha\|_0$. *In case* $\alpha = \mu$ *and* $\beta = a\mu + b\lambda$ *with* $b > 0$, *we have* $|r_\gamma - r_\beta| \leq c/b\|\alpha\|_0$.

If M *is hyperbolic, the same statement also holds for the canonical norm* $\|\cdot\|_M$.

Theorem 1 and Corollary 2 can be applied to the study of relations between boundary slopes and cyclic, finite, or Seifert slopes. A *small Seifert manifold* is a 3-manifold which admits the structure of a Seifert fibred space whose base orbifold is S^2 with at most three cone points. A small Seifert manifold is irreducible if and only if it is not $S^1 \times S^2$, and Haken if and only if it has infinite first homology.

Corollary 3. *Let* M *be a knot exterior and* α *and* β *be distinct non-zero elements in* $L = H_1(\partial M, \mathbb{Z})$ *which span a sublattice* $\tilde{L}$. *Suppose* $X(M)$ *contains a norm curve*.
(1) *Suppose* β *has minimal norm. Then there is a strict boundary class* γ *with* $|s_{\tilde{\gamma}}| \leq 1$.

Suppose further that M *is hyperbolic and* $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$. *Suppose* α *and* β *are cyclic classes and are not strict boundary classes*. *Then there are two distinct strict boundary classes* γ *and* δ *such that* $|s_{\tilde{\gamma}}| < 1$ *and* $|s_{\tilde{\delta}}| > 1$. *In case* $\alpha = \mu$ *and* $\beta = a\mu + \lambda$, *we have* $|r_\gamma - r_\beta| < 1$ *and* $|r_\delta - r_\beta| > 1$.
(2) *Suppose* M *is hyperbolic and* β *is a finite class. Then there is a strict boundary class* γ *with* $|s_{\tilde{\gamma}}| \leq 3$. *In case* $\alpha = \mu$ *and* $\beta = a\mu + b\lambda$ *with* $b > 0$, *we have* $|r_\gamma - r_\beta| \leq 3/b$.
(3) *Suppose that there is a class* δ *in* L *such that* $\mathrm{Hom}(\pi_1(M(\delta)), PSL(2, \mathbb{C}))$ *contains only diagonalisable representations*. *Suppose* $M(\beta)$ *is an irreducible non-Haken small Seifert manifold*. *Then there is a strict boundary class* γ *with* $|s_{\tilde{\gamma}}| \leq 1 + 2A/s_0$, *where* A *is the number of characters* $\chi_\rho \in X_0$ *of non-abelian representations* $\rho \in R_0$ *with* $\rho(\beta) = \pm I$. *In case* $\alpha = \mu$ *and* $\beta = a\mu + b\lambda$ *with* $b > 0$, *we have* $|r_\gamma - r_\beta| \leq (1 + 2A/s_0)/b$.

Note that if M is the exterior of a (hyperbolic) knot in S^3 then it satisfies the conditions of Corollary 3, i.e., $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\mathrm{Hom}(\pi_1(S^3), PSL(2, \mathbb{C}))$ contains only diagonalisable representations. Corollary 3 (1) improves a result of Dunfield [12] who showed that for a cyclic slope r there is a boundary slope in $(r - 1, r + 1)$.

If M is the exterior of a hyperbolic knot in a homotopy 3-sphere, we can take a preferred meridian-longitude pair for (μ, λ). Then $r_\beta = a/b$ is the usual slope. In this case, by [4, Theorem 1.1], b in Corollary 3 (2) is either 1 or 2 and, for the fillings in Corollary 3 (2) and (3), $b = 1$ is conjectured. (See Conjecture A in problem 1.77 of [17].)

Next we consider a relationship between the diameter of the set of strict boundary slopes and the norms of these slopes. Let $\mathcal{B}$ be the set of strict boundary slopes associated to $\mathcal{I}$ with respect to a basis (μ, λ) of L. As in [10], if $\infty \notin \mathcal{B}$, let $\operatorname{diam}\mathcal{B}$ denote the *diameter* of $\mathcal{B}$, which is defined to be the difference between the greatest and least elements of $\mathcal{B}$. From Theorem 1 we obtain the following corollary.

Corollary 4. *Let M be a knot exterior with $\infty \notin \mathcal{B}$. Suppose $X(M)$ contains a norm curve X_0 with the norm $\|\cdot\|_0$. Let β be a strict boundary class associated to an ideal point of $\mathcal{I}$ with $r_\beta = a/b$. Then* $\operatorname{diam}\mathcal{B} > \|\beta\|_0/b\|\mu\|_0$.

Note that if M is hyperbolic, by Proposition 9.1 of [5] we have $\|\beta\|_0/b\|\mu\|_0 = \|\beta\|_M/b\|\mu\|_M$.

In [10] Culler and Shalen showed that if M is the exterior of a non-trivial, non-cable knot in an orientable 3-manifold with cyclic fundamental group, then $\operatorname{diam}\mathcal{B} \geq 2$.

The structure of the paper is as follows. In the next section we give a brief introduction to the character variety and the Culler-Shalen norm and how they apply to the study of exceptional surgeries. We prove Theorem 1 and Corollaries 2, 3, and 4 in Section 3. Finally, in Section 4 we discuss examples: the $(-2, 3, n)$ pretzel knots and the twist knots.

2. Character variety, Culler-Shalen norm, and exceptional surgery

In this section we recall the definitions of character varieties and Culler-Shalen norms. The main references are the first chapter of [7] and [8]. Applications to finite surgery are developed in [4, 5].

Let $R(M)$ denote the set of representations $\rho\colon \pi_1(M) \to SL(2, \mathbb{C})$. It is easy to show that $R(M)$ is a complex affine algebraic set. The *character* of an element $\rho \in R(M)$ is the function $\chi_\rho\colon \pi_1(M) \to \mathbb{C}$ defined by the trace map $\chi_\rho(\gamma) = \mathrm{trace}(\rho(\gamma))$. The set of characters of the representations in $R(M)$ is also a complex affine algebraic set [8]. We call it the *character variety* of $\pi_1(M)$ and denote it by $X(M)$.

For $\gamma \in \pi_1(M)$ we define the regular function $I_\gamma\colon X(M) \to \mathbb{C}$ by $I_\gamma(\chi_\rho) = \mathrm{trace}(\rho(\gamma))$. The Hurewicz isomorphism induces an isomorphism $H_1(\partial M, \mathbb{Z}) \simeq \pi_1(\partial M) \subset \pi_1(M)$. So we can identify $L = H_1(\partial M, \mathbb{Z})$ with a subgroup of $\pi_1(M)$, well defined up to conjugacy. Thus each element $\gamma \in L$ determines a regular function I_γ. A *norm curve* X_0 is a one-dimensional irreducible component of $X(M)$ such that, for each $\gamma \in L \setminus \{0\}$, I_γ is not constant on X_0. By [9, Proposition 2], any irreducible component of $X(M)$ containing the character of a discrete, faithful representation is a norm curve of $X(M)$. If M is hyperbolic, $X(M)$ contains the character of a discrete, faithful representation, namely the holonomy representation, so there will be at least one norm curve in the character variety.

In [8], Culler and Shalen proved that each ideal point detects an essential surface which is not a fibre over S^1. They called a boundary slope ‘strict’ if it was the boundary class of such a non-fibre essential surface, and many papers in the field used this terminology. Recently, in [11], Culler and Shalen proved that the detected essential surface cannot be a semi-fibre and they again used the term ‘strict’ to describe an essential surface which is not a semi-fibre. Since the papers to which we refer only use the fact that the essential surface detected by an ideal point is strict, we can replace the meaning of ‘strict’ used in [8] with that of [11]. So, in this paper we use the word ‘strict’ in this new sense.

The *Culler-Shalen norm* is a norm, on the real vector space $H_1(\partial M, \mathbb{R})$, associated to a norm curve X_0 in the following manner. Let $f_\gamma = I_\gamma^2 - 4$. Since this function is regular, it can be pulled back to $\tilde{X}_0$, where $\tilde{X}_0$ is the smooth projective completion of X_0. We again denote the pull-back by f_γ. For $\gamma \in L$, define $\|\gamma\|_0$ to be the degree of $f_\gamma \colon \tilde{X}_0 \to \mathbb{CP}^1$. It is shown in [7, Proposition 1.1.2] that there exists a norm $\|\gamma\|_0$ on $H_1(\partial M, \mathbb{R})$ satisfying (i) $\|\gamma\|_0 = \text{degree}\, f_\gamma$ when $\gamma \in L$, and (ii) the unit ball is a finite-sided polygon whose vertices are rational multiples of strict boundary classes in L. We call this norm the Culler-Shalen norm.

Let β be a finite class which is not a strict boundary class. Following a classification of Milnor [19], Boyer and Zhang [4] say that β falls into one of six types C, D, T, O, I, or Q. The notation refers to the fact that $\pi_1(M(r))$ is an extension of a Cyclic, Dihedral, Tetrahedral, etc. group.

By [7, Corollary 1.1.4], a cyclic or C-type class which is not a boundary class realizes the minimal norm on $L = H_1(\partial M, \mathbb{Z})$. In general, for a finite slope $r = r_\beta$ which is not a boundary slope, β realizes the minimal norm on a sublattice $\tilde{L}$ of some index q. This is Proposition 9.3 of [5] which we restate here:

Theorem 5 (Proposition 9.3 [5]). *Let M be a hyperbolic knot exterior. Let $s_M = \min\{\|\gamma\|_M ; \gamma \in L = H_1(\partial M, \mathbb{Z}), \gamma \neq 0\}$. Suppose that β is a finite class and not a strict boundary class. Then there is an integer $q \in \{1, \dots, 5\}$ and an index q sublattice $\tilde{L}$ of L such that $\|\beta\|_M \leq \|\gamma\|_M$ for all $0 \neq \gamma \in \tilde{L}$.*

Moreover,

(1) *if β is C-type, then $\|\beta\|_M = s_M$, i.e., $q = 1$;*
(2) *if β is D-type or Q-type, then $\|\beta\|_M \leq 2s_M$ and $q \leq 2$;*
(3) *if β is T-type, then $\|\beta\|_M \leq s_M + 4$ and $q \leq 3$;*

(4) *if* β *is I-type, then* $\|\beta\|_M \leq s_M + 8$ *and* $q \in \{1, 2, 3, 5\}$; *and*
(5) (a) *if* β *is O-type and* $H_1(M, \mathbb{Z})$ *has no non-trivial even torsion, then* $\|\beta\|_M \leq s_M + 6$ *and* $q \in \{2, 4\}$ *and* (b) *if* β *is O-type and* $H_1(M, \mathbb{Z})$ *has non-trivial even torsion, then* $\|\beta\|_M \leq s_M + 12$ *and* $q \leq 3$.

Note that there may be more than one choice of q for a given finite slope. For example, the O-type surgery 22 of the $(-2, 3, 9)$ pretzel knot realizes the minimal norm on sublattices of index $q = 2, 3,$ and 4.

Next we refer to a result of Boyer and Ben Abdelghani.

Theorem 6 (Theorem C [1]). *Let* M *be a knot exterior. Suppose that there is a class* δ *in* L *such that* $\mathrm{Hom}(\pi_1(M(\delta)), PSL(2, \mathbb{C}))$ *contains only diagonalisable representations. Suppose, for a non-boundary class* β, $M(\beta)$ *is an irreducible non-Haken small Seifert manifold. Then* $\|\beta\|_0 = s_0 + 2A$, *where* A *is the number of characters* $\chi_\rho \in X_0$ *of non-abelian representations* $\rho \in R_0$ *with* $\rho(\beta) = \pm I$.

3. Proofs

Fix a norm curve X_0 in the character variety X with set of ideal points $\mathcal{I}$. Let $\Pi_x(f_\alpha)$ denote the order of the pole of f_α at $x \in \mathcal{I}$.

We start by stating the main tool of our proof.

Proposition 7. *Let* M *be a knot exterior and* α *and* β *be elements in* $L = H_1(\partial M, \mathbb{Z})$. *Suppose* $X(M)$ *contains a norm curve* X_0 *with the Culler-Shalen norm* $\|\cdot\|_0$. *Then either*
(1) *there are two distinct ideal points* x *and* y *such that* $\Pi_x(f_\alpha)/\|\alpha\|_0 < \Pi_x(f_\beta)/\|\beta\|_0$ *and* $\Pi_y(f_\alpha)/\|\alpha\|_0 > \Pi_y(f_\beta)/\|\beta\|_0$, *or*
(2) *for any ideal point* z, *we have* $\Pi_z(f_\alpha)/\|\alpha\|_0 = \Pi_z(f_\beta)/\|\beta\|_0$.
If M *is hyperbolic, the same statement also holds for the canonical norm* $\|\cdot\|_M$.

Proof. From the definition of the norm $\|\cdot\|_0$, we have, $\|\alpha\|_0 = \sum_{x\in\mathcal{I}} \Pi_x(f_\alpha)$. Hence we have $\sum_{x\in\mathcal{I}} \Pi_x(f_\alpha)/\|\alpha\|_0 = \sum_{x\in\mathcal{I}} \Pi_x(f_\beta)/\|\beta\|_0$. Hence if (2) does not hold, then (1) holds.

The same argument applies to $\|\cdot\|_M$. □

Lemma 8. *Suppose* M *has a norm curve* X_0 *with the norm* $\|\cdot\|_0$. *Then there are two ideal points in* $\mathcal{I}$ *whose associated strict boundary classes are distinct*.

Proof. Suppose, for a contradiction, that there is at most one strict boundary class associated to $\mathcal{I}$. If there is no strict boundary class associated to $\mathcal{I}$, we have $\|\gamma\|_0 = 0$ for any element $\gamma \in L$. If each strict boundary class associated to $\mathcal{I}$ is equal to $\gamma \in L$, then we have $\|\gamma\|_0 = 0$. In both cases we have contradictions to the fact that $\|\cdot\|_0$ is a norm. □

Proof of Theorem 1. Suppose (1) of Proposition 7 holds. Then we have ideal points x and y satisfying the inequalities described in the proposition.

First suppose $\Pi_x(f_\alpha) > 0$. We have $\Pi_x(f_\beta)/\Pi_x(f_\alpha) > \|\beta\|_0/\|\alpha\|_0$ and $\Pi_y(f_\beta)/\Pi_y(f_\alpha) < \|\beta\|_0/\|\alpha\|_0$. Let γ and δ be the strict boundary classes associated to the ideal points y and x respectively. Then, using the proof of [7, Lemma 1.4.1], we see that the number $\Pi_x(f_\beta)/\Pi_x(f_\alpha)$ (resp., $\Pi_y(f_\beta)/\Pi_y(f_\alpha)$) is equal to $|s_{\tilde{\gamma}}|$ (resp., $|s_{\tilde{\delta}}|$).

Next suppose $\Pi_x(f_\alpha) = 0$. This happens only in case $\Pi_x(f_\alpha)/\|\alpha\|_0 < \Pi_x(f_\beta)/\|\beta\|_0$. Then α is a strict boundary slope and satisfies the desired condition $|s_{\tilde{\alpha}}| = \infty > \|\beta\|_0/\|\alpha\|_0$.

In case $\alpha = \mu$ and $\beta = a\mu + b\lambda$, by changing coordinates we have $s_\gamma = b(r_\gamma - r_\beta)$. Hence we have the conclusions of Theorem 1 (1).

Suppose (2) of Proposition 7 holds. Note that there is an ideal point x such that $\Pi_x(f_\alpha) > 0$, otherwise $\|\alpha\|_0 = 0$ and this is a contradiction to the definition of the Culler-Shalen norm. Hence for the strict boundary slope γ associated to the ideal point x, we have $|s_{\tilde{\gamma}}| = \|\beta\|_0/\|\alpha\|_0$. There are at least two distinct strict boundary classes associated to ideal points in $\mathcal{I}$ by Lemma 8. Hence there are exactly two distinct boundary classes, say γ and δ, such that $-s_{\tilde{\gamma}} = \|\beta\|_0/\|\alpha\|_0$ and $s_{\tilde{\delta}} = \|\beta\|_0/\|\alpha\|_0$. (Here we assumed without loss of generality that $r_\delta > 0$.) In case $\alpha = \mu$ and $\beta = a\mu + b\lambda$, by changing coordinates we have the conclusion.

The proof for the canonical norm when M is hyperbolic is exactly the same. □

Next we prove Corollary 3. We will prove the three parts separately. In each case we calculate the ratio $t = \|\beta\|_0/\|\alpha\|_0 = \|\beta\|_M/\|\alpha\|_M$ and apply Theorem 1 and Corollary 2.

First we remark that Theorem 1 (2) does not occur when there are two distinct cyclic classes. Indeed, Dunfield proved the following result.

Lemma 9 (Lemma 4.4 and 4.5 [12]). *Suppose M is hyperbolic and $H^1(M, \mathbb{Z}_2) = \mathbb{Z}_2$. Let α and β be cyclic classes. Then f_β/f_α cannot be constant on X_0.*

He then proved that $|s_{\tilde{\gamma}}| < 1$. Our Corollary 3 (1) asserts additionally the opposite inequality $|s_{\tilde{\delta}}| > 1$.

Proof of Corollary 3 (1). If β has minimal norm, then $t \leq 1$ and we have a proof of the first assertion. If α and β are both cyclic, then $t = 1$. Due to Proposition 1.1.3 of [7], the function f_β/f_α cannot have poles except at ideal points. Hence if (2) of Theorem 1 occurs then the function f_β/f_α also has no poles at the ideal points and is, therefore, constant. However this contradicts Lemma 9. Thus (1) of Theorem 1 holds. Hence, we can find two distinct strict boundary classes γ and δ with $|s_{\tilde{\gamma}}| < 1$ and $|s_{\tilde{\delta}}| > 1$. □

For the finite slope case, we quote a lemma of [5].

Lemma 10 (Lemma 9.1 [5]). *If M is hyperbolic,* $4 \leq 2|H_1(M, \mathbb{Z}_2)| \leq s_M$ *holds.*

Proof of Corollary 3 (2). If β is a strict boundary class, then β satisfies the conclusion. Hence we assume that β is not a strict boundary class. First note that for a sublattice $\tilde{L}$ of index q, $q\alpha$ is in $\tilde{L}$ for any element $\alpha \in L$ and if β realizes the minimal norm on a $\tilde{L}$, then $\|\beta\|_M \leq \|q\alpha\|_M = q\|\alpha\|_M$. Hence we have $t \leq q$. By Lemma 10 we have $s_M \geq 4$. Then by using this fact and Theorem 5: if β is C-type, $t \leq 1$; if β is D-type, $t \leq 2$; if β is T-type, from $\|\beta\|_M \leq s_M + 4$ we have $t \leq 2$; if β is I-type, from $\|\beta\|_M \leq s_M + 8$ we have $t \leq 3$; if β is O-type and $H_1(M, \mathbb{Z})$ has no non-trivial even torsion, from $\|\beta\|_M \leq s_M + 6$ we have $t \leq 5/2$; finally, if β is O-type and $H_1(M, \mathbb{Z})$ has non-trivial even torsion, then $t \leq q \leq 3$. Hence we have $t = \|\beta\|_M / \|\alpha\|_M \leq 3$. □

Proof of Corollary 3 (3). We assume that β is not a strict boundary class. By Theorem 6, we have $\|\beta\|_0 = s_0 + 2A$. Hence we have $t \leq 1 + 2A/s_0$. □

Proof of Corollary 4. We take α to be μ and β to be a strict boundary slope associated to an ideal point, say x. Since $\infty \notin \mathcal{B}$, we have $\Pi_y(f_\mu) > 0$ for any ideal point $y \in \mathcal{I}$. Since $\Pi_x(f_\beta) = 0$, we have $\Pi_x(f_\beta)/\|\beta\|_0 < \Pi_x(f_\mu)/\|\mu\|_0$, i.e., case (1) in Theorem 1 always holds. Then we can find a strict boundary class δ with $|r_\delta - r_\beta| > \|\beta\|_0 / b\|\mu\|_0$. Since $\operatorname{diam} \mathcal{B} \geq |r_\delta - r_\beta|$, we have the conclusion. □

4. Examples

Corollary 3 shows that a cyclic, finite, or Seifert slope lies near a strict boundary slope r_γ. We verify this conclusion for the twist knots and the $(-2, 3, n)$ pretzel knots by taking α to be the meridian and β to be one of these exceptional classes. We will also verify the second assertion of Theorem 1 (1) by evaluating $t = \|\beta\|_0/\|\alpha\|_0$ and observing that $|r_\gamma - r_\beta| < t$. (The exceptional slopes are all integral, so that $b = 1$.)

For each of these knots, there is only one norm curve X_0 in the character variety. Moreover, with the exception of the figure eight knot (which is a kind of twist knot) the ideal points of $\tilde{X}_0$ are associated to three different strict boundary slopes. This means that (1) of Proposition 7 holds, since (2) would imply that there are exactly two distinct strict boundary slopes associated to the norm curve (see Theorem 1). For each knot we determine the strict boundary slopes associated to the ideal points x and y of Proposition 7 (1).

In addition, we calculate the diameter of $\mathcal{B}$ for each knot and compare it with the best estimate obtained from Corollary 4.

4.1. The (−2, 3, *n*) pretzel knots. We will assume n is odd and $n \neq 1, 3, 5$ so that the $(-2, 3, n)$ pretzel knot is hyperbolic. The Culler-Shalen seminorms of this knot are worked out explicitly in [18] where it is shown that there is only one norm curve X_0 in the character variety. We first examine the $(-2, 3, 7)$ and $(-2, 3, 9)$ knots which have cyclic and finite slopes before turning to the remaining pretzel knots which have Seifert slopes.

4.1.1. The (−2, 3, 7) pretzel knot. The finite surgeries of the $(-2, 3, 7)$ pretzel knot are classified in [4, Example 10.1]. There are cyclic surgeries on the meridian, and at slopes 18 and 19, as well as an I-type finite surgery at slope 17. The boundary slopes are given in [15] as $0, 16, 37/2$, and 20. The longitude 0 is the slope of a fibre in a fibration [14] while the remaining boundary slopes correspond to ideal points of $\tilde{X}_0$ and are therefore strict. Indeed, the calculation of [18] shows that there are ideal points u and v of $\tilde{X}_0$ associated to the slopes 16 and 20 respectively with $\Pi_u(f_\mu) = \Pi_v(f_\mu) = 2$.

For the boundary slope $37/2$, the result of [18] shows that $\Pi_x(f_\mu)$ summed over the ideal points associated with $37/2$ is eight. Furthermore, using well-known methods for the toric compactification of a plane curve with respect to its Newton polygon (see [21]), we can conclude that there are two ideal points w_1 and w_2 associated to the slope $37/2$. In particular, they satisfy $\Pi_{w_1}(f_\mu) = \Pi_{w_2}(f_\mu) = 4$ (by [4, Lemma 6.2 (1)], $4 \mid \Pi(f_\mu)$ for any ideal point associated to the slope $37/2$).

Given $\Pi_x(f_\mu)$, the order of pole of any other f_γ is determined by the formula (see [4, Lemma 6.2 (1)])

$$\Pi_x(f_\gamma) = \frac{\Delta(r_\gamma, r_\beta)}{\Delta(r_\mu, r_\beta)} \Pi_x(f_\mu), \tag{1}$$

where β is the boundary class associated to x and $\Delta(a/b, c/d) = |ad - bc|$ is the minimal geometric intersection of the two slopes. For example, Table 1 gives the degree of pole of various functions at the four ideal points.

Let $\alpha = \mu$ (then [18], $\|\alpha\|_0 = s_0 = 12$) and let β be one of the other cyclic or finite classes. We will determine the boundary slope associated to the ideal points x

Ideal point	Associated boundary slope	$\Pi(f_\mu)$	$\Pi(f_{17})$	$\Pi(f_{18})$	$\Pi(f_{19})$
u	16	2	2	4	6
v	20	2	6	4	2
w_1	37/2	4	6	2	2
w_2	37/2	4	6	2	2

Table 1. Order of pole at ideal points of $\tilde{X}_0$ for the $(-2, 3, 7)$ pretzel knot

and y of Proposition 7 (1).

If $r_\beta = 19$, $\|\beta\|_0 = 12$ so that $t = \|\beta\|_0/\|\alpha\|_0 = 1$. Here, $\Pi_u(f_\alpha)/\|\alpha\|_0 < \Pi_u(f_\beta)/\|\beta\|_0$ while $\Pi_{w_i}(f_\alpha)/\|\alpha\|_0 > \Pi_{w_i}(f_\beta)/\|\beta\|_0$ for $i = 1, 2$. Thus, in the Proposition, $x = u$ and $y = w_i$. Moreover, the boundary slopes of Corollary 3 are the associated boundary slopes $r_\gamma = 37/2$ and $r_\delta = 16$. These also verify the second assertion of Theorem 1 (1) since $|r_\gamma - r_\beta| = |37/2 - 19| = 1/2 < t = 1$ and $|r_\delta - r_\beta| = |16 - 19| = 3 > t = 1$.

For $r_\beta = 18$, again, $\|\beta\|_0 = 12$ and $t = 1$. Here, $\Pi_u(f_\alpha)/\|\alpha\|_0 < \Pi_u(f_\beta)/\|\beta\|_0$ and $\Pi_v(f_\alpha)/\|\alpha\|_0 < \Pi_v(f_\beta)/\|\beta\|_0$, while $\Pi_{w_i}(f_\alpha)/\|\alpha\|_0 > \Pi_{w_i}(f_\beta)/\|\beta\|_0$ for $i = 1, 2$. Therefore, in Corollary 3, we again have $r_\gamma = 37/2$ while 16 and 20 are both valid choices for r_δ. That is, $|r_\gamma - r_\beta| = 1/2 < t = 1$ and $|r_\delta - r_\beta| = 2 > t = 1$.

Finally, $r_\beta = 17$ has norm $\|\beta\|_0 = 20$ [18] so that $t = 20/12 = 5/3$. Here, $\Pi_v(f_\alpha)/\|\alpha\|_0 < \Pi_v(f_\beta)/\|\beta\|_0$ while $\Pi_u(f_\alpha)/\|\alpha\|_0 > \Pi_u(f_\beta)/\|\beta\|_0$ and $\Pi_{w_i}(f_\alpha)/\|\alpha\|_0 > \Pi_{w_i}(f_\beta)/\|\beta\|_0$ for $i = 1, 2$. Hence, 37/2 and 16 are strict boundary slopes r_γ near the finite slope $r_\beta = 17$. Note that $|r_\gamma - r_\beta| < 5/3 = t$ in both cases. In other words, for finite slopes, t/b will often give us a better estimate than the bound of $3/b$ stated in Corollary 3.

The diameter of the set of strict boundary slopes is $20-16 = 4$. Using Corollary 4, we obtain the lower bound $\|16\|_0/\|\mu\|_0 = 28/12 = 7/3$.

4.1.2. The (−2, 3, 9) pretzel knot. The finite surgeries of the $(-2, 3, 9)$ pretzel knot are classified in [18]. There are a cyclic meridional surgery, an O-type finite surgery at slope 22, and an I-type finite surgery at slope 23. The boundary slopes may be calculated using the methods of [15, 13] as $0, 16, 67/3$ and 24, but 0 is not strict [14].

The calculation of [18] shows that there are ideal points u and v associated to the slopes 16 and 24. The slope 67/3 also has ideal points and, using the knowledge of toric compactification again, we can conclude that there are two ideal points w_1, w_2 associated to the slope 67/3. Note that the A-polynomial of the $(-2, 3, 9)$-pretzel knot can be obtained by using the formula in [22].

Let $\alpha = \mu$, $r_{\beta_1} = 22$ and $r_{\beta_2} = 23$. Then [18], $\|\alpha\|_0 = 16$, $\|\beta_1\|_0 = 20$ and $\|\beta_2\|_0 = 24$ so that $t_1 = \|\beta_1\|_0/\|\alpha\|_0 = 5/4$ and $t_2 = 3/2$.

Ideal point	Associated boundary slope	$\Pi(f_\mu)$	$\Pi(f_{22})$	$\Pi(f_{23})$
u	16	2	12	14
v	24	2	4	2
w_1	67/3	6	2	4
w_2	67/3	6	2	4

Table 2. Order of pole at ideal points of $\tilde{X}_0$ for the $(-2, 3, 9)$ pretzel knot

For β_1, $\Pi_u(f_\alpha)/\|\alpha\|_0 < \Pi_u(f_{\beta_1})/\|\beta_1\|_0$ and $\Pi_v(f_\alpha)/\|\alpha\|_0 < \Pi_v(f_{\beta_1})/\|\beta_1\|_0$ while $\Pi_{w_i}(f_\alpha)/\|\alpha\|_0 > \Pi_{w_i}(f_{\beta_1})/\|\beta_1\|_0$ for $i = 1, 2$. In other words, we can choose $y = w_i$ in Proposition 7 while $x = u$ and $x = v$ are both valid choices. In Corollary 3, we have $r_\gamma = 67/3$. Note that $|r_\gamma - r_{\beta_1}| = |67/3 - 22| = 1/3 < 5/4 = t_1$.

For β_2, $x = u$ while $y = v$ and $y = w_i$ ($i = 1, 2$) are both correct in Proposition 7. Consequently, $r_\gamma = 67/3$ and $r_\gamma = 24$ both satisfy Corollary 3. Again, these in fact satisfy the stronger inequality $|r_\gamma - r_{\beta_2}| < 3/2 = t_2$.

The diameter of the set of strict boundary slopes is $24-16 = 8$. Using Corollary 4, we obtain the lower bound $\|16\|_0/\|\mu\|_0 = 92/16 = 23/4$.

4.1.3. Pretzel knots with Seifert slopes. For n odd and $n \neq 1, 3, 5, 7, 9$, the $(-2, 3, n)$ pretzel knot admits Seifert surgeries at slopes $r_{\beta_1} = 2n + 4$ and $r_{\beta_2} = 2n + 5$ (see [2]). The number of ideal points may be quite large for these knots, so we will work with a slight reformulation of Proposition 7. Using Equation 1,

$$\frac{\Pi_x(f_\alpha)}{\|\alpha\|_0} > \frac{\Pi_x(f_\beta)}{\|\beta\|_0} \quad \text{if and only if} \quad \frac{\Delta(r_\alpha, r_\delta)}{\|\alpha\|_0} > \frac{\Delta(r_\beta, r_\delta)}{\|\beta\|_0} \tag{2}$$

where δ is the strict boundary class associated to the ideal point x. The boundary slopes and norm of the $(-2, 3, n)$ pretzel knot differ depending on the sign of n, so we consider the two cases separately.

If $n \geq 11$, the boundary slopes are [15, 13] $0, 16, 2n{+}6$, and $(n^2{-}n{-}5)/((n{-}3)/2)$, but 0 is not strict [14]. The norm depends on whether or not $3 \mid n$: $\|\alpha\|_0 = \|\mu\|_0 = s_0 = 3(n-3)$ (respectively, $3n-11$), $\|\beta_1\|_0 = 6(n-5)$ (resp., $6n-34$), and $\|\beta_2\|_0 = 7n-37$ (resp., $7n-39$) when $3 \nmid n$ (resp., $3 \mid n$). So, $t_1 = 2(n-5)/(n-3)$ (respectively, $(6n-34)/(3n-11)$) and $t_2 = (7n-37)/(3(n-3))$ (resp., $(7n-39)/(3n-11)$). Thus, $\Delta(r_\mu, r_\delta)/\|\mu\|_0 > \Delta(r_{\beta_1}, r_\delta)/\|\beta_1\|_0$ only for the boundary slope $\delta = (n^2-n-5)/((n-3)/2)$.

Indeed, the number of non-abelian characters for this Seifert slope is $A_1 = (3/2)(n-7)$ (respectively, $(1/2)(3n-23)$) when $3 \nmid n$ (resp., $3 \mid n$), and, for $r_\gamma = (n^2-n-5)/((n-3)/2)$, $|r_\gamma - r_{\beta_1}| = 2/(n-3) < t_1 = 1 + 2A_1/s_0$ in agreement with Corollary 3. For the other Seifert slope, $\beta_2 = 2n+5$, the inequality of Proposition 7, $\Delta(r_\mu, r_\delta)/\|\mu\|_0 > \Delta(r_{\beta_2}, r_\delta)/\|\beta_2\|_0$, holds for both $\delta = 2n+6$ and $(n^2-n-5)/((n-3)/2)$. Indeed, the number of non-abelian characters is $A_2 = 2(n-7)$ and $|r_\gamma - r_{\beta_2}| \leq 1 < t_2 = 1 + 2A_2/s_0$ for both $r_\gamma = 2n+6$ and $r_\gamma = (n^2-n-5)/((n-3)/2)$.

When $n \geq 11$, the diameter of the set of strict boundary slopes of the $(-2, 3, n)$ pretzel knot is $2n+6-16 = 2(n-5)$. Using Corollary 4, we obtain the lower bound $\|16\|_0/\|\mu\|_0 = (6n^2-56n+126)/(3n-9)$ (respectively, $(6n^2-60n+146)/(3n-11)$) when $3 \nmid n$ (resp., $3 \mid n$). Thus, the difference, $(\operatorname{diam} \mathcal{B} - \|16\|_0/\|\mu\|_0)$, is $13/6$ when $n = 11$ and increases towards $8/3$ as n tends to infinity.

If $n < 0$, the boundary slopes are [15, 13] $0, 10, 2n+6$, and $2(n+1)^2/n$. The longitude 0 is not strict unless $n = -1$ or $n = -3$ [14]. Again, the norm depends on

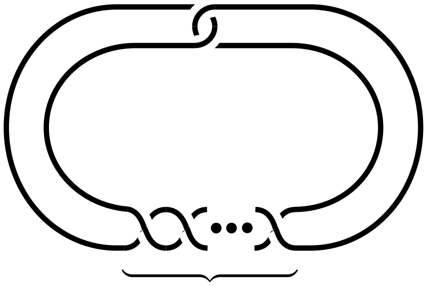

Fig. 1. The twist knot K_n.

whether or not $3 \mid n$ (see [18]): $\|\mu\|_0 = s_0 = 3(1-n)$ (respectively, $1-3n$), $\|\beta_1\|_0 = 6(3-n)$ (resp., $2(7-3n)$), and $\|\beta_2\|_0 = 15-7n$ (resp., $13-7n$) when $3 \nmid n$ (resp., $3 \mid n$). So, $t_1 = 2(3-n)/(1-n)$ (respectively, $2(7-3n)/(1-3n)$) and $t_2 = (15-7n)/(3(1-n))$ (resp., $(13-7n)/(1-3n)$). Thus, both $\delta = 2n+6$ and $\delta = 2(n+1)^2/n$ will satisfy the Proposition 7 inequality (see Equation 2) $\Delta(r_\mu, r_\delta)/\|\mu\|_0 > \Delta(r_\beta, r_\delta)/\|\beta\|_0$ for the Seifert slopes $\beta_1 = 2n+4$ and $\beta_2 = 2n+5$. Indeed, for β_1, the number of non-abelian characters is $A_1 = (3/2)(5-n)$ (respectively, $(1/2)(13-3n)$) when $3 \nmid n$ (resp., $3 \mid n$) and $|r_\gamma - r_{\beta_1}| < t_1 = 1 + 2A_1/s_0$ for both $r_\gamma = 2n+6$ and $r_\gamma = 2(n+1)^2/n$. For β_2, we have $A_2 = 2(3-n)$ and, again, both choices of r_γ verify Corollary 3: $|r_\gamma - r_{\beta_2}| < t_2 = 1 + 2A_2/s_0$.

When $n < 0$, the diameter of the set of strict boundary slopes of the $(-2,3,n)$ pretzel knot is $10 - 2(n+1)^2/n = 6 - 2n - 2/n$. Using Corollary 4, we obtain the lower bound $\|10\|_0/\|\mu\|_0 = (6n^2 - 18n + 8)/(3-3n)$ (respectively, $(6n^2 - 14n)/(1-3n)$) when $3 \nmid n$ (resp., $3 \mid n$). Thus, the difference, $(\operatorname{diam} \mathcal{B} - \|10\|_0/\|\mu\|_0)$, is $14/3$ when $n = -1$ and decreases towards 2 as n tends to negative infinity.

4.2. The twist knot K_n. Fig. 1 shows the twist knot K_n. We will assume $n \neq 0, 1$ so that the twist knot K_n is hyperbolic. These knots have Seifert slopes at $-1, -2$, and -3. Burde [6] showed that the character variety has only one norm curve X_0 and the associated Culler-Shalen seminorm is determined in [3]. Ohtsuki [20] has enumerated the ideal points x of these knots and demonstrated that $\Pi_x(f_\mu) = 2$ at each ideal point. Since the norm and boundary slopes depend on the sign of n, we consider two cases.

If $n \geq 2$, the boundary slopes are [16] $0, -4$, and $-(4n+2)$ and these are all strict [3, 20]. The norms are $\|\mu\|_0 = s_0 = 4n-2$, $\|-1\|_0 = 2(8n-3)$, $\|-2\|_0 = 8(2n-1)$, and $\|-3\|_0 = 2(8n-5)$. For each of the Seifert slopes β, the inequality of Proposition 7 (see Equation 2), $\Delta(r_\mu, r_\delta)/\|\mu\|_0 > \Delta(r_\beta, r_\delta)/\|\beta\|_0$, obtains when δ is either of the boundary slopes 0 or -4. Indeed, the number of non-abelian characters is $A = 6n-2$ (respectively, $6n-3, 6n-4$) for the Seifert slope -1 (resp., $-2, -3$) so that $|r_\gamma - r_\beta| \leq$

$t = 1 + 2A/s_0$ whenever r_β is one of the three Seifert slopes and r_γ is one of the boundary slopes 0 or -4, in agreement with Corollary 3.

The diameter of $\mathcal{B}$ is $4n+2$ when $n \geq 2$. Using Corollary 4, we obtain the lower bound $\|-(4n+2)\|_0/\|\mu\|_0 = 16n(n-1)/(4n-2) = 8n(n-1)/(2n-1)$. The difference between the diameter and this bound is $14/3$ when $n = 2$ and decreases towards 4 as n tends to infinity.

If $n \leq -1$, the boundary slopes are [16] 0, -4 and $-4n$ and these are strict as long as $n \leq -2$. For the figure eight knot, K_{-1}, 0 is not a strict boundary slope (but ± 4 are). The minimal norm is [3] $\|\mu\|_0 = -4n$ and the Seifert slopes $-1, -2$, and -3 all have norm $-16n$. Again, the boundary slopes 0 and -4 satisfy the inequality of Proposition 7, $\Delta(r_\mu, r_\delta)/\|\mu\|_0 > \Delta(r_\beta, r_\delta)/\|\beta\|_0$, for each of the Seifert slopes β. Indeed, the number of non-abelian characters is $A = -6n$ for each of the three Seifert slopes so that $|r_\gamma - r_\beta| \leq 4 = t = 1 + 2A/s_0$, in accord with Corollary 3, whenever r_β is Seifert and r_γ is one of the boundary slopes 0 or -4.

The diameter of $\mathcal{B}$ is $4 - 4n$ when $n \leq -1$. Using Corollary 4, we obtain the lower bound $\|-4n\|_0/\|\mu\|_0 = 16n^2/(-4n) = -4n$.

The figure eight knot, K_{-1}, is of special interest as it provides an example of Theorem 1 (2) and Proposition 7 (2). For this knot, the norm curve $\tilde{X}_0$ has only two associated strict boundary slopes 4 and -4. Let $r_\alpha = a/b$ and $r_\beta = c/d$. Then α and β will satisfy part 2 of Theorem 1 and Proposition 7 provided $16bd = ac$. For example, let $r_\alpha = 1/0$ (so that $\alpha = \mu$) and $r_\beta = 0/1$ ($\beta = \lambda$). Then, $\|\beta\|_0/\|\alpha\|_0 = 16/4 = 4$, so that $r_\beta - r_\gamma = \|\beta\|_0/\|\alpha\|_0$ for $r_\gamma = -4$, and $r_\delta - r_\beta = \|\beta\|_0/\|\alpha\|_0$ for $r_\delta = 4$ (compare Theorem 1 (2)). For ideal points u associated to the slope 4, we have $\Delta(r_\alpha, 4)/\|\alpha\|_0 = 1/4$ and $\Delta(r_\beta, 4)/\|\beta\|_0 = 4/16 = 1/4$. Therefore, (compare Equation 2) $\Pi_u(f_\alpha)/\|\alpha\|_0 = \Pi_u(f_\beta)\|\beta\|_0$ in accord with Proposition 7 (2). Similarly, at any ideal point v associated to slope -4, $\Pi_v(f_\alpha)/\|\alpha\|_0 = \Pi_v(f_\beta)/\|\beta\|_0$ since $\Delta(r_\alpha, -4)/\|\alpha\|_0 = 1/4$ and $\Delta(r_\beta, -4)/\|\beta\|_0 = 4/16 = 1/4$.

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