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A PRIORI UPPER BOUNDS OF SOLUTIONS SATISFYING A CERTAIN DIFFERENTIAL INEQUALITY ON COMPLETE MANIFOLDS

KENSHO TAKEGOSHI

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Abstract

In this article we study a priori upper bounds of subsolutions satisfying a certain differential inequality $(*)$ below on a non-compact complete Riemannian manifold (M, g) without any Ricci curvature condition. Our method depends on a volume estimate of open subsets where those solutions satisfy a certain strong subharmonicity. Several applications in conformal deformation of metrics and value distribution of harmonic maps are given.

1. Introduction

Let (M, g) be a connected Riemannian manifold of dimension m and Δ_g the Laplacian defined by $\Delta_g u := \mathrm{Trace}_g \nabla\nabla u$ for a smooth function u on M. Throughout this article (M, g) is always assumed to be *non-compact complete* and *connected* unless otherwise stated. We are interested in a priori upper bounds of a non-negative smooth function u satisfying the following differential inequality:

$$(*) \qquad \Delta_g u + ku - lu^{a+1} \geq 0$$

on M where k and l are continuous functions on M, and $a > 0$ is a constant respectively. A differential geometric interpretation of a priori upper bounds of such a subsolution appears in conformal deformation of metrics and value distribution of harmonic maps and has been studied under a certain curvature condition of g. Nevertheless our method does not depend on any curvature condition of g and depends only on a volume estimate of an open subset where u satisfies a certain strong subharmonicity. It can be stated as follows.

Theorem 1.1. *Let a smooth function u on M satisfy the following differential inequality*

$$\Delta_g u \geq \frac{Cu^{a+1}}{(1 + r_*)^b}$$

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on an open subset $\{u > \delta\} \neq \emptyset$ *for certain constants* $C > 0$, $a > 0$ *and* $\delta > 0$, *where* r_* *is the distance function from a fixed point* x_* *of* M. *If* $b < 2$ (*resp.* $b = 2$), *then*

$$\liminf_{r\to+\infty} \frac{\log V_{x_*}(r)}{r^{2-b}} = +\infty \quad \left(resp.\ \liminf_{r\to+\infty} \frac{\log V_{x_*}(r)}{\log r} = +\infty\right),$$

where $V_{x_*}(r)$ *is the volume of the geodesic ball* $B_{x_*}(r)$ *centered at a fixed point* $x_* \in M$ *and of radius* $r > 0$.

This result is a refinement of Theorem 1.1 in [18] and plays a crucial role to show the following a priori upper estimate of u satisfying $(*)$.

Theorem 1.2. *Let* u *be a non-negative smooth function satisfying the differential inequality* $(*)$ *on* M *and the functions* k *and* l *in* $(*)$ *satisfy the following condition*

$$k \leq Hl \quad \textit{for some constant} \quad H \geq 0$$

and

$$l \geq \frac{L}{(1+r_*)^b} \quad \textit{for certain constants} \quad L > 0 \quad \textit{and} \quad b \in \mathbf{R}$$

on M *respectively*. *Suppose the following volume growth condition either*

$$\liminf_{r\to+\infty} \frac{\log V_{x_*}(r)}{r^{2-b}} < +\infty \quad \textit{if} \quad b < 2, \tag{1}$$

or

$$\liminf_{r\to+\infty} \frac{\log V_{x_*}(r)}{\log r} < +\infty \quad \textit{if} \quad b = 2. \tag{2}$$

Then

$$\sup_M u \leq H^{1/a}.$$

Especially u *vanishes identically if* $H = 0$.

REMARK 1.1. The above volume growth condition is weaker than a decay condition of Ricci curvature studied in [14, 15] (cf. [14], Theorem A and [15], Theorem 0.2). Actually in view of the Laplacian comparison theorem (cf. [6]), if there exist constants $C \geq 0$ and $b \leq 2$ such that

$$\mathrm{Ric}_g(x) \geq -C(1 + r_*(x))^{2(1-b)}$$

for any $x \in M$, then one can see that

$$\limsup_{r\to+\infty} \frac{\log V_{x_*}(r)}{r^{2-b}} < +\infty \quad \textit{if} \quad b < 2 \quad \left(\text{resp. } \limsup_{r\to+\infty} \frac{\log V_{x_*}(r)}{\log r} < +\infty \quad \text{if} \quad b = 2\right).$$

In case $b = 2$ the above pointwise lower bound condition of Ricci curvature can be replaced by the following weaker condition. Namely if the negative part $R_{M,-}$ of the Ricci curvature of (M, g) satisfies the following

$$\int_{B_{x_*}(r)} R_{M,-}^p \, dv_g = O(r^k)$$

for any $r > 0$ and positive integers p, k with $p > m - 1$ and $p/(2p + k)(m - 1) \ll 1$, then the condition (2) is satisfied (cf. [7], Theorem 1.1 and Corollary 1.2).

As an applications of Theorem 1.2, we can show the following.

Theorem 1.3. *Under the condition either* (1) *or* (2) *of* Theorem 1.2 *for* $b \leq 2$, *suppose* (M, g) *has dimension* $m \geq 2$ *and the scalar curvature* s_g *of* g *satisfies the following inequality*

$$s_g \leq -\frac{L}{(1 + r_*)^b} \quad \textit{for some constant} \quad L > 0$$

on M. *Then any conformal transformation* f *of* (M, g) *which preserves* s_g *i.e., the scalar curvature* K_{f^*g} *of* f^*g *coincides with* s_g *is an isometry* (*cf*. Corollary 3.1 *and* [14], Corollary 1).

Theorem 1.1 is deeply related to a generalized maximum principle for the Laplacian Δ_g on a complete manifold (M, g). In fact we can show the following in terms of our formulation.

Theorem 1.4. *Suppose the condition either* (1) *or* (2) *of* Theorem 1.2 *is satisfied for* $b \leq 2$, *and a smooth function* u *is bounded from above on* M. *Then for any* $\varepsilon > 0$ *and* $x \in M$, *there exists a point* x_ε *of* M *such that*

(i) $$u(x) \leq u(x_\varepsilon),$$

(ii) $$\Delta_g u(x_\varepsilon) < \frac{\varepsilon}{(1 + r_*(x_\varepsilon))^b}.$$

Furthermore if $0 \leq b \leq 2$ *and there exists a continuous function* λ *on a real line such that* $\Delta_g u \geq \lambda(u)$ *on* M, *then one can take the above point* x_ε *which satisfies* $|\nabla u|(x_\varepsilon) < \varepsilon$ *simultaneously*.

REMARK 1.2. In [18], Theorem 2.3 we have announced that a generalized maximum principle for Δ_g can be induced under the condition (1) for $b = 0$ in Theorem 1.2. However its proof is incomplete and so the problem is still unsolved except the above case (cf. [10]).

By Theorem 1.4 we can restore several results stated in [18], §3, Applications without proof. For instance we get the following.

Theorem 1.5. *Under the condition either* (1) *or* (2) *of* Theorem 1.2 *for* $b \le 2$, *suppose that* $f : (M, g) \to (N, h)$ *is a harmonic map to an Hadamard manifold* (N, h) *and the energy density* $\mathbf{e}(f)$ *of* f *satisfies the following inequality*

$$\mathbf{e}(f) \ge \frac{C}{(1 + r_*)^b} \quad \textit{for some constant} \quad C > 0$$

on M. *Then the image of* f *is unbounded*. *In particular if* (N, h) *is an* n*-dimensional Euclidean space* $(\mathbf{R}^n, g_e)$ *provided with Euclidean metric* g_e *and the condition either* (1) *or* (2) *of* Theorem 1.2 *is satisfied for* $0 \le b \le 2$, *then the image of* f *can not be contained in any non-degenerate cone of* $\mathbf{R}^n$ (*cf*. Corollary 3.6, Theorem 3.7, *and* [9], Theorem B).

In the second section we give the proof of the above results except Theorems 1.3 and 1.5. Their applications including those theorems are given in the third section.

REMARK 1.3. In preparation of this work, an article [11] has been published by S. Pigola, M. Rigoli and A.S. Setti. In the paper they study a priori upper bounds of u satisfying $(*)$ from a view of volume growth condition of complete manifolds and give certain applications related to our results. However their method can not allow us to study the case $b = 2$, i.e., (M, g) has a polynomial volume growth. The upper bound 2 of b is originated from the fact that Δ_g is the 2-Laplacian which is a special case of the p-Laplacian $\Delta_{g,p}$ defined by $\Delta_{g,p} u := \mathrm{div}(|\nabla u|^{p-2} \nabla u)$ for $u \in C^\infty(M)$.

2. A volume estimate for a strong subharmonicity of solutions

Let (M, g) be a complete non-compact Riemannian manifold of dimension m as in the introduction and r_* the distance function from a fixed point $x_* \in M$. We restate Theorem 1.1 in the introduction.

Theorem 2.1. *Let* u *be a smooth function on* (M, g) *satisfying the inequality*

$$\Delta_g u \ge \frac{C_1 u^{a+1}}{(1 + r_*)^b} \quad \textit{on} \quad \{u > \delta\} \ne \emptyset \quad \textit{for} \quad C_1 > 0,\ a > 0 \quad \textit{and} \quad \delta > 0. \tag{2.1}$$

If $b < 2$ (*resp.* $b = 2$), *then*

$$\liminf_{r\to+\infty} \frac{\log V_{x_*}(r)}{r^{2-b}} = +\infty \quad \left(\textit{resp.}\ \liminf_{r\to+\infty} \frac{\log V_{x_*}(r)}{\log r} = +\infty \right).$$

Proof. We may assume $\sup_M u = +\infty$. If $u^* := \sup_M u < +\infty$, then putting $v := 1/(u^* - u)$ (u does not attain u^* on $\{u > \delta\}$ by (2.1)) one can verify that $\Delta_g v \geq C_1\delta^{a+1}v^2/(1+r_*)^b$ on $\{v > \delta_*\}$ with $\delta_* := 1/(u^* - \delta) > 0$. We have only to discuss by replacing u by v for $a = 1$. Since we can assume that $\{u > \delta + C_2\} \neq \emptyset$ for any $C_2 > 0$, we replace u by $u/(\delta + C_2)$ and C_1 by $C := C_1(\delta + C_2)^a > 0$ respectively, and set $k_* := 1/(1+r_*)^b$ in (2.1). The inequality (2.1) can be modified into the following form:

$$\Delta_g u \geq Ck_* u^{a+1} \quad \text{on} \quad M^* := \{u > 1\} \neq \emptyset. \tag{2.2}$$

From the above observation, we can take the constant C arbitrarily large in (2.2). We choose a non-negative smooth convex function λ on a real line $\mathbf{R}$ such that $\lambda(t) \equiv 0$ if $t \leq 1$, $\lambda(t) > 0$, $\lambda'(t) > 0$, $\lambda''(t) \geq 0$ if $t > 1$ and $\lambda'(t) \equiv 1$ if $t > 1+\eta$ for a sufficiently small $\eta > 0$ and a Lipschitz continuous function ω on M such that $0 \leq \omega \leq 1$, $\mathrm{Supp}(\omega) \subset \overline{B_{x_*}(2r)}$, $\omega \equiv 1$ on $B_{x_*}(r)$, and $|\nabla\omega| \leq 1/r$. By using (2.2), a direct calculation shows the following for any p and $q > 0$:

$$\begin{aligned}\mathrm{div}\big(\omega^{2q}\nabla\lambda(u^p)\big) &\geq p\lambda'(u^p)\big\{(p-1)\omega^{2q}u^{p-2}|\nabla u|^2 + \omega^{2q}u^{p-1}\Delta_g u + 2q\omega^{2q-1}u^{p-1}\langle\nabla\omega, \nabla u\rangle\big\} \\ &\geq p\lambda'(u^p)\big\{(p-1)\omega^{2q}u^{p-2}|\nabla u|^2 + C\omega^{2q}k_*u^{p+a} - 2q\omega^{2q-1}u^{p-1}|\nabla u|\,|\nabla\omega|\big\}.\end{aligned}$$

By integrating the both sides and hypothesis, for any $\varepsilon > 0$ we get

$$\begin{aligned}(p-1)\int \omega^{2q}\lambda'(u^p)u^{p-2}|\nabla u|^2\,dv_g + C\int \omega^{2q}k_*\lambda'(u^p)u^{p+a}\,dv_g \\ \leq 2q\int \omega^{2q-1}\lambda'(u^p)u^{p-1}|\nabla u||\nabla\omega|\,dv_g \\ \leq \varepsilon\int \omega^{2q}\lambda'(u^p)u^{p-2}|\nabla u|^2\,dv_g + \frac{q^2}{\varepsilon}\int \omega^{2(q-1)}\lambda'(u^p)u^p|\nabla\omega|^2\,dv_g.\end{aligned}$$

For $\varepsilon = (p-1)/2 > 0$ we obtain

$$\int \omega^{2q}k_*\lambda'(u^p)u^{p+a}\,dv_g \leq \frac{2q^2}{C(p-1)}\int \omega^{2(q-1)}\lambda'(u^p)u^p|\nabla\omega|^2\,dv_g. \tag{2.3}$$

By setting $q = (p+a)/a > 1$ in (2.3), the following holds:

$$\int \omega^{2(p+a)/a}k_*\lambda'(u^p)u^{p+a}\,dv_g \leq \frac{2(p+a)^2}{a^2(p-1)C}\int \omega^{2p/a}u^p\lambda'(u^p)|\nabla\omega|^2\,dv_g. \tag{2.4}$$

The Hölder inequality yields the following:

$$\begin{aligned}&\int \omega^{2p/a} u^p \lambda'(u^p)|\nabla\omega|^2\, dv_g\\ &\le \left(\int \omega^{2(p+a)/a} k_* \lambda'(u^p) u^{p+a}\, dv_g\right)^{p/(p+a)} \left(\int k_*^{-p/a} \lambda'(u^p)|\nabla\omega|^{2(p+a)/a}\, dv_g\right)^{a/(p+a)}.\end{aligned}$$

Since there exists $r_0 \ge 2$ depending on C such that $B_{x_*}(r) \cap M^* \ne \emptyset$ for any $r \ge r_0$, by substituting the above inequality into the right hand side of (2.4), we get

(2.5)

$$\int_{B_{x_*}(r)} k_* \lambda'(u^p) u^{p+a}\, dv_g \le \left(\frac{2(p+a)^2}{a^2(p-1)C}\right)^{(p+a)/a} \int_{B_{x_*}(2r,r)} k_*^{-p/a} \lambda'(u^p)|\nabla\omega|^{2(p+a)/a}\, dv_g.$$

where $B_{x_*}(2r, r) := B_{x_*}(2r) \setminus B_{x_*}(r)$ for any $r \ge r_0$. We set

$$F(r, p) := \int_{B_{x_*}(r)} \lambda'(u^p)\, dv_g > 0$$

for any $r \ge r_0$ and $p > 1$. Since $r^{-2}(1 + r_*)^b \le 2^4(1 + r)^{b-2}$ for $1 \le r \le r_* \le 2r$ and $b \le 2$, the right-hand side of (2.5) can be estimated as follows:

$$\begin{aligned}&\left(\frac{2(p+a)^2}{a^2(p-1)C}\right)^{(p+a)/a} \int_{B_{x_*}(2r,r)} k_*^{-p/a} \lambda'(u^p)|\nabla\omega|^{2(p+a)/a}\, dv_g\\ &\le \alpha_b(r) F(2r, p) \left(\frac{2^5(p+a)^2}{a^2(p-1)C(1+r)^{2-b}}\right)^{(p+a)/a},\end{aligned}$$

where $\alpha_b(r) = (1 + r)^{-b}$ if $b \ge 0$ and $\alpha_b(r) = (1 + 2r)^{-b}$ if $b < 0$. Since $\lambda'(u^p) > 0$ if and only if $u > 1$, by combining this estimate with (2.5), we get for any $r \ge r_0$ and $p > 1$

$$F(r, p) \le \beta_b(r) F(2r, p) \left(\frac{2^5(p+a)^2}{a^2(p-1)C(1+r)^{2-b}}\right)^{(p+a)/a}, \tag{2.6}$$

where $\beta_b(r) = 1$ if $b \ge 0$ and $\beta_b(r) = (1 + 2r)^{-b}$ if $b < 0$. If $r \ge 1$ and $b \le 2$, then we set

$$p(r) := \frac{a^2 C(1+r)^{2-b}}{2^9}.$$

Since we may assume that $p(r) \ge a + 2$ for any $r \ge 1$ by taking C arbitrarily large, we get

$$\frac{2^5(p(r)+a)^2}{a^2(p(r)-1)C(1+r)^{2-b}} \le \frac{1}{2} \quad \text{for any} \quad r \ge 1.$$

By putting $p = p(r)$ and $F(r) := F(r, p(r))$ in (2.6), we have

$$F(r) \le \beta_b(r) F(2r) \left(\frac{1}{2}\right)^{(p(r)+a)/a} \quad \text{for any} \quad r \ge r_0. \tag{2.7}$$

We fix r with $r > 2r_0 \ge 4$ and assume $b < 2$. Since there exists an integer $k \ge 1$ with $2^{-(k+1)} < r_0/r \le 2^{-k}$, by putting $r_j = 2^j r_0$ and using (2.7), we can see

$$\begin{aligned} F(r_0) &\le \beta_b(r)^{\log_2(r/r_0)} \left(\frac{1}{2}\right)^{\left(\sum_{j=0}^{k-1} p(r_j)+a\right)/a} F(r_k) \\ &\le \frac{r_1 \beta_b(r)^{\log_2(r/r_0)}}{r} \left(\frac{1}{2}\right)^{aCr^{2-b}/2^{13-2b}} F(r), \end{aligned}$$

which implies

$$\frac{aC \log 2}{2^{13-2b}} - \frac{\max\{-b, 0\} (\log(1+2r))^2}{r^{2-b} \log 2} \le \frac{\log F(r)}{r^{2-b}}$$

for any $r \ge r(C, a, b)$ with a sufficiently large $r(C, a, b) > r_0$. By taking $r(C, a, b)$ so large again we get

$$\frac{aC \log 2}{2^{14-2b}} \le \frac{\log F(r)}{r^{2-b}}$$

for any $r \ge r(C, a, b)$. Since we can take C arbitrarily large and $F(r) \le V_{x_*}(r)$ by $\sup_{\mathbf{R}} \lambda' = 1$, we attain the conclusion. If $b = 2$, then by $\beta_2(r) \equiv 1$ we get the following by the same argument as above:

$$F(r_0) \le \left(\frac{r_1}{r}\right)^{(aC/2^9)+1} F(r),$$

which implies

$$\frac{aC}{2^9} \le \frac{\log F(r)}{\log r}$$

for any r with $r \ge r(C, a, 2) \gg 0$. Therefore we attain the conclusion similarly. □

Now we are in a position to show Theorem 1.2 stated in the introduction.

Proof of Theorem 1.2. If $\{u > H^{1/a}\} \ne \emptyset$, then taking $\varepsilon > 0$ with $\{u > (H+\varepsilon)^{1/a}\} \ne \emptyset$, u satisfies $\Delta_g u \ge C_1 u^{a+1}/(1+r_*)^b$ on $\{u > \delta\}$ for $C_1 = \varepsilon L/(H+\varepsilon)$ and $\delta = (H+\varepsilon)^{1/a}$. However this contradicts the volume growth condition in view of Theorem 2.1. □

As a corollary of Theorem 1.2 we get the following.

Corollary 2.2. *Let u be a non-negative smooth function u satisfying the differential inequality $(*)$ and the function k (resp. l) in $(*)$ satisfy the following*

$$k \leq \frac{K}{(1+r_*)^c} \quad \left(resp.\ l \geq \frac{L}{(1+r_*)^b}\right) \quad for\ K \geq 0\ (resp.\ L > 0)\ and\ c\ (resp.\ b) \in \mathbf{R}$$

on M. If the condition either (1) *or* (2) *of* Theorem 1.2 *is satisfied for $b \leq \min\{2, c\}$, then*

$$\sup_M u \leq \left(\frac{K}{L}\right)^{1/a}.$$

Especially u vanishes identically on M if $K = 0$.

Proof. Since $k \leq (K/L)l$ if $b \leq c$, the assertion follows from Theorem 1.2 immediately. □

The difference of two solutions of $(*)$ can be estimated as follows (cf. [19], Theorem 4.9).

Corollary 2.3. *Let u_1 and u_2 be non-negative solutions of the equality*

$$\Delta_g u + ku - lu^{a+1} = 0$$

on M, where h and k satisfy the assumption of Theorem 1.2 *respectively. If the condition of either* (1) *or* (2) *of* Theorem 1.2 *is satisfied for $b \leq 2$, then* $\sup_M |u_1 - u_2| \leq H^{1/a}$.

Proof. By setting $w := (u_1 - u_2)^2$, one can verify that w satisfies the inequality $\Delta_g w \geq -2kw + 2lw^{(a/2)+1}$ on M. Hence the conclusion follows from Theorem 1.2 immediately. □

Here we show Theorem 1.4 stated in the introduction.

Proof of Theorem 1.4. We may assume that u does not attain $u^* := \sup_M u < +\infty$ on M. We put $\varepsilon_* := \min\{\varepsilon, u^* - u(x)\}/(1 + \min\{\varepsilon, u^* - u(x)\}) > 0$ for a fixed constant $\varepsilon > 0$ and point $x \in M$ respectively. We set $w := 1/(1 + u^* - u) > 0$ and

$$M_p := \{y \in M; w^p(y) > 1 - \varepsilon_*\} \quad \text{and} \quad \Gamma_p := \left\{y \in M; \Delta_g w^p(y) < \frac{\varepsilon_* w^{2p}(y)}{(1 + r_*(y))^b}\right\}$$

for any positive integer p. One can verify that $M_p \subset M_q$ and $\Gamma_p \subset \Gamma_q$ for any $p > q \geq 1$ in view of the equality $\Delta_g w^p = (p/q)w^{p-q}\Delta_g w^q + p(p-q)w^{p+2}|\nabla u|^2$. By Theorem 2.1 the volume growth condition implies that $\Sigma_p := M_p \cap \Gamma_p$ is a non-empty, and unbounded subset of M for any p otherwise w^p satisfies $1-\varepsilon_* < \sup_{\Sigma_p} w^p < 1$ and $\Delta_g w^p \geq \varepsilon_* w^{2p}/(1+r_*)^b$ on $\{w^p > \sup_{\Sigma_p} w^p\} \neq \emptyset$. Moreover if $y \in \Sigma_p$ for $p \geq 1$, then one can see that $u(x) \leq u(y)$ and

$$\Delta_g u(y) < \frac{\varepsilon_*}{p(1+r_*(y))^b} - (p+1)(1-\varepsilon_*)^{1/p} |\nabla u|^2(y). \tag{2.8}$$

The estimate 2.8 implies that any point of Σ_1 is the desired one. To show the latter half assertion, suppose $|\nabla u| \geq \eta$ on Σ_1 for a constant $\eta > 0$. Clearly we can verify that $\Sigma_p \subset \Sigma_q$ for $p > q \geq 1$, and $\bigcap_{p=1}^{+\infty} \Sigma_p = \emptyset$ because $u < u^*$ on M. Hence for each point $y_p \in \Sigma_p$ we get $\lambda(u^*) = \lim_{p\to+\infty} \lambda\left(u(y_p)\right) \leq \lim_{p\to+\infty} \Delta_g u(y_p) = -\infty$ by (2.8). This is a contradiction. □

As a direct consequence of Theorems 1.4 and 2.1 we can obtain the following similarly to an aspect by Cheng and Yau (cf. [3], Corollary).

Corollary 2.4. *Let u be a smooth function satisfying the inequality*

$$\Delta_g u \geq \frac{\lambda(u)}{(1+r_*)^b}$$

on M, where λ is a continuous function on $\mathbf{R}$ *such that*

$$\lambda(t) \geq C_\varepsilon t^{a+1} \quad \text{for any} \quad t \geq \varepsilon \quad \text{with certain constants} \quad a > 0, \varepsilon > 0 \text{ and } C_\varepsilon > 0.$$

If the condition either (1) *or* (2) *of* Theorem 1.2 *is satisfied for $b \leq 2$, then $\sup_M u \leq \varepsilon$ and $\lambda(\sup_M u) \leq 0$. Especially if $u \geq 0$ and λ satisfies the above property for any small $\varepsilon > 0$, then $u \equiv 0$. Moreover $\inf_M |\nabla u| = 0$ if $0 \leq b \leq 2$.*

REMARK 2.1. As a related topic, Tachikawa showed a non-existence theorem of harmonic maps from $\mathbf{R}^m$ to an Hadamard manifold with negative sectional curvature under a certain non-degenerate condition which is similar to the condition (3.2) below (cf. [17], Theorem 1). His result can be also induced by applying Corollary 2.4 to $\lambda(t) = \sinh \kappa t$ $(\kappa > 0)$ and $b = 2$ (see the inequality (2.2) in [17], p.152).

We can also get the following theorem which is related to a priori bound estimates of solutions for a certain Poisson equation (cf. [19], Corollary 4.3).

Corollary 2.5. *Let u be a smooth solution of the following equation*

$$\Delta_g u = \frac{\lambda(u)}{(1+r_*)^b}$$

on M, where λ is a continuous function on a real line such that $\lambda(t) \geq C_+ t^{a+1}$ (resp. $\lambda(t) \leq C_- t^3$) for $t \geq \alpha_+ \geq 0$, $a > 0$ and $C_+ > 0$ (resp. $t \leq \alpha_- \leq 0$ and

$C_- > 0$). *If the condition either* (1) *or* (2) *of* Theorem 1.2 *is satisfied for* $b \leq 2$, *then* $\alpha_- \leq \inf_M u \leq \sup_M u \leq \alpha_+$. *Especially if* $\alpha_+ = \alpha_- = 0$, *then* $u \equiv 0$. *Moreover* $\inf_M |\nabla u| = 0$ *if* $0 \leq b \leq 2$.

3. Applications in differential geometry

Let (M, g) be a complete non-compact Riemannian manifold of dimension $m \geq 2$ and $f : (M, g) \to (N, h)$ a smooth map to a Riemannian manifold (N, h). $f : (M, g) \to (N, h)$ is said to be a *conformal immersion* if there exists a smooth function $u > 0$ on M satisfying $f^*h = u^{4/(m-2)}g$ (resp. $f^*h = ug$) if $m \geq 3$ (resp. $m = 2$). It is known that u satisfies the following equality on M:

$$
\begin{aligned}
&m \geq 3 \Longrightarrow c_m \Delta_g u - s_g u + K_{f^*h} u^{(m+2)/(m-2)} \equiv 0, \quad c_m := \frac{4(m-1)}{(m-2)} \\
&m = 2 \Longrightarrow \Delta_g \log u - s_g + K_{f^*h} u \equiv 0,
\end{aligned} \tag{3.1}
$$

where s_g (resp. K_{f^*h}) is the scalar curvature of g (resp. the pull back f^*h of h by f). First we state the following theorem (cf. [14], Theorem 1).

Theorem 3.1. *Suppose* $f : (M, g) \to (N, h)$ *is a conformal immersion such that*

$$K_{f^*h} \leq \min\{s_g, 0\} \quad \text{and} \quad K_{f^*h} \leq -\frac{L}{(1 + r_*)^b} \quad \text{for some constant} \quad L > 0$$

on M. *If the condition either* (1) *or* (2) *of* Theorem 1.1 *is satisfied for* $b \leq 2$, *then* f *is distance decreasing, i.e.*, $\sup_M u \leq 1$.

Proof. By applying Theorem 1.2 to $k = -\min\{0, s_g\}/c_m$, $l = -K_{f^*h}/c_m$, $H = 1$ and $a = 4/(m-2)$ (resp. $a = 1$) for $m \geq 3$ (resp. $m = 2$), we can get the conclusion. □

We get the following from Theorem 3.1 immediately (cf. [14], Corollary 1 & the references, and [19], Theorem 4.7).

Corollary 3.2. *Under the condition either* (1) *or* (2) *of* Theorem 1.2 *for* $b \leq 2$, *suppose* (M, g) *has dimension* $m \geq 2$ *and the scalar curvature* s_g *of* g *satisfies the following*:

$$s_g \leq -\frac{L}{(1 + r_*)^b} \quad \text{for some constant} \quad L > 0$$

on M. *Then any conformal transformation* f *of* (M, g) *which preserves* s_g *i.e., the scalar curvature* K_{f^*g} *of* f^*g *coincides with* s_g *is an isometry.*

By applying this to the identity map of M we get the following (cf. [14], Corollary 2).

Corollary 3.3. *Under the same hypothesis as* Corollary 3.2 *suppose h is a conformal metric of g whose scalar curvature coincides with s_g. Then $h = g$.*

Corollary 2.2 yields the following (cf. [13], Corollary 4.2 and [15], Theorem 0.2 & Corollary 0.1).

Corollary 3.4. *Under the condition either* (1) *or* (2) *of* Theorem 1.2 *for $b \leq 2$, suppose the scalar curvature of g is non-negative on M and S is a smooth function satisfying*

$$S \leq -\frac{L}{(1+r_*)^b} \quad \textit{for some constant} \quad L > 0$$

on M. Then the metric g cannot be conformally deformed to any metric of scalar curvature S.

REMARK 3.1. In the above results it is not necessary to control the lower bound of s_g. The reader should see [11] (resp. [14, 15]) which studies the case $-C_1 \leq s_g \leq -C_2/(1+r_*)^b$ (resp. $-C_1/(1+r_*)^{2(b-1)} \leq s_g \leq -C_2/(1+r_*)^b$) for certain constants C_1, C_2 and b with $C_1 \geq C_2 > 0$ and $b \leq 2$ respectively.

REMARK 3.2. If l asymptotically behaves like $-1/(1+r_*)^b$ for $b > 2$ and $k \equiv 0$, then an existence theorem of non-trivial solutions u satisfying the equation $\Delta_{g_e} u = lu^{a+1}$ is known on an $m \geq 3$ dimensional Euclidean space $\mathbf{R}^m$ provided with Euclidean metric g_e (cf. [2], Theorem II).

The rest of this section is devoted to give several applications of Theorem 1.1 related to value distribution of maps. First we begin with the following (cf. [4], Theorem 3.1, [12], Theorem 2.17, and [18], Theorem 3.5).

Theorem 3.5. *Let $f : (M, g) \to (N, h)$ be a smooth map to an Hadamard manifold (N, h) whose sectional curvature is bounded from above by a non-positive constant K. Suppose the energy density $\mathbf{e}(f)$ and tension field $\tau(f)$ of f satisfy the following*

$$\mathbf{e}(f) \geq \frac{C_1}{(1+r_*)^b} \textit{ and } \|\tau(f)\| \leq \frac{C_2}{(1+r_*)^c} \quad \textit{for certain constants} \quad C_1 > 0 \textit{ and } C_2 \geq 0$$

on M respectively. If the condition either (1) *or* (2) *of* Theorem 1.2 *is satisfied for $b \leq \min\{2, c\}$ and $2\sqrt{-K}C_1 > C_2 \geq 0$ (resp. $2C_1 > C_2 \geq 0$) for $K < 0$ (resp. $K = 0$), then f is unbounded, i.e., the image $f(M)$ of M can not be relatively compact in N.*

Proof. By letting r_y be the distance function from a point $y \in N \setminus \overline{f(M)} \neq \emptyset$, we set $u(x) := f^*\lambda(r_y)$ with $\lambda(t) = \cosh(C_3 t)/2$ for $C_3 = \sqrt{-K}$ if $K < 0$ and $C_3 = 1$ if

$K = 0$. By combining the composition law of maps (cf. [5], (2.20), Proposition) with the Hessian comparison theorem (cf. [6], §2), the following estimate holds (cf. [12], (2.22)):

$$\Delta_g u \geq 2C_3 u \left(C_3 \mathbf{e}(f) - \frac{1}{2} \|\tau(f)\| \tanh\left(C_3 f^* r_y\right) \right).$$

By hypothesis we can see

$$\Delta_g u \geq \frac{C_3(2C_3C_1 - C_2)u}{2(1+r_*)^b} > 0$$

on M. If f is bounded, then u is bounded from above and $\inf_M u > 0$. However u does not attain its supremum on M by the above inequality. By putting $w = 1/(\sup_M u - u) > 0$, a direct calculation shows the following:

$$\Delta_g w \geq \frac{C_4 w^2}{(1+r_*)^b} \quad \text{for some constant} \quad C_4 > 0$$

on M. On the other hand Corollary 2.4 implies that w should vanish identically. This is a contradiction. □

Especially by letting $C_2 = 0$ we get the following immediately (cf. [12], Theorem 2.12).

Corollary 3.6. *Under the condition either* (1) *or* (2) *of* Theorem 1.2 *for* $b \leq 2$, *suppose that* $f : (M, g) \to (N, h)$ *is a harmonic map to an Hadamard manifold* (N, h) *and the energy density* $\mathbf{e}(f)$ *of* f *satisfies*

$$\mathbf{e}(f) \geq \frac{C}{(1+r_*)^b} \quad \textit{for some constant} \quad C > 0 \tag{3.2}$$

on M. *Then* f *is unbounded*.

In case $(N, h) = (\mathbf{R}^n, g_e)$, we can show the following which is a more precise result than Corollary 3.6 (cf. [9], Theorem B, [1], Theorem 3 and [18], Theorem 3.3).

Theorem 3.7. *Let* $f : (M, g) \to (\mathbf{R}^n, g_e)$ *be a harmonic map satisfying the condition* (3.2) *for* $0 \leq b \leq 2$. *If the condition either* (1) *or* (2) *of* Theorem 1.2 *is satisfied for* $0 \leq b \leq 2$, *then the image of* f *can not be contained in any non-degenerate cone of* $\mathbf{R}^n$.

Proof. The idea of proof is due to [9], Theorem B (see also [1]). Assume there exists a unit vector ν at the origin of $\mathbf{R}^n$ such that $\langle f(x), \nu \rangle / \| f(x) \| \geq \delta$ for a fixed constant $\delta > 0$ and any $x \in M$. Here $\langle \ , \ \rangle$ (resp. $\| \cdot \|$) is the inner product (resp. the norm) relative to g_e. Let $\mathbf{R}^{n-1}$ be the subspace of $\mathbf{R}^n$ which is orthogonal to ν and $\overline{f}$ the $\mathbf{R}^{n-1}$-component of the position vector f, i.e. $\overline{f} := f - \langle f, \nu \rangle \nu$. We may assume that $\langle f, \nu \rangle^2 - \delta^2 \langle \overline{f}, \overline{f} \rangle \geq 1$ on M. For a constant a with $\delta > a > 0$, we set

$$F_a := -\langle f, \nu \rangle + \sqrt{a^2 \langle \overline{f}, \overline{f} \rangle + 1} \leq 0.$$

Since the set $\{f(x); F_a(x) \geq F_a(x_*)\}$ is contained in a compact set for any a with $0 < a < \delta - \delta'$ and a fixed point $x_* \in M$, there exists a small constant $a > 0$ such that

$$a^2 \langle f(x), f(x) \rangle \leq 1 \tag{3.3}$$

for any $x \in M_a := \{x; F_a(x) \geq F_a(x_*)\}$. We fix such a constant a and put $F := F_a - F_a(x_*)$. Clearly F is bounded from above and $F(x_*) = 0$. A direct calculation shows

$$\langle f_* X, \nu \rangle^2 \leq 2 \left\{ \| F_* X \|^2 + a^2 \left(\| f_* X - \langle f_* X, \nu \rangle \nu \|^2 \right) \| f \|^2 \right\} \tag{3.4}$$

for any $X \in TM_x$ and $x \in M$. The harmonicity of f implies

$$\Delta_g F \geq \frac{a^2 \sum_{i=1}^m \| f_* X_i - \langle f_* X_i, \nu \rangle \nu \|^2}{\left(a^2 \langle \overline{f}, \overline{f} \rangle + 1 \right)^{3/2}} \tag{3.5}$$

for an orthogonal basis $\{X_i\}$ in TM_x and $x \in M$. By applying Theorem 1.4 to $\lambda \equiv \inf_M \Delta_g F \geq 0$ (see (3.5)) and putting $u = F$ in (2.8) there exists a sequence $\{x_n\}$ of points of M such that

(3.6)

$$\text{(i) } F(x_n) > 0, \quad \text{(ii) } |\nabla F|^2(x_n) < \frac{1}{n(1 + r_*(x_n))^b} \quad \text{and} \quad \text{(iii) } \Delta_g F(x_n) < \frac{1}{n(1 + r_*(x_n))^b}.$$

By putting $k_n := (1 + r_*(x_n))^b$, if $k_n \sum_{i=1}^m \| f_* X_i - \langle f_* X_i, \nu \rangle \nu \|^2(x_n)$ tends to zero, then $k_n \sum_{i=1}^m \langle f_* X_i, \nu \rangle^2(x_n)$ also tends to zero by the conditions (3.3), (3.4) and (ii) of (3.6), and so $k_n \sum_{i=1}^m \| f_* X_i \|^2(x_n) = 2k_n \mathbf{e}(f)(x_n)$ tends to zero. However this contradicts (3.2). Hence there exists a constant $C_5 > 0$ such that $k_n \sum_{i=1}^m \| f_* X_i - \langle f_* X_i, \nu \rangle \nu \|^2(x_n) \geq C_5 > 0$ for any n. However this again contradicts the condition (iii) of (3.6) in view of (3.3) and (3.5). □

We can show the following distance decreasing property of holomorphic maps of complex manifolds (cf. [20], Theorem 2, [16], Theorem 1, and [14], Theorem 3).

Theorem 3.8. *Let $f : (M, g) \to (N, h)$ be a holomorphic map from an m-dimensional complete non-compact Kähler manifold (M, g) to a complex hermitian manifold (N, h). Let $R_{M,-}$ (resp. HS_N) be the negative part of the pointwise lower* *bound of the Ricci curvature of* g (*resp. the pointwise upper bound of the holomorphic sectional curvature of* h). *Suppose*

$$R_{M,-} \leq \frac{K}{(1+r_*)^c} \quad \textit{and} \quad HS_N(f) \leq -\frac{L}{(1+r_*)^b}$$

on M *for certain constants* $K \geq 0$, $L > 0$, b *and* c. *If* $b \leq \min\{2, c\}$, *then* $\sup_M \mathbf{e}(f) \leq 2\nu K/(\nu+1)L$, *where* ν *is the maximal rank of* df. *Especially* f *is constant if the Ricci curvature of* g *is non-negativ*.

Proof. Since $b \geq 2(b-1)$ for $b \leq 2$, by hypothesis the Ricci curvature of g can be supported from below by $-K/(1+r_*)^{2(b-1)}$. Hence the condition either (1) or (2) of Theorem 1.2 is satisfied as stated in the introduction. On the other hand since the energy density $\mathbf{e}(f)$ of f satisfies the inequality

$$\Delta_g \log \mathbf{e}(f) \geq -2R_{M,-} - \frac{\nu+1}{\nu} HS_N(f)\mathbf{e}(f),$$

where $\mathbf{e}(f) \neq 0$ (cf. [16], Proposition 4), the conclusion follows by applying Corollary 2.2 to $u = \mathbf{e}(f)$, $k = 2H/(1+r_*)^c$ and $l = K(\nu+1)/\nu(1+r_*)^b$ respectively. □

We can also show the following volume decreasing property of holomorphic maps of complex manifolds (cf. [8], §1, [7], Theorem 3.5 & Corollary 3.6, and [18], Theorem 3.7).

Theorem 3.9. *Let* $f : (M, g) \to (N, h)$ *be a holomorphic map from an* m-*dimensional complete non-compact Kähler manifold* (M, g) *to a complex hermitian manifold* (N, h) *of the same dimension. Let* $S_{M,-}$ *be the negative part of the scalar curvature* S_M *of* g. *Let* u_f *denote the ratio* f^*V_N/V_M *of the volume forms* V_M *relative to* g *and* V_N *relative to* h *respectively. Suppose*

$$S_{M,-} \leq \frac{K}{(1+r_*)^c} \quad \textit{and} \quad \mathrm{Ric}_N(f) \leq -\frac{L}{(1+r_*)^b}$$

on M *for certain constants* $K \geq 0$, $L > 0$, b *and* c. *If the condition either* (1) *or* (2) *of* Theorem 1.2 *is satisfied for* $b \leq \min\{2, c\}$, *then* $\sup_M u_f \leq (2K/mL)^{2m}$.

Proof. By letting $u := u_f^{1/2m}$, u satisfies the following inequality on $\{u > 0\}$ (see [7], the proof of Theorem 3.5):

$$\Delta_g \log u \geq \frac{1}{2m} S_M - \frac{1}{4} \mathrm{Ric}_N(f) u^2.$$

To get the conclusion we have only to apply Corollary 2.2 to $k = -(1/2m)S_{M,-}$ and $l = -(1/4)\,\mathrm{Ric}_N(f)$ respectively. □

If the scalar curvature is non-negative, then the Ricci curvature is bounded from below and so the condition (1) of Theorem 1.2 is satisfied for $b = 1$. By setting $K = 0$ in Theorem 3.9 we obtain the following immediately.

Corollary 3.10. *Let (M, g) be a complete Kähler manifold whose scalar curvature is non-negative. Let $f : (M, g) \to (N, h)$ be a holomorphic map of complex manifolds with the same dimension such that*

$$\mathrm{Ric}_N(f) \leq -\frac{L}{1 + r_*}$$

on M for some constant $L > 0$. Then f degenerates everywhere on M.

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Department of Mathematics
Graduate School of Science
Osaka University
Machikaneyama 1-16, Toyonaka
Osaka, 560-0043, Japan

Current address:
Faculty of Engineering
Kansai University
3-3-35 Yamate-cho, Suita-shi
Osaka 564-8680
Japan
e-mail: tkenshou@ipcku.kansai-u.ac.jp