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ON UNIQUENESS IN THE CAUCHY PROBLEM FOR SYSTEMS WITH PARTIAL ANALYTIC COEFFICIENTS

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Abstract

In this paper, we consider the uniqueness in the Cauchy problem for systems of differential operators with partial analytic coefficients. By proving Carleman's estimate, we obtain the uniqueness theorem which contains the classical Holmgren's theorem for systems of differential operators.

1. Introduction

The problem of the uniqueness in the Cauchy problem is a fundamental problem in a theory of partial differential equation. In this paper, we consider the uniqueness in the Cauchy problem for systems with partial analytic coefficients. In the case when coefficients are analytic, by Holmgren's theorem, the uniqueness holds for any non-characteristic initial hypersurface. On the other hand, it is well known that, when coefficients are merely C^∞ function, the uniqueness is false for some non-characteristic initial hypersurface. For example, Alinhac and Baouendi [1] showed that there exists some second order hyperbolic operator $P = \partial_t^2 - A(t, x, \partial_x)$, where A is a second order elliptic operator with C^∞ coefficients, and some time-like initial hypersurface for which the uniqueness result is false. Here time-like means that, if C^2-hypersurface S is locally defined by $S = \{x \mid \varphi(x) = 0\}$, φ satisfies that

$$\varphi_t'(0)^2 - A(0, 0, \varphi_x'(0)) < 0.$$

In [10], [5], [8], Tataru, Hörmander, Robbiano and Zuily showed that under the assumption of partial analyticity of coefficients, uniqueness result holds for any non-characteristic initial hypersurface. But they considered the uniqueness only for scalar differential operators. In this paper, we consider uniqueness for systems of partial differential operators with partial analytic coefficients. The study of the uniqueness for systems is not so many as for scalar differential equation. For example, in [6], G. Hile and M. Protter considered the uniqueness for first order systems with C^1-coefficients. But their result holds for some restricted differential systems. The purpose of this paper is that, under the assumption of partial analyticity, we prove the uniqueness result for differential systems which can't be proved in [6].

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We introduce some notation. Let n_a, n_b be non negative integers with $n = n_a + n_b \geq 1$. We set $\mathbb{R}^n = \mathbb{R}^{n_a} \times \mathbb{R}^{n_b}$ and, for x, ξ in $\mathbb{R}^n$, $x = (x_a, x_b)$, $\xi = (\xi_a, \xi_b)$. Let $P(x, D_x) = (p_{ij}(x, D_x))_{1\leq i,j\leq N} = \sum_{|\alpha|\leq m} A_\alpha(x) D_x^\alpha$ be a linear differential system with the principal part $P_m(x, \xi) = \sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha$. Let S be a C^2 hypersurface through 0 locally given by

$$S = \{x : \varphi(x) = 0\},$$

where φ satisfies

$$\varphi(0) = 0, \quad \varphi'(0) = (\varphi_a'(0), \varphi_b'(0)) \neq 0.$$

Our result is stated as follows;

Theorem 1.1. *Let $P(x, D_x)$ be a differential system of order m with C^∞ coefficients. We assume that all coefficients of P are analytic with respect to x_a in a neighborhood of* 0, *and that the principal part $P_m(x, \xi)$ satisfies the following conditions*:

1. *For any $\xi_b \in \mathbb{R}^{n_b} \backslash \{0\}$*

$$\det P_m(0, 0, \xi_b) \neq 0. \tag{1}$$

2. *For any $\xi_b \in \mathbb{R}^{n_b}$*

$$\det P_m(0, i\varphi_a'(0), i\varphi_b'(0) + \xi_b) \neq 0. \tag{2}$$

Let V be a neighborhood of 0 *and $u = (u_1, u_2, \ldots, u_N) \in C^\infty(V)^N$, which satisfy*

$$\begin{cases} P(x, D_x)u(x) = 0, & x \in V, \\ \operatorname{supp} u = \bigcup_{k=1}^N \operatorname{supp} u_k \subset \{x \in V : \varphi(x) \leq 0\}. \end{cases}$$

Then there exists a neighborhood W of 0 in which $u \equiv 0$.

We make some remarks on this result. This theorem contains Holmgren's uniqueness theorem. In fact, when we set $n_a = n, n_b = 0$, the condition of $P_m(x, \xi)$ in this theorem means that $P(x, D_x)$ is non-characteristic with respect to the initial hypersurface S. Moreover, by this theorem, we can show that the uniqueness holds in the following differential system.

EXAMPLE 1. We set $(t, x) \in \mathbb{R}^2$. Let $a(t, x), b(t, x), c(t, x) \in C^\infty(\mathbb{R}^2)$, and $A(x) \in C^\infty(\mathbb{R}, M_2(\mathbb{C}))$. We assume that $a = a(t, x)$, $b = b(t, x)$, $c = c(t, x)$ satisfy the following conditions:

1. $a(t, x)$, $b(t, x)$, $c(t, x)$ is analytic with respect to t in some neighborhood of 0.
2. $a_0, b_0, c_0 \in \mathbb{R}$ and $a_0 b_0 \neq 0$, where $a_0 = a(0, 0)$, $b_0 = b(0, 0)$, $c_0 = c(0, 0)$.

Then the equation

$$\partial_t u = \begin{pmatrix} c\partial_x & a\partial_x \\ b\partial_x & 0 \end{pmatrix} u + A(x)u$$

has a unique continuation property with respect to the initial surface $S = \{x \mid \varphi(t,x) = 0\}$, where φ satisfies

1. $\varphi_t'(0)^2 - c_0\varphi_t'(0)\varphi_x'(0) - a_0 b_0 \varphi_x'(0)^2 \neq 0$;
2. $c_0\varphi_t'(0) + 2a_0 b_0 \varphi_x'(0) \neq 0$.

When $\varphi = t$, this equation is not always strictly hyperbolic, so by the method of energy estimate, as in [11], we can't prove the uniqueness of this equation. And because the coefficient of u_x is not normal, the result in [6] doesn't contain this example.

Our proof is based on the Carleman method and the FBI transformation theory, as the proof given in [8]. By Sjöstrand's theory of FBI transformation, we microlocalize the symbol of $P(x, D_x)$ with respect to x_a and, by using semi-classical pseudo differential symbolic calculus and Gårding's inequality, we establish the Carleman estimate of $P(x, D_x)$.

2. Preliminaries

In this section, we prepare the tools for our proof taken from [2], [7], [8], [9].

2.1. From theory of PDO.

DEFINITION 2.1. 1. Let $g \in C^\infty(\mathbb{R}^d : (0,\infty))$. We say that g is an order function if, for any $\alpha \in \mathbb{N}^d$, there exists some $C_\alpha > 0$ such that

$$\left|\partial_x^\alpha g(x)\right| \leq C_\alpha g(x). \tag{3}$$

2. Let g be an order function. For $a(x,\lambda) \in C^\infty(\mathbb{R}^d \times (0,\infty))$, we say that $a(x,\lambda)$ is of the symbol class $S_d(g)$ if, for any $\alpha \in \mathbb{N}^d$, there exists some positive constant C_α such that

$$\left|\partial_x^\alpha a(x,\lambda)\right| \leq C_\alpha g(x). \tag{4}$$

DEFINITION 2.2. Let g be an order function. For $\{a_j\} \subset S_d(g)$, $a \in S_d(g)$, we write

$$\sum_{j\geq 0} \frac{1}{\lambda^j} a_j \sim a \quad \text{in} \quad S_d(g) \tag{5}$$

if, for any $\alpha \in \mathbb{N}^d$ and $N \in \mathbb{N}$, there exist some positive constants $C_{N,\alpha}, \lambda_{N,\alpha}$ such that, for $(x,\lambda) \in \mathbb{R}^d \times [\lambda_{N,\alpha}, \infty)$, $a(x,\lambda)$ satisfies the following inequality:

$$\left| \partial_x^\alpha \left(a(x,\lambda) - \sum_{j=0}^{N} \frac{1}{\lambda^j} a_j(x,\lambda) \right) \right| \le \frac{C_{N,\alpha}}{\lambda^{N+1}} g(x). \tag{6}$$

In the following context, we set $d = 2n$, $\mathbb{R}^d = \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$.

DEFINITION 2.3. For $\sigma \in S_{2n}(\langle \xi \rangle^m)$, we define $\mathrm{Op}_\lambda(\sigma)(x, D_x) = \sigma_\lambda(x, D_x)$ by

$$\sigma_\lambda(x, D_x)u(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi, \tag{7}$$

where the integral means oscillatory integral. Similarly we define $\sigma(x, D_x)$ by

$$\sigma(x, D_x)u(x) = \left(\frac{1}{2\pi}\right)^n \iint e^{i(x-y)\cdot\xi} \sigma\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi. \tag{8}$$

Theorem 2.4. 1. *For $\sigma \in S_{2n}(\langle \xi \rangle^m)$, we have*

$$\sigma_\lambda(x, D_x)^* = \overline{\sigma}_\lambda(x, D_x), \tag{9}$$

where $\sigma_\lambda(x, D_x)^*$ is the formal adjoint of $\sigma_\lambda(x, D_x)$.
2. *For $\sigma_i \in S_{2n}(\langle \xi \rangle^{m_i})$ $(i = 1, 2)$, there exists some $\sigma_3 \in S_{2n}(\langle \xi \rangle^{m_1+m_2})$ such that*

$$\sigma_{1,\lambda}(x, D_x)\sigma_{2,\lambda}(x, D_x) = \sigma_{3,\lambda}(x, D_x). \tag{10}$$

And we have the following expansion formula for σ_3:

$$\sigma_3(x,\xi) \sim \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right) \left(\frac{i}{2\lambda}\right)^k (D_\xi D_y - D_\eta D_x)^k \sigma_1(x,\xi)\sigma_2(y,\eta)|_{(y,\eta)=(x,\xi)} \quad \text{in} \quad S_{2n}(\langle \xi \rangle^{m_1+m_2}). \tag{11}$$

By a slight modification of the proof of Theorem 2.4 in [7], we have the following theorem.

Theorem 2.5. 1. *For $\sigma \in S_{2n}(\langle \xi_b \rangle^m)$, we have*

$$\sigma_\lambda(x, D_x)^* = \overline{\sigma}_\lambda(x, D_x), \tag{12}$$

where $\sigma_\lambda(x, D_x)^*$ is the formal adjoint of $\sigma_\lambda(x, D_x)$.
2. *For $\sigma_i \in S_{2n}(\langle \xi_b \rangle^{m_i})$ $(i = 1, 2)$, there exists some $\sigma_3 \in S_{2n}(\langle \xi_b \rangle^{m_1+m_2})$ such that*

$$\sigma_{1,\lambda}(x, D_x)\sigma_{2,\lambda}(x, D_x) = \sigma_{3,\lambda}(x, D_x). \tag{13}$$

And we have the following expansion formula for σ_3:

$$\sigma_3(x,\xi) \sim \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)\left(\frac{i}{2\lambda}\right)^k (D_\xi D_y - D_\eta D_x)^k \sigma_1(x,\xi)\sigma_2(y,\eta)|_{(y,\eta)=(x,\xi)} \quad \text{in} \quad S_{2n}(\langle\xi_b\rangle^{m_1+m_2}). \tag{14}$$

Theorem 2.6. *Let* $\sigma \in C^\infty(\mathbb{R}^{2n})$ *satisfy the following inequality*:

$$\|\sigma\|_* \equiv \sum_{|\alpha+\beta|\le 2n+1} \left\| D_\xi^\alpha D_x^\beta \sigma \right\|_\infty < +\infty. \tag{15}$$

Then $\sigma_\lambda(x, D_x)$ *is* L^2 *bounded. Moreover we have*

$$\|\sigma_\lambda(x, D_x)\| \le C\|\sigma\|_*, \tag{16}$$

where C *is a positive constant independent of* σ, λ.

2.2. Review on the theory of FBI transformation. In this section, we review on the theory of FBI transformation from [7], [8], [9].

DEFINITION 2.7. We denote the partial FBI transformation by T, that is, for $u \in \mathcal{S}(\mathbb{R}^n)$, we define Tu by

$$Tu(z_a, x_b, \lambda) = K(\lambda)\int e^{-(\lambda/2)(z_a-y_a)^2} u(y_a, x_b)\, dy_a, \tag{17}$$

where $z_a \in \mathbb{C}^{n_a}$, $x_b \in \mathbb{R}^{n_b}$, $\lambda \ge 1$, and $K(\lambda) = 2^{-n_a/2}(\lambda/\pi)^{3n_a/4}$, $z_a^2 = \sum_{j=1}^{n_a} z_{aj}^2$.

We introduce some notations about the FBI transformation.

$$\Phi(z_a) = \frac{1}{2}(\operatorname{Im} z_a)^2, \tag{18}$$

$$\kappa_T(x_a, \xi_a) = (x_a - i\xi_a, \xi_a), \quad (x_a, \xi_a) \in T^*\mathbb{R}^{n_a}. \tag{19}$$

DEFINITION 2.8. For $u \in \mathcal{S}(\mathbb{R}^n)$ and $0 \le \eta \le 1$, we define the partial FBI transformation $T_\eta u$ by

$$T_\eta u(z_a, x_b, \lambda) = K_\eta(\lambda)\int e^{-(\lambda/2)(1+\eta)(z_a-y_a)^2} u(y_a, x_b)\, dy_a, \tag{20}$$

where $z_a \in \mathbb{C}^{n_a}$, $x_b \in \mathbb{R}^{n_b}$, $\lambda \ge 1$, and $K_\eta(\lambda) = 2^{-n_a/2}(\lambda(1+\eta)/\pi)^{3n_a/4}$, $z_a^2 = \sum_{j=1}^{n_a} z_{aj}^2$.

We define $\Phi_\eta(z_a)$ and κ_{T_η} by

$$\Phi_{(1+\eta)}(z_a) = \frac{1}{2}(1+\eta)(\operatorname{Im} z_a)^2, \tag{21}$$

$$\kappa_{T_\eta}(x_a,\xi_a)=\left(x_a-\frac{i\xi_a}{1+\eta},\xi_a\right),\quad (x_a,\xi_a)\in T^*\mathbb{R}^{n_a}. \tag{22}$$

Next we introduce some function spaces where T and T_η operate. For $k\in\mathbb{Z}$ and $0\le\eta\le 1$, we define $L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a},H^k(\mathbb{R}^{n_b}))$ by

$$L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a},H^k(\mathbb{R}^{n_b}))=L^2\big(\big(\mathbb{C}^{n_a},e^{-2(1+\eta)\Phi(z_a)}L(dz_a)\big),H^k(\mathbb{R}^{n_b})\big), \tag{23}$$

where $L(dz_a)$ is Lebesgue measure on $\mathbb{C}^{n_a}$, and $H^k(\mathbb{R}^{n_b})$ is the Sobolev space of order k. In particular, when $k=0$, we denote $L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a},L^2(\mathbb{R}^{n_b}))$ by $L^2_{(1+\eta)\Phi}$. And we define $\mathcal{L}^2_{(1+\eta)\Phi}$ by

$$\mathcal{L}^2_{(1+\eta)\Phi}=L^2_{(1+\eta)\Phi}\cap\mathcal{H}(\mathbb{C}^{n_a}), \tag{24}$$

where $\mathcal{H}(\mathbb{C}^{n_a})$ is the space of all entire functions in $\mathbb{C}^{n_a}$. Then we have the following proposition.

Proposition 2.9. *For $\eta\in[0,1]$, we have*
1. *T_η is an isometry mapping from $L^2(\mathbb{R}^{n_a},H^k(\mathbb{R}^{n_b}))$ into $L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a},H^k(\mathbb{R}^{n_b}))$.*
2. *$T^*_\eta T_\eta$ is an identity on $L^2(\mathbb{R}^n)$, where T^*_η is the formal adjoint of T_η.*
3. *$T_\eta T^*_\eta$ is the projection from $L^2_{(1+\eta)\Phi}$ to $\mathcal{L}^2_{(1+\eta)\Phi}$, in particular, if $\widetilde{v}=Tv$, $v\in\mathcal{S}(\mathbb{R}^n)$, we have $T_\eta T^*_\eta\widetilde{v}=\widetilde{v}$.*

2.3. Microlocalization of PDO whose coefficients are analytic. In this section we prepare some theorems taken from [8]. Let $p(x,\xi)=\sum_{|\alpha|\le m}a_\alpha(x)\xi^\alpha$ be a polynomial in ξ with coefficients in $C_0^\infty(\mathbb{R}^n)$. Assume moreover that there exists some positive constant c_0 such that all $a_\alpha(x)$ are holomorphic with respect to x_a in ω, where $\omega=\{z_a\in\mathbb{C}^{n_a}:|z_a|<c_0\}$.

Theorem 2.10. *For $v\in C_0^\infty(\mathbb{R}^n)$, and $\eta\in[0,1]$, we have $T_\eta p(x,D_x)v=\widetilde{p}_\eta T_\eta v$ where*

$$\widetilde{p}_\eta Tv(x,\lambda)=\left(\frac{\lambda}{2\pi}\right)^{2n}\iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\left(\iint_{\xi_a=-(1+\eta)\,\mathrm{Im}((x+y)/2)}\omega_\eta\right)dy_b\,d\xi_b \tag{25}$$

with ω_η defined by

$$\omega_\eta=e^{i\lambda(x_a-y_a)\cdot\xi_a}p\left(\frac{x_a+y_a}{2}+i\frac{\xi_a}{1+\eta},\frac{x_b+y_b}{2},\lambda\xi\right)T_\eta u(y_a,y_b,\lambda)\,dy_a\wedge d\xi_a. \tag{26}$$

Let d be a positive number such that $13d<c_0$, and let $\chi(z_a,\zeta_a)\in C_0^\infty(\mathbb{C}^{2n_a})$ satisfy

$$\chi(z_a,\zeta_a)=\begin{cases}1 & |z_a|+|\zeta_a|<12d,\\ 0 & |z_a|+|\zeta_a|>13d.\end{cases} \tag{27}$$

Moreover let χ be almost analytic in $\Lambda_{\Phi_\eta} = \kappa(\mathbb{R}^{2n_a})$, which means that, for any positive integer N, there exists some positive number C_N such that

$$\left|\overline{\partial}\chi(z_a, \zeta_a)\right| \le C_N |\zeta_a + (1+\eta)\,\mathrm{Im}\, z_a|^N. \tag{28}$$

Then we have the following theorem.

Theorem 2.11. *Let* $v \in C_0^\infty(\mathbb{R}^n)$ *with* $\mathrm{supp}\, v \subset \{|x| < d\}$, *and let us set, for* $\eta \in (0, 1]$,

$$\widetilde{Q}_\lambda T v(x, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{2n} \iint e^{i\lambda(x_b - y_b)\cdot\xi_b} \left(\iint_{\xi_a = -(1+\eta)\,\mathrm{Im}((x+y)/2)} \chi\left(\frac{x_a + y_a}{2}, \xi_a\right)\omega\right) dy_b\, d\xi_b, \tag{29}$$

where $\omega = \omega_0$ *is defined in* Theorem 2.10. *Then we have*

$$\widetilde{p}_\lambda T v = \widetilde{Q}_\lambda T v + \widetilde{R}_\lambda T v + \widetilde{g}_\lambda \tag{30}$$

with $\widetilde{p}(x, D_x) = \widetilde{p}_0(x, D_x)$ *defined in* Theorem 2.10, *and* $\widetilde{R}_\lambda$ *and* $\widetilde{g}_\lambda$ *satisfy the following properties*: *For any* $N \in \mathbb{N}$, *there exists some positive constant* C_N *such that*

$$\left\|\widetilde{R}_\lambda T v\right\|_{L^2_{(1+\eta)\Phi}} \le C_N \lambda^{-N} \|T v\|_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))}, \tag{31}$$

$$\left\|\widetilde{g}(\,\cdot\,, \lambda)\right\|_{L^2_{(1+\eta)\Phi}} = O\left(e^{-(\lambda/3)\eta d^2} \|v\|_{H^{n_0}(\mathbb{R}^n)}\right), \quad \lambda \to +\infty, \tag{32}$$

where n_0 *depends only on the dimension* n *and the order of* p.

2.4. Extension to the system case. Let us introduce some notation for system. We denote $\mathcal{S}(\mathbb{R}^n)^N$ by

$$\mathcal{S}(\mathbb{R}^n)^N = \overbrace{\mathcal{S}(\mathbb{R}^n) \times \cdots \times \mathcal{S}(\mathbb{R}^n)}^{N}$$

and define $L^2(\mathbb{R}^n)^N$, $L^2_{(1+\eta)\psi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))^N$, $\mathcal{L}^2_{(1+\eta)\psi}{}^N$ similarly. Let $u = {}^t(u_1, u_2, \ldots, u_N)$, $v = {}^t(v_1, v_2, \ldots, v_N)$ be in $L^2(\mathbb{R}^n)^N$, we denote $(u, v)_{L^2}$ by

$$(u, v)_{L^2} = (u_1, v_1)_{L^2} + \cdots + (u_N, v_N)_{L^2}$$

and denote $(u, v)_{L^2_{(1+\eta)\Phi}}$ similarly. For $u \in \mathcal{S}(\mathbb{R}^n)^N$ and $\eta \in [0, 1]$, $T_\eta u$ is denoted by

$$T_\eta u = {}^t(T_\eta u_1, \ldots, T_\eta u_N). \tag{33}$$

Let a matrix valued function $A(x, \xi) = (a_{ij}(x, \xi))_{1 \le i,j \le N}$ on $\mathbb{R}^{2n}$ be in $S_{2n}(\langle\xi\rangle^m)$ with $m \in \mathbb{R}$, which means that, for any i, j, $a_{i,j} \in S_{2n}(\langle\xi\rangle^m)$. For $u \in \mathcal{S}(\mathbb{R}^n)^N$, we define $A_\lambda(x, D_x)u(x)$ by

$$A_\lambda(x, D_x)u(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\xi} A\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi \tag{34}$$

and define $A(x, D_x)u$ similarly. For a differential system $P(x, D_x) = \sum_{|\alpha|\leq m} A_\alpha(x)D_x$, $\sum_{|\alpha|=m} A_\alpha(x)\xi^\alpha$ is called the principal part of $P(x, D_x)$, and denoted by $P_m(x, \xi)$. Then similarly to the scalar case, we have the following theorems.

Theorem 2.12. *For $A(x,\xi) \in S_{2n}(\langle\xi_b\rangle^{m_1})$, and $B(x,\xi) \in S_{2n}(\langle\xi_b\rangle^{m_2})$, with $m_1, m_2 \in \mathbb{R}$, we have*

1. *$A_\lambda(x, D_x)^* = A_\lambda^*(x, D_x)$, where $A^*(x, \xi) = {}^t\overline{A(x, \xi)}$.*
2. *There exists some $C(x, \xi) \in S_{2n}(\langle\xi_b\rangle^{m_1+m_2})$ such that*

$$A_\lambda(x, D_x)B_\lambda(x, D_x) = C_\lambda(x, D_x). \tag{35}$$

Moreover we have

$$C(x, \xi) \sim \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)\left(\frac{i}{2\lambda}\right)^k (D_\xi D_y - D_\eta D_x)^k A(x, \xi)B(y, \eta)|_{(y,\eta)=(x,\xi)} \quad in \quad S_{2n}(\langle\xi_b\rangle^{m_1+m_2}). \tag{36}$$

Theorem 2.13. *Let $A(x, \xi) \in S_{2n}(1)$. Then $A_\lambda(x, D_x)$ is L^2-bounded. Furthermore, we set $\|A\|_* = \left(\sum_{i,j} \|a_{i,j}\|_*^2\right)^{1/2}$. Then, for $u \in \mathcal{S}(\mathbb{R}^n)^N$, we have*

$$\|A_\lambda(x, D_x)u\| \leq C\|A\|_*\|u\| \tag{37}$$

where C is independent of λ, A.

Theorem 2.14. *If $A(x, \xi) = (a_{ij}(x, \xi))_{1\leq i,j\leq N}$, $a_{ij} \in S_{2n}(\langle\xi_b\rangle^m)$ satisfy, for all $(x, \xi) \in \mathbb{R}^{2n}$,*

$$A(x, \xi) \geq 0 \tag{38}$$

then, there exists some constant $C > 0$ such that, for all $u \in \mathcal{S}(\mathbb{R}^n)^N$ and $\lambda > 0$, we have

$$(A_\lambda(x, D_x)u, u) \geq -\frac{C}{\lambda}\left\|\langle D_{x_b}/\lambda\rangle^{m/2}u\right\|^2, \tag{39}$$

where $\langle D_{x_b}/\lambda\rangle = \mathrm{Op}_\lambda(\langle\xi_b\rangle)(x, D_x)$.

Proposition 2.15. *For $\eta \in [0, 1]$, T_η has the following properties.*

1. *T_η is an isometry mapping from $L^2(\mathbb{R}^{n_a}, H^m(\mathbb{R}^{n_b}))^N$ into $L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))^N$.*
2. *$T_\eta^* T_\eta$ is an identity on $L^2(\mathbb{R}^n)^N$.*
3. *$T_\eta T_\eta^*$ is the projection from ${L^2_{(1+\eta)\Phi}}^N$ to ${\mathcal{L}^2_{(1+\eta)\Phi}}^N$. In particular $T_\eta T_\eta^* \widetilde{v} = \widetilde{v}$ if $\widetilde{v} = Tv$, $v \in \mathcal{S}(\mathbb{R}^n)^N$.*

Let $P(x,\xi) = \sum_{|\alpha|\le m} A_\alpha(x)\xi^\alpha$, be a polynomial in ξ whose coefficients are matrix valued function in $C_0^\infty(\mathbb{R}^n : M_N(\mathbb{C}))$. Assume moreover that there exists $c_0 > 0$ such that, for all α, $A_\alpha(x)$ is analytic with respect to x_a in ω_a, where $\omega_a = \{z_a \in \mathbb{C}^{n_a} \mid |z_a| < c_0\}$.

Proposition 2.16. *Let $\psi(x)$ be a real quadratic polynomial. For $u \in C_0^\infty(\mathbb{R}^n)^N$, we have*

$$e^{\lambda\psi} P(x, D_x) e^{-\lambda\psi} u(x) = P_{\lambda,\psi}(x, D_x) u(x), \tag{40}$$

where $P_{\lambda,\psi}(x, D_x)$ is a differential operator defined by

(41)
$$P_{\lambda,\psi}(x, D_x)u(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\xi} P\left(\frac{x+y}{2}, \lambda\left(\xi + i\psi'\left(\frac{x+y}{2}\right)\right)\right) u(y)\, dy\, d\xi.$$

Theorem 2.17. *For $\eta \in [0,1]$ and $v \in C_0^\infty(\mathbb{R}^n)^N$, we have $T_\eta P_{\lambda,\psi}(x, D_x)v = \widetilde{P}_{\lambda,\psi,\eta} T_\eta v$, where*

$$\widetilde{P}_{\lambda,\psi,\eta} T v(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{2n} \iint e^{i\lambda(x_b-y_b)\cdot\xi_b} \left(\iint_{\xi_a = -(1+\eta)\,\mathrm{Im}((x+y)/2)} \omega_\eta\right) dy_b\, d\xi_b. \tag{42}$$

Here ω_η is defined by

$$\omega_\eta = e^{i\lambda(x_a-y_a)\cdot\xi_a} P(X, \lambda(\xi + i\psi'(X))) T_\eta u(y_a, y_b, \lambda)\, dy_a \wedge d\xi_a \tag{43}$$

with $X = ((x_a+y_a)/2 + i(\xi_a/(1+\eta)), (x_b+y_b)/2)$.

In particular, we denote $\widetilde{P}_{\lambda,\psi,0}$ by $\widetilde{P}_{\lambda,\psi}$. Let $\chi \in C_0^\infty$ and $d > 0$ be the same as in Theorem 2.11, then we have the following theorem similar to the scalar case.

Theorem 2.18. *Let $v(x)$ be in $C_0^\infty(\mathbb{R}^n)^N$ with $\operatorname{supp} v \subset \{|x| < d\}$. For $\eta \in (0,1]$, we set*

(44)
$$\widetilde{Q}_\lambda T v(x,\lambda) = \left(\frac{\lambda}{2\pi}\right)^{2n} \iint e^{i\lambda(x_b-y_b)\cdot\xi_b} \left(\iint_{\xi_a = -(1+\eta)\,\mathrm{Im}((x+y)/2)} \chi\left(\frac{x_a+y_a}{2}, \xi_a\right)\omega\right) dy_b\, d\xi_b,$$

where $\omega = \omega_0$ with ω_η defined in Teorem 2.17. *Then we have*

$$\widetilde{P}_{\lambda,\psi} T v = \widetilde{Q}_\lambda T v + \widetilde{R}_\lambda T v + \widetilde{g}_\lambda. \tag{45}$$

Here $\widetilde{g}_\lambda$ and $\widetilde{R}_\lambda T v$ satisfy the following properties. For any $N \in \mathbb{N}$, there exists some positive number C_N such that $\widetilde{R} T v$ and $\widetilde{g}$ satisfy

$$\left\| \widetilde{R}_\lambda T v \right\|_{L^2_{(1+\eta)\Phi}} \le C_N \lambda^{-N} \|T v\|_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))}, \tag{46}$$

$$\left\|\widetilde{g}(\,\cdot\,,\lambda)\right\|_{L^2_{(1+\eta)\Phi}} = O\left(e^{-(\lambda/3)\eta d^2}\|v\|_{H^{n_0}(\mathbb{R}^n)}\right), \quad \lambda \to +\infty, \tag{47}$$

where n_0 depends only on the dimension n and on the order of P.

3. Main estimate

Let v be in $C_0^\infty(\mathbb{R}^n)^N$ and set $w = T_\eta^* T v$. It follows from Proposition 2.15 that

$$T_\eta w(x) = T v(x). \tag{48}$$

Let us set

$$Q_\lambda(x,\xi) = \chi\left(x_a - \frac{i\xi_a}{1+\eta}, \xi_a\right) P(Z, \lambda(\xi + i\psi'(Z))) \tag{49}$$

with $Z = (x_a + (i\eta/(1+\eta))\xi_a, x_b)$, then we deduce from Theorem 2.17

$$\begin{aligned} \widetilde{Q}_\lambda T v(x) &= \widetilde{Q}_\lambda T_\eta w(x) \\ &= T_\eta Q_\lambda w(x), \end{aligned} \tag{50}$$

where Q_λ is the operator defined by

$$Q_\lambda w(x) = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x-y)\cdot\xi} Q_\lambda\left(\frac{x+y}{2}, \xi\right) w(y)\,dy\,d\xi. \tag{51}$$

Moreover setting

$$Q_k(x,\xi) = \sum_{|\alpha|=k} \chi\left(x_a - \frac{i\xi_a}{1+\eta}, \xi_a\right) A_\alpha(Z)(\xi + i\psi'(Z))^\alpha, \tag{52}$$

we have

$$Q_\lambda(x,\xi) = \sum_{k=0}^{m} \lambda^k Q_k(x,\xi). \tag{53}$$

3.1. The estimate of Q_λ. In this section, we prove Carleman estimate for Q_λ. First of all, we precise our choice of ψ. Let φ and P_m satisfy the assumption in Theorem 1.1.

Proposition 3.1. *There exists a quadratic polynomial $\psi(x)$ such that*

1. $\psi(0) = \varphi(0) = 0$, $\psi'(0) = \varphi'(0)$.
2. *There exists some neighborhood W of 0 such that*

$$x \in W \setminus \{0\} \cap \{x \mid \psi(x) = 0\} \Longrightarrow \varphi(x) > 0. \tag{54}$$

Proof. We set

$$\psi(x) = \varphi'(0)x + \frac{1}{2}(\varphi''(0)x, x) + (\varphi'(0)x)^2 - |x|^2. \tag{55}$$

Then ψ obviously satisfies

$$\psi(0) = \varphi(0) = 0, \quad \psi'(0) = \varphi'(0). \tag{56}$$

And, when $\psi(x) = 0$, there exists some $C > 0$ such that

$$\begin{aligned} \varphi'(0)x &= -\frac{1}{2}(\varphi''(0)x, x) - (\varphi'(0)x)^2 + |x|^2 \\ &\leq C|x|^2. \end{aligned}$$

Then, by Taylor's formula, we have

$$\begin{aligned} \varphi(x) &= \varphi'(0)x + \frac{1}{2}(\varphi''(0)x, x) + O(|x|^3) \\ &= -(\varphi'(0)x)^2 + |x|^2 + O(|x|^3) \\ &\geq -(C|x|^2)^2 + |x|^2 + O(|x|^3). \end{aligned}$$

Therefore $\varphi(x) > 0$ when x is small enough and $x \neq 0$. The proposition is proved. □

Lemma 3.2. *There exist positive numbers ε_0, δ such that if $|x| + |\xi_a| < \varepsilon_0$, for any $\xi_b \in \mathbb{R}^{n_b}$, we have*

$$Q_m(x, \xi)^* Q_m(x, \xi) - \delta \langle \xi_b \rangle^{2m} I \geq 0. \tag{57}$$

Proof. Because $|x|+|\xi_a|$ is small enough, we may assume $\chi(x_a+i\xi_a/(1+\eta), \xi_a) = 1$. Then it follows that

$$Q_m(x, \xi)^* Q_m(x, \xi) = P_m(X, \xi + i\psi'(X))^* P_m(X, \xi + i\psi'(X)), \tag{58}$$

where $X = (x_a + i\eta\xi_a/(1 + \eta), x_b)$.

(1) The case of large ξ_b.

By the definition of P_m, we have

$$\begin{aligned} &P_m(X, \xi + i\psi'(X))^* P_m(X, \xi + i\psi'(X)) \\ &= P_m(X, 0, \xi_b)^* P_m(X, 0, \xi_b) + \sum_{|\beta| \leq 2m-1} B_\beta(X, \xi_a)\xi_b^\beta, \end{aligned} \tag{59}$$

where, for any β, $B_\beta(x, \xi_a)$ is an hermitian matrix whose components are in $C^\infty(\mathbb{R}^n_x \times$

$\mathbb{R}^n_\xi$). Then by homogeneity of P_m, for any $w \in \mathbb{R}^N$ with $|w| = 1$, we have

$$
\begin{aligned}
(Q_m(x,\xi)^* Q_m(x,\xi)w, w) &= (P_m(X,0,\xi_b)^* P_m(X,0,\xi_b)w, w) \\
&\quad + \sum_{|\beta| \le 2m-1} \xi_b^\beta (B_\beta(X,\xi_a)w, w) \\
&= |\xi_b|^{2m} \left\{ \left(P_m\left(X, 0, \frac{\xi_b}{|\xi_b|}\right)^* P_m\left(X, 0, \frac{\xi_b}{|\xi_b|}\right) w, w \right) \right. \\
&\quad \left. + \frac{1}{|\xi_b|} \sum_{|\beta| \le 2m-1} \frac{\xi_b^\beta}{|\xi_b|^{2m-1}} (B_\beta(X,\xi_a)w, w) \right\}.
\end{aligned}
$$

On the other hand, it follows from the ellipticity of $P_m(0,0,\xi_b)$ and the continuity of $P_m(X,0,\xi_b)$ that there exist positive constants δ_1, ε_1 such that, if $|\xi_b| = 1$ and $|x| + |\xi_a| < \varepsilon_1$, we have

$$(P_m(X,0,\xi_b)^* P_m(X,0,\xi_b)w, w) > 2\delta_1. \tag{60}$$

And, by the continuity of B_β, there exists some $C_\beta > 0$ which satisfies that, for any (x, ξ_a) with $|x| + |\xi_a| < \varepsilon_1$, we have

$$|B_\beta(X,\xi_a)w| < C_\beta. \tag{61}$$

It follows that if we take $C > 0$ such that $C^{-1} \sum_\beta C_\beta < \delta_1$, for any ξ_b with $|\xi_b| > C$, we have

$$
\begin{aligned}
(Q_m(x,\xi)^* Q_m(x,\xi)w, w) &\ge |\xi_b|^{2m} \left(2\delta_1 - \frac{1}{|\xi_b|} \sum_{|\beta| \le 2m-1} |B_\beta(X,\xi_a)w| \right) \\
&\ge \delta_1 |\xi_b|^{2m}.
\end{aligned}
$$

Therefore if $|\xi_b| > C$ and $|x| + |\xi_a| < \varepsilon_1$,

$$Q_m(x,\xi)^* Q_m(x,\xi) - \delta_1 \langle \xi_b \rangle^{2m} I \tag{62}$$

is positive definite.

(2) The case that ξ_b is bounded.

We argue by contradiction. Assume that there exist sequences $\{x_k\} \subset \mathbb{R}^n$, $\{\xi_{ak}\} \subset \mathbb{R}^{n_a}$, $\{\xi_{bk}\} \subset \mathbb{R}^{n_b}$, $\{w_k\} \subset \mathbb{R}^N$ which satisfy

$$
\begin{gathered}
|x_k| + |\xi_{ak}| < \frac{1}{k}, \quad |\xi_{bk}| \le C, \quad |w_k| = 1, \\
(Q_m(x_k,\xi_{ak},\xi_{bk})^* Q_m(x_k,\xi_{ak},\xi_{bk})w_k, w_k) < \frac{1}{k} \langle \xi_{b_k} \rangle^{2m}.
\end{gathered}
$$

Because $\{\xi_{bk}\}, \{w_k\}$ are bounded, we can take convergent sequences $\{\xi_{bk_j}\}, \{w_{k_j}\}$ with $\lim_{j\to\infty} \xi_{bk_j} = \xi_{b0}$, $\lim_{j\to\infty} w_{k_j} = w_0$. By the fact that $\lim_{j\to\infty} x_{k_j}$, $\lim_{j\to\infty} \xi_{a,k_j}$ are equal to 0, and by the continuity of Q_m, we have

$$(Q_m(0,0,\xi_{b0})^* Q_m(0,0,\xi_{b0})w_0, w_0) = 0, \tag{63}$$

which contradicts that $\det Q_m(0,0,\xi_{b0}) = \det P_m(0, \varphi_a'(0), \xi_{b0} + i\varphi_b'(0)) \neq 0$. thus the lemma is proved. □

Let $h(x_b) \in C_0^\infty(\mathbb{R}^{n_b} : [0,1])$ satisfy

$$h(x_b) = \begin{cases} 1 & |x_b| < \dfrac{\varepsilon_1}{6}, \\ 0 & |x_b| > \dfrac{\varepsilon_1}{3}, \end{cases} \tag{64}$$

where ε_1 is a small parameter, and let $\widetilde{\theta}_0 \in C_0^\infty(\mathbb{C}^{n_a} \times \mathbb{C}^{n_a} : [0,1])$ be almost analytic in $\Lambda_{(1+\eta)\Phi}$. Moreover we can take $\widetilde{\theta}_0$ satisfies

$$\widetilde{\theta}_0(z_a, \zeta_a) = \begin{cases} 1 & |z_a| + |\zeta_a| < \dfrac{\eta}{1+\eta}\dfrac{\varepsilon_1}{3}, \\ 0 & |z_a| + |\zeta_a| > \dfrac{2\varepsilon_1}{3}. \end{cases} \tag{65}$$

When we define $\theta_1(x, \xi_a) \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^{n_a})$ by

$$\theta_1(x, \xi_a) = \widetilde{\theta}_0 \circ \kappa_\eta(x_a, \xi_a) h(x_b), \tag{66}$$

then we have

$$\theta_1(x, \xi_a) = \begin{cases} 1 & |x| + |\xi_a| < \dfrac{\varepsilon_1}{3}, \\ 0 & |x| + |\xi_a| > \varepsilon_1. \end{cases} \tag{67}$$

Lemma 3.3. *There exist positive numbers C_1, C_2, λ_0 such that, for any $\lambda > \lambda_0$ and any $u \in \mathcal{S}(\mathbb{R}^n)^N$, we have*

$$C_1\lambda^{2m}\big(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)u, u\big) + C_2\|Q_\lambda u\|^2 \geq \frac{\lambda^{2m}\delta}{2}\left\|\langle D_{x_b}/\lambda\rangle^m u\right\|^2. \tag{68}$$

Proof. By Lemma 3.2, we can take $C_1 > 0$ large enough such that

$$C_1(1-\theta_1(x,\xi_a))\langle\xi_b\rangle^{2m} + Q_m(x,\xi)^* Q_m(x,\xi) - \delta\langle\xi_b\rangle^{2m} I \geq 0. \tag{69}$$

By this inequality and Theorem 2.14, there exists some positive constant C_3 such that, for any $u \in \mathcal{S}(\mathbb{R}^n)^N$,

$$\begin{aligned} &C_1(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})u, u) + ((Q_m^* Q_m)_\lambda(x, D_x)u, u) - \delta(\langle D_{x_b}/\lambda\rangle^{2m}u, u) \\ &\geq -\frac{C_3}{\lambda}\left\|\langle D_{x_b}/\lambda\rangle^m u\right\|^2 . \end{aligned} \tag{70}$$

On the other hand, by Theorem 2.12, we have

$$(Q_m^* Q_m)_\lambda(x, D_x) = Q_{m\lambda}(x, D_x)^* Q_{m\lambda}(x, D_x) + \frac{1}{\lambda} R_\lambda(x, D_x), \tag{71}$$

where $R_\lambda(x, D_x)$ is a pseudodifferential system whose symbol $R(x, \xi, \lambda)$ be an hermitian matrix in $S_{2n}(\langle\xi_b\rangle^{2m})$. Therefore we have

$$\begin{aligned} &C_1(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)u, u) + \| Q_{m\lambda}(x, D_x)u\|^2 \\ &\geq \left(\delta - \frac{C_3}{\lambda}\right)\left\|\langle D_{x_b}/\lambda\rangle^m u\right\|^2 - \frac{1}{\lambda}(R_\lambda(x, D_x)u, u). \end{aligned} \tag{72}$$

Moreover, by Theorem 2.13, there exists some positive number C_4 such that

$$\begin{aligned} |(R_\lambda(x, D_x)u, u)| &\leq \left\|\langle D_{x_b}/\lambda\rangle^{-m} R_\lambda(x, D_x)u\right\| \left\|\langle D_{x_b}/\lambda\rangle^m u\right\| \\ &\leq C_4 \left\|\langle D_{x_b}/\lambda\rangle^m u\right\|^2 . \end{aligned}$$

Then when we take $\lambda_1 > 0$ with $2C_3/\lambda_1 < \delta/3$, it follows that, for any $\lambda > \lambda_1$,

$$C_1(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)u, u) + \| Q_{m\lambda}(x, D_x)u\|^2 \geq \frac{2}{3}\delta\left\|\langle D_{x_b}/\lambda\rangle^m u\right\|^2 . \tag{73}$$

On the other hand, by the definition of Q_λ, we have

$$\begin{aligned} \| Q_{m\lambda}(x, D_x)u\|^2 &= \left\|\lambda^{-m} Q_\lambda u - \sum_{k=0}^{m-1} \lambda^{k-m} Q_{k\lambda}(x, D_x)u\right\|^2 \\ &\leq 2^m \left(\lambda^{-2m}\| Q_\lambda u\|^2 + \sum_{k=0}^{m-1} \lambda^{2k-2m}\| Q_{k\lambda}(x, D_x)u\|^2\right) \\ &\leq 2^m \lambda^{-2m}\| Q_\lambda u\|^2 + \frac{mC_4}{\lambda^2}\left\|\langle D_x/\lambda\rangle^m u\right\|^2 , \end{aligned}$$

where C_5 is a positive constant independent of u and λ. Then if we take λ_2 with $mC_4/\lambda_2^2 < \delta/6$, for any $\lambda > \lambda_2$, we have

$$\| Q_{m,\lambda}(x, D_x)u\|^2 \leq 2^m \lambda^{-2m}\| Q_\lambda u\|^2 + \frac{\delta}{6}\left\|\langle D_x/\lambda\rangle^m u\right\|^2 . \tag{74}$$

It follows that, if $\lambda > \lambda_0 = \max(\lambda_1, \lambda_2)$, we have

$$C_1((\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^m)(x, D_x)u, u) + 2^m\lambda^{-2m}\|Q_\lambda u\|^2 \geq \frac{\delta}{2}\left\|\langle D_{x_b}/\lambda\rangle^m u\right\|^2, \tag{75}$$

which completes the proof. □

Lemma 3.4. *We can take positive real numbers ε_2, σ_0 which have the following properties: Let $v \in C_0^\infty(\mathbb{R}^n)^N$ satisfy $\operatorname{supp} v \subset \{x : |x| < \varepsilon_2\}$. Then, for any $k \in \mathbb{N}$, there exist positive constants $C_{k,1}, C_{k,2}$ such that*

$$\begin{aligned} &|(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)T_\eta^* T v, T_\eta^* T v)| \\ &\leq \frac{C_{k,1}}{\lambda^k}\left\|\langle D_{x_b}/\lambda\rangle^m u\right\|_{L^2_{(1+\eta)\Phi}} + C_{2,k}e^{-\lambda\sigma_0}\|v\|^2_{H^{n_0}}, \end{aligned} \tag{76}$$

where n_0 is an integer depending only on the dimension n.

Proof. Using Proposition 2.15, we have

$$\begin{aligned} &(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)T_\eta^* T v, T_\eta^* T v) \\ &= (T_\eta \,\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)T_\eta^* T v, T_\eta T_\eta^* T v)_{L^2_{(1+\eta)\Phi}}. \end{aligned} \tag{77}$$

And, by Theorem 2.17, we have

$$\begin{aligned} &T_\eta \,\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)T_\eta^* T v(x) \\ &= \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\langle\xi_b\rangle^{2m}\iint_{\xi_a = -(1+\eta)\operatorname{Im}((x_a+y_a)/2)} w_\eta \, dy_b \, d\xi_b, \end{aligned} \tag{78}$$

where ω_η is defined by

(79)

$$w_\eta = e^{i\lambda(x_a-y_a)\cdot\xi_a}\left(1-\theta_1\left(\frac{x_a+y_a}{2} + \frac{i\xi_a}{1+\eta}, \frac{x_b+y_b}{2}, \xi_a\right)\right) T_\eta T_\eta^* T v(y_a, y_b)\, dy_a \wedge d\xi_a.$$

By the definition of θ_1, we have

$$\begin{aligned} \theta_1\left(\frac{x_a+y_a}{2} + \frac{i\xi_a}{1+\eta}, \frac{x_b+y_b}{2}, \xi_a\right) &= \theta_1\left(\kappa_\eta^{-1}\left(\frac{x_a+y_a}{2}, \xi_a\right), \frac{x_b+y_b}{2}\right) \\ &= \widetilde{\theta}_0\left(\frac{x_a+y_a}{2}, \xi_a\right) h\left(\frac{x_b+y_b}{2}\right) \end{aligned} \tag{80}$$

and by Proposition 2.15,

$$T_\eta T_\eta^* T v = T v. \tag{81}$$

Therefore

$$w_\eta = e^{i\lambda(x_a-y_a)\cdot\xi_a}\left(1-\widetilde{\theta}_0\left(\frac{x_a+y_a}{2}, \xi_a\right) h\left(\frac{x_b+y_b}{2}\right)\right) T v(y_a, y_b)\, dy_a \wedge d\xi_a. \tag{82}$$

We define $\widetilde{S}Tv$ by

$$\widetilde{S}Tv = \left(\frac{\lambda}{2\pi}\right)^n \iint e^{i\lambda(x_b-y_b)\cdot\xi_b}\langle\xi_b\rangle^{2m}\iint_{\xi_a=-(1+\eta)\,\mathrm{Im}((x_a+y_a)/2)} w_\eta\,dy_b\,d\xi_b. \tag{83}$$

By the same method in [8], we can show the following inequality: There exists some $\varepsilon_2 > 0$ such that if $v \in C_0^\infty(\mathbb{R}^n)^N$ satisfy $\operatorname{supp} v \subset \{x : |x| < \varepsilon_2\}$, then, for any positive integer k, we can take $\sigma_0, C_{k,1}, C_{k,2} > 0$ which satisfy

$$\|\widetilde{S}Tv\|_{L^2_{(1+\eta)\Phi}} \le \frac{C_{k,1}}{\lambda^k}\,\|\langle D_{x_b}/\lambda\rangle^m Tv\|_{L^2_{(1+\eta)\Phi}} + C_{k,2}e^{-\lambda\sigma_2}\|v\|_{H^{n_0}}, \tag{84}$$

where n_0 is a positive integer dependent only on the dimension n. From this inequality, for any positive integer k, we have

$$\begin{aligned}
&|(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)T_\eta^*Tv, T_\eta^*Tv)| \\
&= \left|\left(\widetilde{S}Tv, Tv\right)_{L^2_{(1+\eta)\Phi}}\right| \\
&\le \|\widetilde{S}Tv\|_{L^2_{(1+\eta)\Phi}}\|Tv\|_{L^2_{(1+\eta)\Phi}} \\
&\le \frac{C_{k,1}}{\lambda^k}\,\|\langle D_{x_b}/\lambda\rangle^m Tv\|^2_{L^2_{(1+\eta)\Phi}} + C_{k,2}e^{\lambda\sigma}\|v\|_{H^{n_0}}\|Tv\|_{L^2_{(1+\eta)\Phi}},
\end{aligned} \tag{85}$$

which completes the proof because $\|Tv\|_{L^2_{(1+\eta)\Phi}} \le \|Tv\|_{L^2_\Phi} = \|v\|_{L^2} \le \|v\|_{H_0^n}$. □

Theorem 3.5. *For* $\widetilde{P}_{\lambda,\psi}$ *defined in* Theorem 2.17, *there exist some positive constants* $C_1, C_2, n_0, \lambda_0, \varepsilon, \sigma$ *such that, for any* $v \in C_0^\infty(\mathbb{R}^n)^N$ *with* $\operatorname{supp} v \subset \{x : |x| < \varepsilon\}$ *and* $\lambda \ge \lambda_0$, *we have*

$$C_1\lambda\|\widetilde{P}_{\lambda,\psi}Tv\|^2_{L^2_{(1+\eta)\Phi}} + C_2e^{-\lambda\sigma}\|v\|^2_{H^{n_0}} \ge \|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))}. \tag{86}$$

Proof. By Proposition 2.15, we have

$$\begin{aligned}
\|Q_\lambda T_\eta^*Tv\| &= \|T_\eta Q_\lambda T_\eta^*Tv\|_{L^2_{(1+\eta)\Phi}} \\
&= \|\widetilde{Q}_\lambda T_\eta T_\eta^*Tv\|_{L^2_{(1+\eta)\Phi}} \\
&= \|\widetilde{Q}_\lambda Tv\|_{L^2_{(1+\eta)\Phi}}.
\end{aligned}$$

Moreover, by choosing ε_1 with $\varepsilon_1 < d$, it follows from Theorem 2.18 that there exist some constants $C_{m,1}, C_{m,2} > 0$ such that

$$\begin{aligned}
\|\widetilde{Q}_\lambda Tv\|_{L^2_{(1+\eta)\Phi}} &= \|\widetilde{P}_{\lambda,\psi}Tv - \widetilde{R}_\lambda Tv - \widetilde{g}_\lambda\|_{L^2_{(1+\eta)\Phi}} \\
&\le \|\widetilde{P}_{\lambda,\psi}Tv\|_{L^2_{(1+\eta)\Phi}} + \|\widetilde{R}_\lambda Tv\|_{L^2_{(1+\eta)\Phi}} + \|\widetilde{g}_\lambda\|_{L^2_{(1+\eta)\Phi}} \\
&\le \|\widetilde{P}_{\lambda,\psi}Tv\|_{L^2_{(1+\eta)\Phi}} + \frac{C_{m,1}}{\lambda^m}\|Tv\|_{L^2_{(1+\eta)\Phi}} + C_{m,2}e^{-(1/3)\eta d^2\sigma_0}\|v\|_{H^{n_0}}.
\end{aligned}$$

On the other hand, if we take $\varepsilon > 0$ satisfying $\varepsilon < \varepsilon_2$, by Lemma 3.4, there exist some positive constants $C_{m,3}, C_{m,4}$ such that

$$\begin{aligned}&\lambda^{2m}\left|\left(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)T_\eta^* T v, T_\eta^* T v\right)\right| \\ &\leq C_{m,3}\left\|\langle D_{x_b}/\lambda\rangle^m T v\right\|^2_{L^2_{(1+\eta)\Phi}} + C_{m,4}\lambda^{2m}e^{-\lambda\sigma_0}\|v\|^2_{H^{n_0}}.\end{aligned} \tag{87}$$

Then, because $\lambda^{2m}e^{-\lambda\sigma_0/2} < 1$ with λ large enough, we have

$$\begin{aligned}&\lambda^{2m}\left|\left(\mathrm{Op}_\lambda((1-\theta_1)\langle\xi_b\rangle^{2m})(x, D_x)T_\eta^* T v, T_\eta^* T v\right)\right| \\ &\leq C_{m,3}\left\|\langle D_{x_b}/\lambda\rangle^m T v\right\|^2_{L^2_{(1+\eta)\Phi}} + C_{m,4}e^{-\lambda\sigma_0/2}\|v\|^2_{H^{n_0}}.\end{aligned} \tag{88}$$

It follows from these inequalities and Lemma 3.3 that, if λ is large enough, there exist positive constants C_5, C_6, C_7, C_8 such that

$$\begin{aligned}&C_5\left\|\widetilde{P}_\lambda T v\right\|^2_{L^2_{(1+\eta)\Phi}} + \left(C_6 e^{-(\lambda/2)\sigma_0} + C_7 e^{-(2/3)\lambda\eta d^2}\right)\|v\|_{H^{n_0}} \\ &\geq \left(\frac{\delta}{2}\lambda^{2m} - C_8\right)\left\|\langle D_{x_b}/\lambda\rangle^m T v\right\|^2_{L^2_{(1+\eta)\Phi}}.\end{aligned} \tag{89}$$

Therefore setting $\sigma = \min(\sigma_0/2, (2/3)\eta d^2)$, we have

$$C_5\left\|\widetilde{P}_\lambda T v\right\|^2_{L^2_{(1+\eta)\Phi}} + (C_6 + C_7)e^{-\sigma\lambda}\|v\|_{H^{n_0}} \geq \frac{\delta}{4}\lambda^{2m}\left\|\langle D_{x_b}/\lambda\rangle^m T v\right\|^2_{L^2_{(1+\eta)\Phi}}, \tag{90}$$

which completes the proof. □

4. Proof of Main Theorem

In this section, using the estimate proved in the previous section, we prove our main theorem. Let us assume that P satisfies the condition of Theorem 1.1, that u is a C^∞-solution near the origin of the equation $Pu = 0$ with $\operatorname{supp} u \subset \{x \in \mathbb{R}^n : \varphi(x) \leq 0\}$, and that ψ is introduced in Proposition 3.1. Let $\chi \in C_0^\infty(\mathbb{R} : [0, 1])$ satisfy

$$\chi(t) = \begin{cases} 1 & t \geq -\dfrac{\varepsilon_2}{2}, \\ 0 & t \leq -\varepsilon_2, \end{cases} \tag{91}$$

where ε_2 is a positive number small enough. We set

$$u_1(x) = \chi(\psi(x))u(x). \tag{92}$$

Then if ε_2 is small enough, it follows that there exists some neighborhood V_0 of 0, and some positive number C such that

$$\operatorname{supp} u_1 \cap V_0 \subset \{x : |x|^2 < C\varepsilon_2\}. \tag{93}$$

If ε_2 is small enough so that $\varepsilon \geq \sqrt{C\varepsilon_2}$, we have

$$\text{supp } u_1 \cap V_0 \subset \{x : |x| < \varepsilon\}. \tag{94}$$

Moreover we set $Pu_1 = f$. Because $Pu = 0$, f satisfies

$$\text{supp } f \subset \left\{x : -\varepsilon \leq \psi(x) \leq -\frac{\varepsilon}{2}\right\}. \tag{95}$$

Let us set

$$v = e^{\lambda\rho\psi} u_1, \tag{96}$$

where ρ is a positive parameter. Then we have

$$\widetilde{P}_{\lambda,\rho,\psi} T v = T(e^{\lambda\rho\psi} f).$$

By Theorem 3.5, there exist positive constants $\lambda_0, \sigma, C_1, C_2$ such that, for $\lambda > \lambda_0$, we have

$$\|Tv\|_{L^2_{(1+\eta)\Phi}(\mathbb{C}^{n_a}, H^m(\mathbb{R}^{n_b}))} \leq C_1 \lambda \|T(e^{\lambda\rho\psi} f)\|_{L^2_{(1+\eta)\Phi}} + C_2 e^{-\lambda\sigma} \|v\|_{H^{n_0}}. \tag{97}$$

By using (95), we can show the following estimate:

$$\|T(e^{\lambda\rho\psi} f)\|_{L^2_{(1+\eta)\Phi}} = O(e^{-(\lambda/3)\varepsilon\rho}), \quad \lambda \to +\infty. \tag{98}$$

Therefore, when λ is large enough, there exists some $C_{\rho,\varepsilon} > 0$ such that

$$\|Tv\|_{L^2_{(1+\eta)\Phi}} \leq C_{\rho,\varepsilon} e^{-\delta\lambda}, \tag{99}$$

where $\delta = \min(\varepsilon\rho/3, \sigma)$. By the same argument of the last section in [8], if ρ satisfies the inequality:

$$\rho \|\psi''(0)\| < \frac{1}{4}, \tag{100}$$

there exists some neighborhood V of 0 where $u = 0$, which completes the proof of the main theorem.

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