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A LEVI-FLAT IN A KUMMER SURFACE WHOSE COMPLEMENT IS STRONGLY PSEUDOCONVEX

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Abstract

It is shown that certain Kummer surface admits a Levi flat hypersurface whose complement is strongly pseudoconvex.

Introduction

It has long been know that some compact complex manifolds admit Levi-flats, which are by definition smooth real hypersurfaces separating the ambient manifolds locally into two Stein open subsets. For instance, for any compact complex manifold M, $\mathbb{CP}^1$-bundles over M with structure group $PSL(2,\mathbb{R})$ admit subbundles with fiber $\mathbb{RP}^1$ as Levi-flats. Since the complements of Levi-flats are locally pseudoconvex in the ambient manifolds, their function theoretic properties are of interest. In the case of $\mathbb{CP}^1$-bundles over compact Riemann surfaces, it is known that the complement of such a Levi-flat is a proper modification of a Stein space, or equivalently a strongly pseudoconvex manifold by Grauert's theorem [2], if and only if the $PSL(2,\mathbb{R})$-bundle is *not* $PSL(2,\mathbb{C})$-equivalent to a $U(1)$-bundle (cf. [1]).

Recently, Y.-T. Siu [4] established a remarkable result which says that there exist no Levi-flats of class C^8 in $\mathbb{CP}^n$ if $n \geq 2$. Generalizing his method, the author classified the real analytic Levi-flats in complex tori of dimension two into two types, i.e. holomorphically flat ones and Levi scrolls, the latter being with Stein complements remarkably (cf. [3]). In particular, it turned out that there exist Levi-flats in the product of two elliptic curves such that their complements are Stein.

In view of these facts, it seems natural to make an effort toward classifying the Levi-flats in other complex surfaces.

The purpose of the present note is to suggest that such an effort might be rewarding by showing that there exists a Levi-flat with strongly pseudoconvex complement in a Kummer surface, or equivalently there exists a Levi scroll in the product of two elliptic curves which is invariant under the involution $(p,q) \mapsto (-p,-q)$ and free from the fixed points.

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1. Construction of the Levi-flat

Let us recall the following dichotomy for Levi-flats in the two dimensional tori.

Theorem 1.1 (cf. [3]). *Let T be a complex torus of dimension two and let $S \subset T$ be a real analytic Levi-flat. Then one of the following holds.*
1) *S is holomorphically flat, i.e. S is the union of flatly embedded complex submanifolds of codimension one in T.*
2) *There exists an elliptic curve C, a holomorphic submersion π from T onto C, and fibers $C_1, \dots, C_{2n}$ of π for some $n \in \mathbb{N}$ such that $\pi|S$ is surjective and has critical fibers $C_1, \dots, C_{2n}$. (S is called a Levi scroll in this case.)*

For simplicity we shall restrict ourselves to the case where the torus T is the product $C \times C$ and π is the projection to the second factor.

Then we put

$$C = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \quad \operatorname{Im}\tau > 0,$$
$$\Sigma = \{[z] \in C \mid [z] = [-z]\} \quad ([z] := z + \mathbb{Z} + \mathbb{Z}\tau)$$

and

$$\Sigma' = \left\{ [z] \in C \;\middle|\; [z] = \left[-z + \frac{1+\tau}{2} \right] \right\}.$$

We define $\sigma, \sigma' \in \operatorname{Aut} C$ by

$$\sigma([z]) = [-z], \quad \sigma'([z]) = \left[-z + \frac{1+\tau}{2} \right].$$

Clearly $\sigma^2 = \sigma'^2 = \mathrm{id}$ and Σ (resp. Σ') is the set of fixed points of σ (resp. σ'). Note that σ acts on Σ'.

Lemma. *There exists a meromorphic 1-form ω on C with poles at Σ' such that $\sigma^*\omega = -\omega$ and $\operatorname{Res}_P \omega \in \{1, -1\}$ for any $P \in \Sigma'$.*

Proof. Let $\Sigma'' = \Sigma + 1/4$, let σ'' be the involution with fixed point set Σ'', and let $f : C \to C/\{\mathrm{id}, \sigma''\}$ be the natural projection to the factor space by the action of $\{\mathrm{id}, \sigma''\}$. Let ζ be the inhomogeneous coordinate of $C/\{\mathrm{id}, \sigma''\}$ ($\simeq \mathbb{CP}^1$) such that $\zeta^{-1}(\{0, \infty\}) = \Sigma'$. Since σ interchanges the zeros and the poles of ζ, we have

$$\omega_0 := \sigma^* f^*(d \log \zeta) - f^*(d \log \zeta) \neq 0.$$

Multiplying a nonzero constant to ω_0, we obtain the desired ω.

We put

$$A = \left\{ a \in \mathbb{C} \;\middle|\; \frac{\tau}{2\pi i} \int_0^1 (\omega + a\, dz) \in \mathbb{R} \right\}$$
$$B = \left\{ b \in \mathbb{C} \;\middle|\; \frac{\tau}{2\pi i} \int_0^{\tau/2} (\omega + b\, dz) \in \mathbb{R} + \mathbb{Z}\tau \right\}.$$

Clearly $A \cap B \neq \emptyset$.

Now we take $c \in A \cap B$, fix a point $z_0 \in \Sigma$, and define a closed real hypresurface S_0 in $C \times (C \setminus \Sigma')$ by

$$S_0 = \left\{ ([w], [z]) \;\middle|\; \operatorname{Im}\left(w + \frac{\tau}{2\pi i} \int_{z_0}^{z} (\omega + c\, dz) \right) = \frac{\operatorname{Im}\tau}{4} \text{ or } \frac{3 \operatorname{Im}\tau}{4} \right\}.$$

By the period condition on $\omega + c\, dz$, S_0 is well defined. S_0 is invariant under σ because of the antisymmetricity of ω and dz. It is easy to see that $\overline{S_0}$ is smooth in T because of the residue condition on ω.

Since $\overline{S_0}$ is a Levi scroll and $\overline{S_0} \cap (\Sigma \times \Sigma) = \emptyset$ we are done. Namely, we have a compact real analytic Levi-flat $S = \overline{S_0} \big/ \{\mathrm{id}, \sigma\}$ in the regular part of the complex space $T/\{\mathrm{id}, \sigma\}$, so that a Levi-flat in a Kummer surface as the preimage of the desingularization of $T/\{\mathrm{id}, \sigma\}$. Strong pseudoconvexity of the complement is obvious. □

2. Notes and remarks

1. The above method is obviously applicable to construct invariant Levi scrolls in non-simple Abelian surfaces, or more generally in elliptic fiber bundles over compact Riemann surfaces.
2. There exist obvious invariant Levi-flats in $C \times C$ which are holomorphically flat. But they are not so interesting at least in their own right.
3. Classify the (real analytic) Levi-flats in Kummer surfaces.
4. The author does not know how to prove or disprove the existence of (nontrivial) Levi-flats in general elliptic $K3$ surfaces.

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ADDED IN PROOF. Quite recently, M. Brunella found a serious gap in Siu's paper [4], which means that the classifications of Levi-flats in $\mathbb{CP}^2$ and complex 2-tori are not yet complete. In particular, we must take back the dichotomy in Theorem 1.1. However, this does not affet the validity of the result in the present article.

References

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