

## NUMERICAL ALGORITHM FOR FINDING BALANCED METRICS

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### Abstract

The purpose of this paper is to give an explicit statement with respect to a numerical algorithm for finding balanced metrics, which has already been pointed out by Donaldson [3].

### 1. Introduction

First of all, we shall recall the definition of balanced metrics. Let  $(X, L)$  be a compact complex manifold of complex dimension  $n$  with a very ample line bundle. Let  $h$  be a Hermitian metric on  $L$ , which defines a Kähler form  $\omega_h := (\sqrt{-1}/(2\pi))\Theta(h)$  on  $X$ , where  $\Theta(h)$  is the curvature form induced by an unitary connection with respect to  $h$ . The metric  $h$  and the Kähler form  $\omega_h$  define an inner product  $\langle \cdot, \cdot \rangle_h$  on the  $(N + 1)$ -dim vector space  $E := H^0(X, L)$  by

$$\langle S, S' \rangle_h := \int_X h(S, S') dV_{\omega_h}$$

where  $dV_{\omega_h} := (1/(n!))\omega_h^n$  is the volume form with respect to  $h$ . Let  $\{S_0, \dots, S_N\}$  be an orthonormal basis with respect to the above inner product. The metric form  $\omega_h$  is called *balanced* when the Bergman kernel

$$B_{\omega_h}(p) := \sum_{k=0}^N |S_k(p)|_h^2$$

is constant on  $X$ .

The existence of balanced metrics is equivalent to the existence of the specific projective embedding of  $X$ . For a basis  $\{S_i\}$  of  $E$ , Kodaira embedding theorem implies a projective embedding

$$\iota_{\{S_i\}} : X \hookrightarrow \mathbf{CP}^N$$

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by  $t_{\{S_i\}}(p) := [S_0(p) : \cdots : S_N(p)]$ . The embedding  $t_{\{S_i\}}$  is called *balanced* when

$$\frac{N+1}{V} \int_X \frac{S_i \bar{S}_j}{\sum_{k=0}^N |S_k|^2} dV_{t^* \omega_{FS}} = \delta_{ij},$$

where  $V$  and  $\omega_{FS}$  denote the volume of  $X$  and the Fubini-Study metric form on  $\mathbf{CP}^N$  respectively. At this moment, we find that  $t^* \omega_{FS}$  is balanced.

The existence of balanced metrics is closely related to stability of the projective variety  $t(X) \subset \mathbf{CP}^N$  in the sense of Geometric Invariant Theory (cf. [6], [9], [12]). Using the asymptotics of the Bergman kernel, Donaldson [2] and Mabuchi [7] proved that the existence of a constant scalar curvature metric in  $c_1(L)$  implies asymptotic stability of  $(X, L)$ .

The purpose of this paper is to carry out explicitly a numerical algorithm for finding balanced metrics. This algorithm has already been obtained by Donaldson in Remark at the end of [3]. However, in view of the importance of the problem, it would be interesting to carry out the procedure very explicitly. For that purpose, we shall use two maps defined by Donaldson as follows. Let  $\mathcal{K}$  and  $M$  be the set of metrics  $h$  on  $L$  such that  $\omega_h$  is a positive  $(1, 1)$  form on  $X$  and the set of Hermitian metrics on  $E$  respectively. We fix a base point  $h_0$  in  $\mathcal{K}$ . We can identify  $\mathcal{K}$  as the set of Kähler potentials, that is,

$$\mathcal{K} = \left\{ \phi \in C^\infty(X) \mid \omega_{h_0} - \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi > 0 \right\},$$

and  $M$  as the set of  $(N+1) \times (N+1)$  positive Hermitian matrices with respect to a fixed basis of  $E$  respectively. We define two maps on  $\mathcal{K}$  and  $M$ .

- Define a map

$$\text{Hilb}: \mathcal{K} \rightarrow M$$

by

$$\|S\|_{\text{Hilb}(h)}^2 := \frac{N+1}{V} \int_X |S|_h^2 dV_{\omega_h}.$$

- For a given  $H \in M$ , let  $\{S_i^H\}$  be an orthonormal basis of  $E$  with respect to  $H$ , which is occasionally abbreviated as  $H$ -ONB. So we define a map

$$FS: M \rightarrow \mathcal{K}$$

by

$$FS(H) := \log \frac{1}{\sum_{k=0}^N |S_k^H|_{h_0}^2}.$$

Of course,  $FS(H)$  is independent of the choice of an  $H$ -ONB and

$$(1.1) \quad \sum_{k=0}^N |S_k^H|_{FS(H)}^2 \equiv 1.$$

We note that the induced form  $\omega_{FS(H)}$  equals to  $\iota_{\{S_i^H\}}^* \omega_{FS}$ .

DEFINITION 1.1. A point  $h \in \mathcal{K}$  (resp.  $H \in M$ ) are called balanced if and only if

$$FS \circ \text{Hilb}(h) = h \quad (\text{resp. } \text{Hilb} \circ FS(H) = H).$$

By definitions, a balanced metric corresponds to a balanced pair  $(h, H)$  up to scalar multiplication. Donaldson pointed out that the iteration of the map  $\text{Hilb} \circ FS: M \rightarrow M$  converges to a balanced point which is the minimiser of a functional  $\tilde{Z}: M \rightarrow \mathbf{R}$  defined in [3]. We denote  $\text{Aut}(X, L)$  be the group of automorphisms of the pair  $(X, L)$  and the trivial automorphisms  $\mathbf{C}^*$  act by constant scalar multiplication on the fibre of  $L$ . Our main result is the following.

**Theorem 1.2.** *Fix a point  $H_0 \in M$ . Let  $H_l := \text{Hilb} \circ FS(H_{l-1})$ . Suppose that  $\text{Aut}(X, L)/\mathbf{C}^*$  is discrete. If there is a balanced point  $H_\infty \in M$ , then there is a constant  $\alpha > 0$  such that  $\{H_l\}$  converges to  $\alpha H_\infty$  as  $l \rightarrow \infty$ . Moreover,*

$$(1.2) \quad \tilde{Z}(H_\infty) = \min_{H \in M} \tilde{Z}(H) = \lim_{l \rightarrow \infty} \tilde{Z}(H_l).$$

REMARK 1.3. The convergence stated above makes sense by virtue of the operator norm when we regard  $H \in M$  as a positive Hermitian matrix with respect to an  $H_0$ -ONB.

## 2. A functional $\tilde{Z}$ on $M$

Donaldson [3] proved that a balanced point in  $M$  is characterized as a critical point of a functional  $\tilde{Z}$  on  $M$ . In this section, we shall run through the definition and the property of  $\tilde{Z}$  (see [3] for the full details).

Fix a basis  $\{S_i\}$  of  $E$ . We regard any  $H \in M$  as a positive Hermitian matrix with respect to  $\{S_i\}$ . Thus we have a map

$$\log \det: M \rightarrow \mathbf{R}.$$

A different choice of a basis just changes this map by the addition of a constant. Correspondingly, we introduce a functional on  $\mathcal{K}$ . For a path  $h_t = e^{\phi_t} h_0$  with  $\phi_0 = 0$  in

$\mathcal{K}$ , we define

$$I: \mathcal{K} \rightarrow \mathbf{R}$$

by

$$I_{h_0}(h_1) := \int_0^1 dt \int_X \dot{\phi}_t dV_{\omega_{h_t}}.$$

This functional is equivalent to the well-known functional (cf. [4])

$$F_{\omega_{h_0}}^0(-\phi_1) := J_{\omega_{h_0}}(-\phi_1) + \frac{1}{V} \int_X \phi_1 \omega_{h_0}^n$$

where

$$\begin{aligned} J_{\omega_{h_0}}(-\phi_1) &:= \frac{1}{V} \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_X \phi_1 \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi_1 \wedge \omega_{h_0}^k \wedge \omega_{h_1}^{n-k-1} \\ &= -\frac{1}{V} \int_0^1 dt \int_X \dot{\phi}_t (\omega_{h_0}^n - \omega_{h_t}^n), \end{aligned}$$

hence the functional  $I$  is independent of the choice of a path.

We now define maps on  $\mathcal{K}$  and  $M$  respectively as follows:

$$\begin{aligned} \mathcal{L} &:= \log \det \circ \text{Hilb}: \mathcal{K} \rightarrow \mathbf{R}, & \tilde{\mathcal{L}} &:= \mathcal{L} - \frac{N+1}{V} I, \\ Z &:= -I \circ FS: M \rightarrow \mathbf{R}, & \tilde{Z} &:= Z + \frac{V}{N+1} \log \det. \end{aligned}$$

We note that  $\tilde{\mathcal{L}}$  and  $\tilde{Z}$  are invariant by scalar multiplication on  $\mathcal{K}$  and  $M$  respectively. Donaldson [3] proved:

- (1) A point  $h^* \in \mathcal{K}$  (resp.  $H^* \in M$ ) is balanced if and only if it is a critical point of  $\tilde{\mathcal{L}}$  (resp.  $\tilde{Z}$ ) on  $\mathcal{K}$  (resp.  $M$ ). Moreover, if there is a balanced point in  $\mathcal{K}$  (resp.  $M$ ), then it is an absolute minimum of the  $\tilde{\mathcal{L}}$  (resp.  $\tilde{Z}$ ) on  $\mathcal{K}$  (resp.  $M$ ).
- (2) The map  $\text{Hilb} \circ FS: M \rightarrow M$  decreases  $\tilde{Z}$ , i.e., for any  $H \in M$

$$(2.1) \quad \tilde{Z}(H) \geq \tilde{Z}(\text{Hilb} \circ FS(H)).$$

Because of these two results, it seems reasonable that  $H_l$  in the main theorem would converge to the minimiser of  $\tilde{Z}$  if balanced points exist.

### 3. Proof of Theorem 1.2

Fix a point  $H_0$  in  $M$  and let  $H_l := \text{Hilb} \circ FS(H_{l-1})$ . With respect to an  $H_0$ -ONB, we identify  $M$  as the set of  $(N+1) \times (N+1)$  positive Hermitian matrices. Then, we can discuss the *boundedness* of a subset  $\mathcal{U} = \{H\} \subset M$ .

DEFINITION 3.1. A subset  $\mathcal{U} \subset M$  is bounded if and only if there is a positive constant  $R > 1$  satisfying the following condition: For any  $H \in \mathcal{U}$ , there is a constant  $\gamma_H > 0$  such that

$$(3.1) \quad \frac{\gamma_H}{R} \leq \min_{\xi \neq 0} \frac{|H(\xi)|}{|\xi|} \leq \max_{\xi \neq 0} \frac{|H(\xi)|}{|\xi|} \leq \gamma_H R.$$

Here  $\max_{\xi \neq 0} (|H(\xi)|/|\xi|)$  is so-called the operator norm  $\|H\|_{op}$  which is the largest eigenvalue of  $H$ , and  $\min_{\xi \neq 0} (|H(\xi)|/|\xi|)$  is the smallest one.

Note that when  $\mathcal{U}$  is bounded with respect to a basis of  $E$ , it is bounded with respect to a different one, too.

We shall give a sufficient condition for the existence of balanced points in  $M$ .

**Proposition 3.2.** *If the sequence  $\{H_l\}$  is bounded,  $\{H_l\}$  has a subsequence which converges to some point  $H_\infty \in M$  such that*

$$(3.2) \quad \tilde{Z}(H) \geq \tilde{Z}(H_\infty) = \inf \tilde{Z}(H_l)$$

for any  $H \in M$ . In particular,  $H_\infty$  is balanced.

Proof. From the boundedness of  $\{H_l\}$ , we find that there is a subsequence of  $\{\gamma_l^{-1}H_l\}$  which converges to some point in  $M$ . Since  $\tilde{Z}(H_l) = \tilde{Z}(\gamma_l^{-1}H_l)$  decreases monotonously with respect to  $l$ , so  $\tilde{Z}$  is bounded from below on  $\{H_l\}$ .

Now we shall show that the map  $\text{Hilb} \circ FS: M \rightarrow M$  decreases  $Z$  and  $\log \det$  respectively. (This can be also found from Lemma 4 and Lemma 5 in [3].) For any  $H \in M$ , let  $H' := \text{Hilb} \circ FS(H)$ .

$$\begin{aligned} & \frac{n!}{V} (Z(H) - Z(H')) \\ &= -F_{\omega_0}^0(-FS(H)) + F_{\omega_0}^0(-FS(H')) \\ &= F_{\omega_{FS(H)}}^0(FS(H) - FS(H')) \\ & \text{(because of the cocycle property of } F_\omega^0) \\ &= F_{\omega_{FS(H)}}^0 \left( \log \left( \frac{V}{N+1} B_{\omega_{FS(H)}} \right) \right) \\ &= J_{\omega_{FS(H)}} \left( \log \left( \frac{V}{N+1} B_{\omega_{FS(H)}} \right) \right) - \frac{n!}{V} \int_X \log \left( \frac{V}{N+1} B_{\omega_{FS(H)}} \right) dV_{\omega_{FS(H)}} \\ &\geq J_{\omega_{FS(H)}} \left( \log \left( \frac{V}{N+1} B_{\omega_{FS(H)}} \right) \right) \\ & \text{(since the logarithm is a concave function)} \\ &\geq 0. \end{aligned}$$

The last inequality is one of the well-known properties of the functional  $J_\omega$  ([1], [11]). From the definition of  $\text{Hilb} \circ FS$ , we get

$$\begin{aligned} \text{Tr}(H'H^{-1}) &= \sum_{i=0}^N \frac{N+1}{V} \int_X \frac{|S_i^H|_{h_0}^2}{\sum_{j=0}^N |S_j^H|_{h_0}^2} dV_{\omega_{FS(H)}} \\ &= N+1, \end{aligned}$$

where  $\{S_i^H\}$  is an  $H$ -ONB. So we find that  $\log \det(H'H^{-1}) \leq 0$ . We remark that  $\text{Tr}(H'H^{-1})$  and  $\log \det(H'H^{-1})$  are independent of the choice of a basis of  $E$ .

Therefore, we find that  $Z(H_l)$  and  $\log \det H_l$  decrease monotonously, respectively. Since  $\tilde{Z}(H_l)$  is bounded from below, so we find that

$$(3.3) \quad \det(H_{l+1}H_l^{-1}) \rightarrow 1$$

as  $l \rightarrow \infty$  and that  $\det H_l$  is bounded. So,  $\{\gamma_l\}$  is also bounded. Consequently, we get that there is a subsequence of  $\{H_l\}$  which converges to some point  $H_\infty \in M$ .

To prove (3.2), it suffices to show that for any  $H \in M$  and  $\varepsilon > 0$

$$(3.4) \quad \tilde{Z}(H) \geq \tilde{Z}(H_l) - \varepsilon$$

for  $l$  sufficiently large. Since  $H_{l+1}H_l^{-1}$  is a Hermitian matrix w.r.t. any  $H_l$ -ONB, we can take an  $H_l$ -ONB  $\{S_i^l\}$  of  $E$  such that  $H_{l+1}H_l^{-1}$  is regarded as a diagonal matrix by

$$\frac{N+1}{V} \text{diag} \left( \int_X \frac{|S_0^l|_{h_0}^2}{\sum_k |S_k^l|_{h_0}^2} dV_{\omega_{FS(H_l)}}, \dots, \int_X \frac{|S_N^l|_{h_0}^2}{\sum_k |S_k^l|_{h_0}^2} dV_{\omega_{FS(H_l)}} \right),$$

where  $h_0$  is a fixed base point in  $\mathcal{K}$ . So (3.3) implies that

$$(3.5) \quad \left| \frac{N+1}{V} \int_X \frac{|S_i^l|_{h_0}^2}{\sum_k |S_k^l|_{h_0}^2} dV_{\omega_{FS(H_l)}} - 1 \right| < \varepsilon$$

for any  $l$  sufficiently large and  $i = 0, \dots, N$ . The boundedness of  $\{H_l\}$  implies that there is a constant  $R_H > 1$  independent of  $l$  such that

$$(3.6) \quad \frac{\tau_l}{R_H} \leq \min_{\xi \neq 0} \frac{|H_l(\xi)|}{|\xi|} \leq \max_{\xi \neq 0} \frac{|H_l(\xi)|}{|\xi|} \leq \tau_l R_H$$

where  $H_l$  is regarded as a matrix with respect to an  $H$ -ONB and  $\tau_l$  is some positive constant. Then, we find that  $H$  can be written by a diagonal matrix

$$(3.7) \quad \text{diag}(e^{a_0}, \dots, e^{a_N}), \quad |a_k + \log \tau_l| \leq \log R_H$$

with respect to a suitable  $H_l$ -ONB  $\{\hat{S}_i^l\}$  (which may be *not* the same as  $\{S_i^l\}$  in (3.5)). By the one-parameter subgroup of  $GL(N + 1, \mathbf{C})$

$$\lambda(t) := \text{diag} (e^{ta_0/2}, \dots, e^{ta_N/2}), \quad 0 \leq t \leq 1,$$

we denote a path  $H(t) := \overline{^T \lambda(t)} \lambda(t)$  in  $M$  from  $H_l$  to  $H$ , so  $H(0) = H_l$  and  $H(1) = H$ . Let  $f(t) := \tilde{Z}(H(t))$ . We have

$$\begin{aligned} \frac{d}{dt} f(t) &= \frac{d}{dt} (-I \circ FS(H(t))) + \frac{V}{N + 1} \left( \sum_{k=0}^N a_k \right) \\ (3.8) \quad &= - \int_X \left( \frac{d}{dt} FS(H(t)) \right) dV_{\omega_{FS(H(t))}} + \frac{V}{N + 1} \left( \sum_{k=0}^N a_k \right) \\ &= - \int_X \frac{\sum a_k e^{-ta_k} |\hat{S}_k^l|_{h_0}^2}{\sum e^{-ta_k} |\hat{S}_k^l|_{h_0}^2} dV_{\omega_{FS(H(t))}} + \frac{V}{N + 1} \left( \sum_{k=0}^N a_k \right). \end{aligned}$$

The equality (3.8) holds, since  $\{e^{-ta_i/2} \hat{S}_i^l\}$  is an  $H(t)$ -ONB. Therefore we get

$$\begin{aligned} \frac{d}{dt} f(t) \Big|_{t=0} &= - \int_X \frac{\sum a_k |\hat{S}_k^l|_{h_0}^2}{\sum |\hat{S}_k^l|_{h_0}^2} dV_l + \frac{V}{N + 1} \left( \sum_{k=0}^N a_k \right) \\ (3.9) \quad &= - \int_X \frac{\sum (a_k + \log \tau_l) |\hat{S}_k^l|_{h_0}^2}{\sum |\hat{S}_k^l|_{h_0}^2} dV_l + \frac{V}{N + 1} \sum_{k=0}^N (a_k + \log \tau_l) \end{aligned}$$

where  $dV_l$  denotes  $dV_{\omega_{FS(H_l)}}$ . From (3.5), (3.7) and (3.9), we have

$$(3.10) \quad \frac{d}{dt} f(t) \Big|_{t=0} \geq -\varepsilon V \log R_H.$$

Donaldson proved that  $\tilde{Z}$  is convex along geodesics in  $M$  (see Proposition 1 in [3], and also [2], [9], and [12]), therefore we get

$$\tilde{Z}(H) \geq \tilde{Z}(H_l) - \varepsilon V \log R_H$$

for  $l$  large enough. Since (3.2) means that  $H_\infty$  is a minimiser (in particular, a critical point) of  $\tilde{Z}$  on  $M$ , so  $H_\infty$  is balanced. Hence, the proof is complete.  $\square$

We shall prove the boundedness of  $\{H_l\}$  if balanced points exist.

**Proposition 3.3.** *Suppose that  $\text{Aut}(X, L)/\mathbf{C}^*$  is discrete. If there is a balanced point  $H_\infty$  in  $M$ ,  $\{H_l\}$  is bounded in  $M$ .*

Proof. The existence of balanced points in  $M$  implies that  $\tilde{Z}$  is bounded from below on  $M$ . In the same manner as the proof of Proposition 3.2, we find that  $\det H_I$  is bounded. Now we normalize  $M$  (with respect to a fixed basis  $\{S_i\}$  of  $E$ ) to

$$M_{SL} := \{H_{SL} := (\det H)^{-1/(N+1)}H \mid H \in M\} \subset M.$$

We note that this normalization unchanges  $\tilde{Z}$ , i.e.,  $\tilde{Z}(H) = \tilde{Z}(H_{SL})$ , and that  $M_{SL}$  can be identified as  $SL(N + 1, \mathbf{C})/SU(N + 1)$ .

The functional  $\tilde{Z}(H_{SL})$  on  $M_{SL}$  equals to  $Z(H_{SL})$  which also equals to  $-(V/(n!))F_{\omega_{h_0}}^0(-FS(H_{SL}))$ . Let  $\sigma \in SL(N + 1, \mathbf{C})$  with  $H_{SL} = {}^T\bar{\sigma}\sigma$  and

$$\phi_{\sigma^{-1}} := \log \frac{{}^T\bar{S} {}^T\bar{\sigma}^{-1}\sigma^{-1}S}{{}^T\bar{S}S},$$

where  $S(p) := {}^T(\varphi \circ S_0(p), \dots, \varphi \circ S_N(p))$  is the column vector of the values of holomorphic sections of  $L$  at  $p \in X$  in a local trivialization  $\varphi$ . It is clear that  $\phi_{\sigma^{-1}}$  is well-defined and independent of the choice of local trivialization. Then, by using the cocycle property of  $F_{\omega}^0$ , we get

$$(3.11) \quad -F_{\omega_{h_0}}^0(-FS(H_{SL})) = -F_{\iota^*\omega_{FS}}^0(\phi_{\sigma^{-1}}) + C$$

for some constant  $C$ , where  $\iota$  is the projective embedding of  $X$  to  $\mathbf{CP}^N$  induced by  $\{S_i\}$ . It is known by many authors that  $-F_{\iota^*\omega_{FS}}^0(\phi_{\sigma^{-1}})$  is equivalent to a specific norm  $\|\cdot\|_{CH}$  on the set of *Chow points*, that is to say,

$$(3.12) \quad -V(n + 1)F_{\iota^*\omega_{FS}}^0(\phi_{\sigma^{-1}}) = \log \frac{\|\sigma^{-1} \cdot \text{CH}(X)\|_{CH}^2}{\|\text{CH}(X)\|_{CH}^2}$$

where  $\text{CH}(X)$  is the Chow point of  $\iota(X)$  (See Theorem 5 in [9] and [12]. See also Section 3 in [10] for the detail of the norm mentioned above).

Since it is known that the assumption of Proposition 3.3 implies the uniqueness of balanced metrics (see Theorem 1 in [2]), so we find that for any one parameter subgroup  $\{\sigma_t\}$  of  $SL(N + 1, \mathbf{C})$

$$(3.13) \quad \tilde{Z}(H_t) := \tilde{Z}({}^T\bar{\sigma}_t\sigma_t) \rightarrow \infty \quad \text{as } t \rightarrow \infty$$

due to the convexity of  $\tilde{Z}$  along  $\{\sigma_t\}$ . From (3.11), (3.12) and (3.13), the numerical criterion of GIT (cf. [5]) implies that  $\iota(X) \subset \mathbf{CP}^N$  is *Chow stable*. In other words,  $\tilde{Z}$  is proper when we regard it as the functional on  $SL(N+1, \mathbf{C})$ . Since  $\tilde{Z}(H_t)$  is bounded, so  $\{(H_t)_{SL}\}$  is bounded in  $M$ . Therefore we find that  $\{H_t\}$  is also bounded due to the boundedness of  $\det H_t$ . The proof is complete. □

From Proposition 3.2 and the proof of Proposition 3.3, we obtain a necessary and sufficient condition for Chow stability of  $\iota(X) \subset \mathbf{CP}^N$ .



**Corollary 3.4.** *Let  $(X, L)$  be a complex manifold with a very ample line bundle. Suppose that  $\text{Aut}(X, L)/\mathbf{C}^*$  is discrete. The projective variety  $X \subset P(H^0(X, L)^*)$  is Chow stable if and only if the sequence  $\{H_l\}$  as above is bounded.*

We shall prove Theorem 1.2.

Proof of Theorem 1.2. The assumption that  $\text{Aut}(X, L)/\mathbf{C}^*$  is discrete, implies the uniqueness of balanced metrics, so the sequence  $\{H_l\}$  has a subsequence which converges to  $\alpha H_\infty$  for some positive constant  $\alpha$  due to Proposition 3.2 and Proposition 3.3. Since  $\det H_l$  decreases monotonously with respect to  $l$  and is bounded from below, so

$$(3.14) \quad \det H_l \rightarrow \alpha(\det H_\infty)$$

as  $l \rightarrow \infty$ . Because of the proof of Proposition 3.2, the uniqueness of balanced metrics and (3.14), we find that any subsequence of  $\{H_l\}$  has a subsequence which converges to  $\alpha H_\infty$ . This means that  $\{H_l\}$  converges to  $\alpha H_\infty$ . The equality (1.2) follows from (3.2). The proof is complete.  $\square$

#### 4. Remarks

Donaldson proved (Theorem 2 in [2]): Let  $(X, L)$  be a complex manifold with a positive line bundle. Suppose that  $\text{Aut}(X, L)/\mathbf{C}^*$  is discrete. If  $(X, L^k)$  admits a balanced metric  $\omega_k \in kc_1(X)$  for all sufficiently large  $k$  and  $(1/k)\omega_k$  converges in  $C^\infty$ , its limit  $\omega_\infty$  has constant scalar curvature.

Hence it may be natural that the algorithm of balanced metrics proved in this paper would be considered as the algorithm of constant scalar curvature metrics (if it exists). In particular, the author guesses that this algorithm would substitute for Continuity method in the case of Kähler-Einstein metrics. He also hope that this prospect would be one of the approaches for trying the extension of the relationship between the existence of Kähler-Einstein metrics and the non-existence of multiplier ideal sheaves defined by Nadel [8] to the case of constant scalar curvature metrics.

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