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HERMITE CONSTANT AND VORONOÏ THEORY OVER A QUATERNION SKEW FIELD

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Abstract

The Hermite constant $\gamma_n(D)$ of a quaternion skew field D over a global field is defined and studied. We obtain an upper bound of $\gamma_n(D)$. In the case that the base field is a number field, we introduce the notion of quaternionic Humbert forms over D. Then $\gamma_n(D)$ is characterized as a critical value of the Hermite invariants for n-ary quaternionic Humbert forms. We extend Voronoï's theorem on extreme forms to quaternionic Humbert forms.

Introduction

An analogue of Hermite's constant for a division algebra over a number field was first studied in [10] as a typical case of generalized Hermite constants of linear algebraic groups. But the definition of this constant given in [10] was not canonical in the sense that it depends on the choice of a splitting field of the division algebra in question. After this work, the second author introduced the notion of the fundamental Hermite constant associated to a pair of a connected reductive algebraic group and its maximal parabolic subgroup both defined over a global field (cf. [11]). This notion especially yields a canonical definition of Hermite constant for a division algebra over a global field.

To be more precise, let D denote a central division algebra over a global field k, $V = e_1D + \cdots + e_nD$ a right D-vector space, $G(k) = \mathrm{Aut}_D(V)$ the group of D-linear automorphisms of V, and $Q(k)$ the stabilizer in $G(k)$ of the line e_1D. As an algebraic group, Q is a maximal k-parabolic subgroup of the affine algebraic k-group G. We write $G(\mathbb{A})$ and $Q(\mathbb{A})$ for the adele groups of G and Q, respectively, and write $G(\mathbb{A})^1$ for the subgroup consisting of all $g \in G(\mathbb{A})$ whose reduced norm satisfies $|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}} = 1$. If we fix a maximal compact subgroup K of $G(\mathbb{A})$ such that $G(\mathbb{A})$ possesses an Iwasawa decomposition $Q(\mathbb{A})K$, we can define the height function $H_Q\colon G(\mathbb{A}) \to \mathbb{R}_{>0}$ by $H_Q(gh) = |\widehat{\alpha}_Q(g)|_{\mathbb{A}}^{-1}$ for $g \in Q(\mathbb{A})$ and $h \in K$, where $\widehat{\alpha}_Q$ denotes some basis of the $\mathbb{Z}$-module of k-rational characters of Q modulo the center

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of G. Then the Hermite constant of D is defined as

$$\gamma_n(D) = \max_{g\in G(\mathbb{A})^1} \min_{x\in Q(k)\backslash G(k)} H_Q(xg).$$

This definition of $\gamma_n(D)$ is intrinsic and does not depend on a splitting field of D. If k is a number field, one can see a relation between $\gamma_n(D)$ and a generalized Hermite constant defined in [10] in the Remark following Lemma 3.4 in §3.

In the case that D is a quaternion skew field, we can express $\gamma_n(D)$ in terms of the twisted heights on the vector space V, i.e., we will show in §2 the following equality:

$$\gamma_n(D) = \max_{g\in G(\mathbb{A})^1} \min_{x\in V-\{0\}} H_g(x)^{2n}. \tag{0}$$

Here H_g denotes the twisted height on V for $g \in G(\mathbb{A})^1$, whose precise definition will be given in §2. As explained by Liebendörfer [3], it is not easy to define appropriately the twisted heights on V. Liebendörfer has studied recently heights on D-vector spaces in details, but only in the case where D is a definite quaternion skew field over the rational number field $\mathbb{Q}$. At least, our definition of H_g is more general, and coincides with hers in that case. The aim of this paper is to study $\gamma_n(D)$ more closely, based on the equation (0). In the first half, we shall yield an upper bound of $\gamma_n(D)$ for any quaternion skew field D over any global field, and in the second half, a Voronoï type theory of *quaternionic Humbert forms* will be developed in connection with $\gamma_n(D)$, provided that k is a number field.

Because of the difficulty of definition of the twisted heights, we restrict ourselves to the case of a quaternion skew field in this paper. However, in a subsequent paper, we will remove this restriction, i.e., we will give a definition of the twisted heights on a vector space over any division algebra D, and then we will study a generalization of successive minima and Minkowski's theorem with respect to the twisted heights. In this work, the Hermite constant $\gamma_n(D)$ will play a crucial role, and an estimate of $\gamma_n(D)$ will have an application to Siegel's lemma over D.

In the rest of this introduction, we briefly explain the results of this paper. An upper bound of some generalized Hermite constant was already given in [10, Theorem 3] in the number field case. However, this theorem (or even its proof) can not be applied to the present case. Thus we have to make a different approach to get an upper bound of $\gamma_n(D)$. We first realize D as a cyclic algebra $(L/k, u) = 1 \cdot L + \mathbf{i} \cdot L$, where L/k is a separable quadratic extension contained in D, u is an element in $k^\times$ and $\mathbf{i}$ is an element in D such that $\mathbf{i}^2 = u$. Regarding V as an L-vector space, one can define the twisted height ${}^L\widehat{F}_\xi : V \wedge V \to \mathbb{R}_{\geq 0}$ for $\xi \in G(\mathbb{A}_L)$, where $\mathbb{A}_L = \mathbb{A} \otimes_k L$, (see §3 for details) and the twisted height ${}^L H_\xi : V \to \mathbb{R}_{\geq 0}$: ${}^L H_\xi$ is just defined by ${}^L H_\xi(x) = {}^L\widehat{F}_\xi(x \wedge x\mathbf{i})^{1/4}$ for $x \in V$. Then $\gamma_n(D)$ has a description of the form

$$\gamma_n(D) = \frac{1}{{}^L H_\eta(e_1)^{2n}} \max_{g\in G(\mathbb{A})^1} \min_{x\in V-\{0\}} {}^L H_{\eta g}(x)^{2n},$$

where η is an element in $G(\mathbb{A}_L)$ determined from the maximal compact subgroups of $G(\mathbb{A})$ and $G(\mathbb{A}_L)$. (In general, η is not contained in $G(\mathbb{A})$. This is a reason why [10, Theorem 3] does not work well for $\gamma_n(D)$.) By making use of Hadamard's inequality and some arguments of geometry of numbers, one can estimate the minimum of ${}^L H_{\eta g}(x)$ (Lemmas 3.3 and 3.6). In this estimate, the function $\psi(\xi) = \omega_V(\mathbf{S} \cap \xi \mathbf{S})/\omega_V(\mathbf{S})$ in $\xi \in G(\mathbb{A}_L)$ occurs, where ω_V is a Haar measure of the adele space $V \otimes_L \mathbb{A}_L$ and **S** is "a unit ball" in $V \otimes_L \mathbb{A}_L$. The point is an explicit computation of $\psi(\xi)$ at $\xi = \overline{\eta} J_u \eta^{-1}$ (Lemma 3.7, see §3 for notations). This leads us to an explicit upper bound of the minimum of ${}^L H_{\eta g}(x)$, and hence of $\gamma_n(D)$ (Theorem 3.8).

If k is a number field, the expression (0) of $\gamma_n(D)$ leads us to the notion of n-ary quaternionic Humbert forms over D. Let $k_\infty = k \otimes_{\mathbb{Q}} \mathbb{R} = \prod_v k_v$, where v runs over all infinite places of k. For $g_v \in G(k_v)$, the matrix $S_v = g_v \overline{g}'_v$ is a positive definite symmetric, Hermitian or quaternionic Hermitian matrix according as $D \otimes_k k_v \cong M_2(\mathbb{R})$, $M_2(\mathbb{C})$ or the Hamilton quaternion $\mathbb{H}$. This S_v defines a form on $V \otimes_k k_v$. We call a system $S = (S_v) = (g_v \overline{g}'_v)$ of forms for $(g_v) \in G(k_\infty)$ an n-ary quaternionic Humbert form over D. The set $P_{n,D}$ of all n-ary quaternionic Humbert forms over D becomes a Riemannian symmetric space. If we fix a maximal order $\mathfrak{O}$ of D and representatives $\Lambda_1, \ldots, \Lambda_h$ of equivalent classes of full $\mathfrak{O}$-lattices in V, then *the Hermite invariant* $\mu_i(S)$ for $S \in P_{n,D}$ is defined to be

$$\mu_i(S) = \frac{1}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_i)} \frac{m_i(S)^n}{\mathrm{Det}\, S}, \quad \text{where} \quad m_i(S) = \min_{u \in \Lambda_i - \{0\}} \frac{S[u]}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)},$$

for each $i = 1, 2, \ldots h$. Here $S[u]$ denotes the value of S at u, $\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)$ the norm of some integral $\mathfrak{O}$-ideal defined from $u \in \Lambda_i$ and $\mathrm{Det}\, S$ "the determinant" of S, see §4 for their precise definitions. Then the equation (0) implies

$$\gamma_n(D) = \max_{1 \le i \le h} \max_{S \in P_{n,D}} \mu_i(S)$$

(§4, Proposition 4.5). Thus $\gamma_n(D)$ is characterized as a critical value of the Hermite invariants μ_i. An investigation of the critical values of such "Hermite invariants" is known as a Voronoï type theory. The second subject of this paper is to develop a Voronoï type theory for the Hermite invariants μ_i. As usual, a quaternionic Humbert form S is said to be *μ_i-extreme* if S achieves a local maximum of μ_i. To define the notion of *μ_i-perfection* and *μ_i-eutaxy* for quaternionic Humbert forms, we make use of Bavard's fundamental work [1] on a Voronoï type theory. Some equivalent conditions for μ_i-perfection and μ_i-eutaxy will be given in §6, Proposition 6.1. Then we will prove the following Voronoï type theorem: *A quaternionic Humbert form S is μ_i-extreme if and only if it is μ_i-perfect and μ_i-eutactic* (§6, Theorem 6.3).

The interest of this Voronoï type characterization is twofold: first it allows to prove that $\gamma_n(D)$ is algebraic, having noticed that μ_i-perfect forms are algebraic (§6, Proposition 6.4). Secondly, a classification of μ_i-perfect (resp. μ_i-eutactic) forms, if possible, allows the computation of $\gamma_n(D)$. In the case of the classical Hermite invariant, such a classification is obtained as a by-product of the so-called Voronoï's algorithm. Unfortunately, this algorithm does not generalize easily to our situation. Nevertheless, we can prove that there are finitely many perfect quaternionic Humbert forms in a given dimension (§6, Theorem 6.7), which is the first required property if one looks for a classification.

In the last part of §4, we treat, as an example, the case of binary quaternionic Humbert forms over Euclidean quaternion fields, and compute the corresponding Hermite constants. This is in fact an easy case, and does not actually require the use of the Voronoï type characterization of extremality.

Notations

Let k be a global field, i.e., an algebraic number field of finite degree over $\mathbb{Q}$ or an algebraic function field of one variable over a finite field. We denote by $\mathfrak{V}, \mathfrak{V}_\infty$ and $\mathfrak{V}_f$ the sets of all places of k, all infinite places of k and all finite places of k, respectively. For $v \in \mathfrak{V}$, let k_v be the completion of k at v and $|\cdot|_{k_v}$ be the absolute value of k_v normalized so that $|a|_{k_v} = \mu_v(aC)/\mu_v(C)$, where μ_v is a Haar measure of k_v and C is an arbitrary compact subset of k_v with nonzero measure. If v is finite, $\mathfrak{o}_{k_v}$ denotes the ring of integers in k_v. The adele ring of k is denoted by $\mathbb{A}$ and its idele norm is denoted by $|\cdot|_{\mathbb{A}}$, i.e., $|\cdot|_{\mathbb{A}} = \prod_{v\in\mathfrak{V}} |\cdot|_{k_v}$. We will write k_∞ and $\mathbb{A}_f$ for the infinite part and the finite part of $\mathbb{A}$, respectively. The restrictions of $|\cdot|_{\mathbb{A}}$ to $k_\infty^\times$ and $\mathbb{A}_f^\times$ are denoted by $|\cdot|_{k_\infty}$ and $|\cdot|_{\mathbb{A}_f}$, respectively. If k is an algebraic number field, then $\mathfrak{o}_k$ denotes the ring of integers in k.

For a unital k-algebra R and positive integers m and n, $M_{m,n}(R)$ stands for the set of m by n matrices with components in R. The transpose of a matrix $A \in M_{m,n}(R)$ is denoted by A'. The unit group of the total matrix algebra $M_n(R) = M_{n,n}(R)$ is denoted by $GL_n(R)$. In general, for a given algebraic k-group $\mathfrak{G}$, $\mathfrak{G}(R)$ stands for the group of R-rational points of $\mathfrak{G}$. If R is a finite dimensional central division k-algebra, $\mathrm{Nr}_{M_n(R)/k}$ stands for the reduced norm of the central simple k-algebra $M_n(R)$ and $\mathrm{Tr}_{M_n(R)/k}$ for the reduced trace.

1. Fundamental Hermite constants of $GL_n(D)$

We fix integers $d \geq 1$ and $n \geq 2$. Throughout this section, D denotes a central division k-algebra of degree d and G the affine algebraic k-group defined by $G(R) = GL_n(D \otimes_k R)$ for any k-algebra R. Let P be the minimal k-parabolic subgroup of G which consists of upper triangular matrices in G. Then the standard maximal k-parabolic subgroups Q_m, $1 \leq m \leq n-1$, of G are given as follows:

$$Q_m(k) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in GL_m(D),\ \ b \in M_{m,n-m}(D),\ \ d \in GL_{n-m}(D) \right\}.$$

In this section, we recall the fundamental Hermite constants $\gamma(G, Q_m, k)$ and $\widetilde{\gamma}(G, Q_m, k)$ introduced in [11].

In the following, we fix m and write Q for Q_m. Let U_Q be the unipotent radical of Q and M_Q the Levi subgroup of Q given by

$$M_Q(k) = \left\{ \mathrm{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in GL_m(D),\ \ b \in GL_{n-m}(D) \right\}.$$

Denote by Z_G and Z_Q the central maximal k-split tori of G and M_Q, respectively, i.e.,

$$Z_G(k) = \{\lambda I_n : \lambda \in k^\times\} \quad \text{and} \quad Z_Q(k) = \{\mathrm{diag}(\lambda I_m, \mu I_{n-m}) : \lambda, \mu \in k^\times\}.$$

We define the k-rational characters $\alpha_Q : Z_Q \to GL_1$ and $\widehat{\alpha}_Q : M_Q \to GL_1$ as follows:

$$\alpha_Q(\mathrm{diag}(\lambda I_m, \mu I_{n-m})) = \lambda\mu^{-1}$$

for $\mathrm{diag}(\lambda I_m, \mu I_{n-m}) \in Z_Q(k)$ and

$$\widehat{\alpha}_Q(\mathrm{diag}(a, b)) = \mathrm{Nr}_{M_m(D)/k}(a)^{(n-m)/\gcd(m,n-m)}\, \mathrm{Nr}_{M_{n-m}(D)/k}(b)^{-m/\gcd(m,n-m)}$$

for $\mathrm{diag}(a, b) \in M_Q(k)$. Then α_Q (resp. $\widehat{\alpha}_Q$) is trivial on Z_G and forms a $\mathbb{Z}$-basis of the module $\mathbf{X}_k^*(Z_G\backslash Z_Q)$ (resp. $\mathbf{X}_k^*(Z_G\backslash M_Q)$) of k-rational characters of $Z_G\backslash Z_Q$ (resp. $Z_G\backslash M_Q$). The index $[\mathbf{X}_k^*(Z_G\backslash Z_Q) : \mathbf{X}_k^*(Z_G\backslash M_Q)]$ is equal to $dm(n-m)/\gcd(m, n-m)$.

Define the unimodular subgroups $G(\mathbb{A})^1$, $M_Q(\mathbb{A})^1$ and $Q(\mathbb{A})^1$ as follows:

$$\begin{aligned} G(\mathbb{A})^1 &= \{g \in G(\mathbb{A}) : |\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}} = 1\}, \\ M_Q(\mathbb{A})^1 &= \{\mathrm{diag}(a, b) \in M_Q(\mathbb{A}) : |\mathrm{Nr}_{M_m(D)/k}(a)|_{\mathbb{A}} = |\mathrm{Nr}_{M_{n-m}(D)/k}(b)|_{\mathbb{A}} = 1\}, \\ Q(\mathbb{A})^1 &= U_Q(\mathbb{A})M_Q(\mathbb{A})^1. \end{aligned}$$

Let K be a maximal compact subgroup of $G(\mathbb{A})$ such that $G(\mathbb{A})$ possesses an Iwasawa decomposition $G(\mathbb{A}) = U_Q(\mathbb{A})M_Q(\mathbb{A})K$. Then the height function $H_Q : G(\mathbb{A}) \to \mathbb{R}_{>0}$ is well defined by

$$\begin{aligned} H_Q(u \cdot \mathrm{diag}(a, b) \cdot h) &= |\widehat{\alpha}_Q(\mathrm{diag}(a, b))|_{\mathbb{A}}^{-1} \\ &= |\mathrm{Nr}_{M_m(D)/k}(a)|_{\mathbb{A}}^{-(m-n)/\gcd(m,n-m)} |\mathrm{Nr}_{M_{n-m}(D)/k}(b)|_{\mathbb{A}}^{m/\gcd(m,n-m)} \end{aligned}$$

for $u \in U_Q(\mathbb{A})$, $\mathrm{diag}(a, b) \in M_Q(\mathbb{A})$ and $h \in K$. By definition, H_Q is left $Z_G(\mathbb{A})Q(\mathbb{A})^1$ and right K invariant.

We set $X_Q = Q(k)\backslash G(k)$ and $Y_Q = Q(\mathbb{A})^1\backslash G(\mathbb{A})^1$. Then X_Q is a subset of Y_Q and the natural map $Y_Q \to (Z_G(\mathbb{A})Q(\mathbb{A})^1)\backslash G(\mathbb{A})$ is injective. Thus the height function H_Q is restricted to Y_Q. Then the Hermite constants $\gamma(G, Q, k)$ and $\widetilde{\gamma}(G, Q, k)$ are defined to be

$$\gamma(G, Q, k) = \max_{g \in G(\mathbb{A})^1} \min_{x \in X_Q} H_Q(xg), \qquad \widetilde{\gamma}(G, Q, k) = \max_{g \in G(\mathbb{A})} \min_{x \in X_Q} H_Q(xg).$$

If k is an algebraic number field, then $\gamma(G, Q, k)$ equals $\widetilde{\gamma}(G, Q, k)$ as $Z_G(\mathbb{A})G(\mathbb{A})^1 = G(\mathbb{A})$. In the case of $m = 1$, we write $\gamma_n(D)$ and $\widetilde{\gamma}_n(D)$ for $\gamma(G, Q_1, k)$ and $\widetilde{\gamma}(G, Q_1, k)$, respectively.

2. $\gamma_n(D)$ for a quaternion skew field D

Hereafter, throughout this paper, let D be a quaternion division k-algebra. In this section, we describe $\gamma_n(D)$ and $\widetilde{\gamma}_n(D)$ in terms of a height on a projective space. These descriptions will be used in the latter sections.

We write $D_{\mathbb{A}}$ and $D_{\mathbb{A}_f}$ for $D \otimes_k \mathbb{A}$ and $D \otimes_k \mathbb{A}_f$, respectively. For each $v \in \mathfrak{V}$, $D_v = D \otimes_k k_v$ is a quaternion algebra over k_v. Let $\varepsilon_v \in 2^{-1}\mathbb{Z}/\mathbb{Z}$ be the Brauer-Hasse invariant of D_v, namely ε_v is equal to 0 or $1/2$ modulo $\mathbb{Z}$ according as $D_v \cong M_2(k_v)$ or not. Then the set $\mathfrak{V}$ is divided into two subsets $\mathfrak{V}' = \{v \in \mathfrak{V} \colon \varepsilon_v = 1/2 \mod \mathbb{Z}\}$ and $\mathfrak{V}'' = \{v \in \mathfrak{V} \colon \varepsilon_v = 0 \mod \mathbb{Z}\}$. The set $\mathfrak{V}'$ is a finite set and its cardinality is even. We write $\mathfrak{V}'_\infty$, $\mathfrak{V}''_\infty$, $\mathfrak{V}'_f$ and $\mathfrak{V}''_f$ for $\mathfrak{V}_\infty \cap \mathfrak{V}'$, $\mathfrak{V}_\infty \cap \mathfrak{V}''$, $\mathfrak{V}_f \cap \mathfrak{V}'$ and $\mathfrak{V}_f \cap \mathfrak{V}''$, respectively.

Let $\mathfrak{O}$ be a maximal order of D. For $v \in \mathfrak{V}_f$, the completion $\mathfrak{O}_v$ of $\mathfrak{O}$ in D_v is a maximal order of D_v. For each $v \in \mathfrak{V}''$, we fix an isomorphism $\iota_v \colon D_v \to M_2(k_v)$ such that $\iota_v(\mathfrak{O}_v) = M_2(\mathfrak{o}_{k_v})$ if v is finite. Then we define elements e_v, e'_v and J_v of D_v by

$$e_v = \iota_v^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right), \quad e'_v = \iota_v^{-1}\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right), \quad J_v = \iota_v^{-1}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

Let $V = e_1 D + \cdots + e_n D$ be a right D-vector space with the standard basis $e_1, \ldots, e_n$. We define the local height H^v on $V_v = V \otimes_k k_v$ for each $v \in \mathfrak{V}$ as follows.

(i) The case of $v \in \mathfrak{V}'$. In this case, D_v is division and V_v is a right D_v-vector space. The local height $F_v = H^v$ is defined to be

$$F_v\left(\sum_{1 \le i \le n} e_i x_i\right) = H^v\left(\sum_{1 \le i \le n} e_i x_i\right) = \begin{cases} \left(\displaystyle\sum_{1 \le i \le n} |\mathrm{Nr}_{D/k}(x_i)|_{k_v}\right)^{1/2} & (v \in \mathfrak{V}'_\infty). \\ \displaystyle\sup_{1 \le i \le n} \left(|\mathrm{Nr}_{D/k}(x_i)|_{k_v}^{1/2}\right) & (v \in \mathfrak{V}'_f). \end{cases}$$

(ii) The case of $v \in \mathfrak{V}''$. In this case, V_v is a free right D_v-module of rank n and decomposes into a direct sum of k_v-vector subspaces $V_v e_v$ and $V_v e'_v$. We write W_v for $V_v e_v$. As a k_v-vector space, W_v is of dimension $2n$. From $J_v e'_v J_v = e_v$, it follows $V_v e'_v J_v = W_v$. Put $f^v_{2i-1} = e_i e_v$ and $f^v_{2i} = e_i e'_v J_v$ for $1 \le i \le n$. Then $\{f^v_1, f^v_2, \ldots, f^v_{2n}\}$

forms a k_v-basis of W_v. We define the norms F_v on W_v and $\widehat{F}_v$ on the wedge product $W_v \wedge W_v$ as follows:

$$F_v\left(\sum_{1\le i\le 2n} f_i^v \lambda_i\right) = \begin{cases} \left(\sum_{1\le i\le 2n} |\lambda_i|_{k_v}^2\right)^{1/2} & (v \in \mathfrak{V}''_\infty,\ \ k_v = \mathbb{R}), \\ \sum_{1\le i\le 2n} |\lambda_i|_{k_v} & (v \in \mathfrak{V}''_\infty,\ \ k_v = \mathbb{C}), \\ \sup_{1\le i\le 2n} (|\lambda_i|_{k_v}) & (v \in \mathfrak{V}''_f). \end{cases}$$

$$\widehat{F}_v\left(\sum_{1\le i<j\le 2n} \left(f_i^v \wedge f_j^v\right)\lambda_{ij}\right) = \begin{cases} \left(\sum_{1\le i<j\le 2n} |\lambda_{ij}|_{k_v}^2\right)^{1/2} & (v \in \mathfrak{V}''_\infty,\ \ k_v = \mathbb{R}), \\ \sum_{1\le i<j\le 2n} |\lambda_{ij}|_{k_v} & (v \in \mathfrak{V}''_\infty,\ \ k_v = \mathbb{C}), \\ \sup_{1\le i<j\le 2n} (|\lambda_{ij}|_{k_v}) & (v \in \mathfrak{V}''_f). \end{cases}$$

Then the local height H^v on V_v is defined to be

$$H^v(x) = \widehat{F}_v\left((xe_v) \wedge \left(xe'_v J_v\right)\right)^{1/2}$$

for $x \in V_v$.

Lemma 2.1. *Let $v \in \mathfrak{V}$. Then*

$$H^v(xa) = |\mathrm{Nr}_{D/k}(a)|_{k_v}^{1/2} H^v(x)$$

holds for all $x \in V_v$ and $a \in D_v^\times$.

Proof. This is obvious by definition if $v \in \mathfrak{V}'$. Thus we assume $v \in \mathfrak{V}''$. Let $\iota_v(a) = \left(\begin{smallmatrix} \lambda & \lambda' \\ \mu & \mu' \end{smallmatrix}\right)$. Then

$$\begin{aligned} xa &= \left(xe_v + xe'_v\right)a = \left(xe_v + xe'_v\right)ae_v + \left(xe_v + xe'_v\right)ae'_v \\ &= \left\{x(e_v a e_v) + x\left(e'_v a e_v\right)\right\} + \left\{x\left(e_v a e'_v\right) + x\left(e'_v a e'_v\right)\right\} \\ &= \left\{xe_v\lambda + xe'_v J_v \mu\right\} + \left\{xe_v J_v \lambda' + xe'_v \mu'\right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} xae_v \wedge xae'_v J_v &= \left\{xe_v\lambda + xe'_v J_v\mu\right\} \wedge \left\{xe_v\lambda' + xe'_v J_v \mu'\right\} \\ &= xe_v \wedge xe'_v J_v\left(\lambda\mu' - \lambda'\mu\right), \end{aligned}$$

and hence, $H^v(xa) = \widehat{F}_v(xae_v \wedge xae'_v J_v)^{1/2} = |\mathrm{Nr}_{D/k}(a)|_{k_v}^{1/2} H^v(x)$. □

In any case, the subgroup

$$K_v = \{g \in G(k_v) : F_v(gx) = F_v(x) \quad (x \in V_v)\}$$

is a maximal compact subgroup of $G(k_v) = GL_n(D_v)$. If $v \in \mathfrak{V}_f$, then K_v is the stabilizer of the free $\mathfrak{O}_v$-lattice $e_1\mathfrak{O}_v + \cdots + e_n\mathfrak{O}_v$. We fix, once and for all, a maximal compact subgroup K of $G(\mathbb{A})$ as

$$K = \prod_{v \in \mathfrak{V}} K_v,$$

and then define H_Q for $Q = Q_1$ as in §1.

For $g = (g_v) \in G(\mathbb{A})$, the global twisted height $H_g : V \to \mathbb{R}_{\geq 0}$ is defined to be

$$H_g(x) = \prod_{v \in \mathfrak{V}} H^v(g_v x).$$

It is easy to see that

$$H_{\lambda I_n \cdot h \cdot g} = |\lambda|_{\mathbb{A}} H_g \tag{1}$$

for all $\lambda I_n \in Z_G(\mathbb{A})$, $h \in K$ and $g \in G(\mathbb{A})$. We define the function $\Phi : G(\mathbb{A}) \to \mathbb{R}_{>0}$ by

$$\Phi(g) = \frac{H_g(e_1)}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}^{1/(2n)}}.$$

The stabilizer of the right D-subspace spanned by e_1 in G is the maximal parabolic subgroup $Q = Q_1$. By (1), Φ is left K and right $Z_G(\mathbb{A})Q(\mathbb{A})^1$ invariant.

Lemma 2.2. *The equality $\Phi(g)^{2n} = H_Q(g^{-1})$ holds for all $g \in G(\mathbb{A})$.*

Proof. Since both $\Phi(g)$ and $H_Q(g^{-1})$ are left K and right $Z_G(\mathbb{A})Q(\mathbb{A})^1$ invariant, it is sufficient to prove $\Phi(\mathrm{diag}(a, b))^{2n} = H_Q(\mathrm{diag}(a, b)^{-1})$ for all $\mathrm{diag}(a, b) \in M_Q(\mathbb{A})$, where $a \in GL_1(D_{\mathbb{A}})$ and $b \in GL_{n-1}(D_{\mathbb{A}})$. On the one hand, it follows from §1 that

$$H_Q(\mathrm{diag}(a, b))^{-1} = |\mathrm{Nr}_{D/k}(a)|_{\mathbb{A}}^{n-1} |\mathrm{Nr}_{M_{n-1}(D)/k}(b)|_{\mathbb{A}}^{-1}.$$

On the other hand,

$$\Phi(\mathrm{diag}(a, b))^{2n} = \frac{\prod_{v \in \mathfrak{V}} H^v(e_1 a_v)^{2n}}{|\mathrm{Nr}_{D/k}(a)|_{\mathbb{A}} |\mathrm{Nr}_{M_{n-1}(D)/k}(b)|_{\mathbb{A}}}.$$

By Lemma 2.1, $H^v(e_1 a_v)^{2n} = |\mathrm{Nr}_{D/k}(a_v)|_{k_v}^n$. Then we obtain

$$\Phi(\mathrm{diag}(a, b))^{2n} = |\mathrm{Nr}_{D/k}(a)|_{\mathbb{A}}^{n-1} |\mathrm{Nr}_{M_{n-1}(D)/k}(b)|_{\mathbb{A}}^{-1}. \qquad \square$$

By this lemma and $G(k)e_1 = V - \{0\}$, we have the following expressions of $\gamma_n(D)$ and $\widetilde{\gamma}_n(D)$:

$$\gamma_n(D) = \max_{g\in G(\mathbb{A})^1} \min_{x\in V-\{0\}} H_g(x)^{2n},$$

$$\widetilde{\gamma}_n(D) = \max_{g\in G(\mathbb{A})} \min_{x\in V-\{0\}} \frac{H_g(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}}.$$

Note that $\gamma_n(D) = \widetilde{\gamma}_n(D)$ if k is a number field.

3. An upper bound of $\widetilde{\gamma}_n(D)$

In this section, we give an upper bound of $\widetilde{\gamma}_n(D)$. For that purpose, we need to fix a realization of D as a cyclic algebra. Namely we fix a separable quadratic extension $L = k(\theta)$ of k and $u \in k^\times$ such that $u \notin N_{L/k}(L^\times)$ and

$$D = (L/k, u) = 1 \cdot L + \mathbf{i} \cdot L, \quad \mathbf{i}^2 = u, \quad \mathbf{i}\lambda = \overline{\lambda}\mathbf{i} \quad (\lambda \in L),$$

where $\lambda \mapsto \overline{\lambda}$ denotes the Galois automorphism of L/k. The reduced norm of $a = \lambda + \mathbf{i}\mu$, $\lambda, \mu \in L$, is equal to

$$\mathrm{Nr}_{D/k}(a) = \mathrm{Nr}_{D/k}(\lambda + \mathbf{i}\mu) = \lambda\overline{\lambda} - u\mu\overline{\mu}.$$

We sometime write $\mathbf{j}$ for θ. The map $\iota\colon D \otimes_k L \to M_2(L)$ defined by

$$\iota(\mathbf{i}) = \begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}, \quad \iota(\mathbf{j}) = \begin{pmatrix} \theta & 0 \\ 0 & \overline{\theta} \end{pmatrix}$$

gives an algebra isomorphism. By using this realization of D, we first give another expression of $\widetilde{\gamma}_n(D)$, and then we will make use of this expression to obtain an upper bound of $\widetilde{\gamma}_n(D)$.

Let $\mathfrak{W}$, $\mathfrak{W}_\infty$ and $\mathfrak{W}_f$ be the sets of all places of L, all infinite places of L and all finite places of L, respectively. For $w \in \mathfrak{W}$, L_w stands for the completion of L at w. The normalized valuation of L_w is denoted by $|\cdot|_{L_w}$. If $w \in \mathfrak{W}_f$, $\mathfrak{o}_{L_w}$ stands for the valuation ring of L_w. The adele ring of L and its idele norm are denoted by $\mathbb{A}_L$ and $|\cdot|_{\mathbb{A}_L}$, respectively. As in §2, we let $V = e_1 D + \cdots + e_n D$ be a right D-vector space. We fix an L-basis of V as follows:

$$\mathbf{f}_{2i-1} = e_i, \quad \mathbf{f}_{2i} = e_i\mathbf{i} \quad (1 \le i \le n).$$

For $x = \mathbf{f}_1\lambda_1 + \cdots + \mathbf{f}_{2n}\lambda_{2n} \in V$, $\lambda_1, \ldots, \lambda_{2n} \in L$, the conjugate $\overline{x}$ of x with respect to L/k is defined by

$$\overline{x} = \mathbf{f}_1\overline{\lambda}_1 + \cdots + \mathbf{f}_{2n}\overline{\lambda}_{2n}.$$

As an L-vector space, the wedge product $\widehat{V}_L := V \wedge V$ has a basis $\mathbf{f}_i \wedge \mathbf{f}_j$, $1 \leq i < j \leq 2n$. For each $w \in \mathfrak{W}$, the L_w-vector space $V \otimes_L L_w$ is denoted by V_{L_w} to avoid confusion with V_v defined in §2. We write $\widehat{V}_{L,w}$ for $\widehat{V}_L \otimes_L L_w = V_{L_w} \wedge V_{L_w}$. By a similar fashion to §2, the local heights ${}^L F_w \colon V_{L_w} \to \mathbb{R}_{\geq 0}$ and ${}^L \widehat{F}_w \colon \widehat{V}_{L,w} \to \mathbb{R}_{\geq 0}$ are defined by

$$
{}^L F_w\left(\sum_{1\leq i\leq 2n} \mathbf{f}_i \lambda_i\right) = \begin{cases} \left(\displaystyle\sum_{1\leq i\leq 2n} |\lambda_i|_{L_w}^2\right)^{1/2} & (w \in \mathfrak{W}_\infty,\ \ L_w = \mathbb{R}), \\ \displaystyle\sum_{1\leq i\leq 2n} |\lambda_i|_{L_w} & (w \in \mathfrak{W}_\infty,\ \ L_w = \mathbb{C}), \\ \displaystyle\sup_{1\leq i\leq 2n} (|\lambda_i|_{L_w}) & (w \in \mathfrak{W}_f) \end{cases}
$$

and

$$
{}^L \widehat{F}_w\left(\sum_{1\leq i<j\leq 2n} (\mathbf{f}_i \wedge \mathbf{f}_j)\lambda_{ij}\right) = \begin{cases} \left(\displaystyle\sum_{1\leq i<j\leq 2n} |\lambda_{ij}|_{L_w}^2\right)^{1/2} & (w \in \mathfrak{W}_\infty,\ \ L_w = \mathbb{R}), \\ \displaystyle\sum_{1\leq i<j\leq 2n} |\lambda_{ij}|_{L_w} & (w \in \mathfrak{W}_\infty,\ \ L_w = \mathbb{C}), \\ \displaystyle\sup_{1\leq i<j\leq 2n} (|\lambda_{ij}|_{L_w}) & (w \in \mathfrak{W}_f). \end{cases}
$$

Then the global heights ${}^L F \colon V \to \mathbb{R}_{\geq 0}$ and ${}^L \widehat{F} \colon \widehat{V}_L \to \mathbb{R}_{\geq 0}$ are defined to be

$$
{}^L F(x) = \prod_{w\in\mathfrak{W}} {}^L F_w(x), \quad {}^L \widehat{F}(X) = \prod_{w\in\mathfrak{W}} {}^L \widehat{F}_w(X)
$$

for $x \in V$ and $X \in \widehat{V}_L$. More generally, we can define the global twisted height ${}^L F_\xi$ and ${}^L \widehat{F}_\xi$ for $\xi = (\xi_w) \in GL_{2n}(\mathbb{A}_L)$ by

$$
{}^L F_\xi(x) = {}^L F(\xi x) := \prod_{w\in\mathfrak{W}} {}^L F_w(\xi_w x), \quad {}^L \widehat{F}_\xi(X) = {}^L \widehat{F}(\xi X) := \prod_{w\in\mathfrak{W}} {}^L \widehat{F}_w(\xi_w X).
$$

Lemma 3.1. *For $x \in V$ and $\xi \in GL_{2n}(\mathbb{A}_L)$, we have ${}^L F\left(\overline{\xi x}\right) = {}^L F(\xi x)$.*

Proof. This is easy by the definition of ${}^L F$. □

We take a maximal compact subgroup ${}^L K$ of $GL_{2n}(\mathbb{A}_L)$ as

$$
{}^L K = \prod_{w\in\mathfrak{W}} {}^L K_w, \quad {}^L K_w = \{g \in GL_{2n}(L_w) \colon {}^L F_w(gx) = {}^L F_w(x) \quad (x \in V_{L_w})\}.
$$

Then both ${}^L F$ and ${}^L\widehat{F}$ are left ${}^L K$ invariant.

Let $\mathbf{P}\widehat{V}_L$ be the projective space of $\widehat{V}_L$ and $\mathbf{P}_D V$ be the set of 1-dimensional right D-subspaces of V. By definition, ${}^L\widehat{F}$ gives rise to the height $\mathbf{P}\widehat{V}_L \to \mathbb{R}_{>0}$. For $x \in V - \{0\}$, the subspace $xD \in \mathbf{P}_D V$ spanned by x is the same as the 2-dimensional L-subspace spanned by $x, x\mathbf{i}$. The correspondence $xD \mapsto (x \wedge x\mathbf{i})L$ yields an injection $\mathbf{P}_D V \hookrightarrow \mathbf{P}\widehat{V}_L$. Thus we can define the height ${}^L H$ on V, more generally the twisted height ${}^L H_\xi$ for $\xi \in GL_{2n}(\mathbb{A}_L)$, by

$$ {}^L H(x) = {}^L\widehat{F}(x \wedge x\mathbf{i})^{1/4}, \quad {}^L H_\xi(x) = {}^L\widehat{F}_\xi(x \wedge x\mathbf{i})^{1/4} $$

for $x \in V$. Since ${}^L H_\xi$ factors through $\mathbf{P}_D V$, the equality ${}^L H_\xi(xa) = {}^L H_\xi(x)$ holds for all $a \in D^\times$ and $x \in V$.

Lemma 3.2. *For $a = (a_v) \in D_{\mathbb{A}}^\times$ and $x \in V$, one has*

$$ {}^L H_\xi(xa) = |\mathrm{Nr}_{D/k}(a)|_{\mathbb{A}}^{1/2} \cdot {}^L H_\xi(x). $$

Proof. Let $a = \lambda + \mathbf{i}\mu$, $\lambda, \mu \in \mathbb{A}_L$. Then

$$ \begin{aligned} (xa) \wedge (xa\mathbf{i}) &= (x\lambda + x\mathbf{i}\mu) \wedge \left(x\mathbf{i}\overline{\lambda} + xu\overline{\mu}\right) = (x \wedge x\mathbf{i})\left(\lambda\overline{\lambda} - u\mu\overline{\mu}\right) \\ &= (x \wedge x\mathbf{i})\,\mathrm{Nr}_{D/k}(a). \end{aligned} $$

Therefore,

$$ {}^L H_\xi(xa) = {}^L\widehat{F}_\xi((x \wedge x\mathbf{i})\,\mathrm{Nr}_{D/k}(a))^{1/4} = |\mathrm{Nr}_{D/k}(a)|_{\mathbb{A}_L}^{1/4} \cdot {}^L H_\xi(x) = |\mathrm{Nr}_{D/k}(a)|_{\mathbb{A}}^{1/2} \cdot {}^L H_\xi(x). \quad \square $$

Lemma 3.3. *For $\xi = (\xi_w) \in GL_{2n}(\mathbb{A}_L)$ and $x \in V$, one has*

$$ {}^L\widehat{F}_\xi(x \wedge (x\mathbf{i})) \le {}^L F_\xi(x)\,{}^L F_{\overline{\xi} J_u}(x), $$

where

$$ J_u = \begin{pmatrix} 0 & u & & & & & 0 \\ 1 & 0 & & & & & \\ & & 0 & u & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & u \\ 0 & & & & & 1 & 0 \end{pmatrix} \in GL_{2n}(k). $$

Proof. By Hadamard's inequality,

$$ {}^L\widehat{F}_\xi(x \wedge (x\mathbf{i})) \le {}^LF_\xi(x)\, {}^LF_\xi(x\mathbf{i}). $$

Let

$$ x = \sum_{i=1}^n \mathbf{f}_{2i-1}\lambda_{2i-1} + \sum_{i=1}^n \mathbf{f}_{2i}\lambda_{2i}, \quad (\lambda_1, \dots, \lambda_{2n} \in L). $$

From the relations $\mathbf{f}_{2i-1}\mathbf{i} = \mathbf{f}_{2i}$, $\mathbf{f}_{2i}\mathbf{i} = \mathbf{f}_{2i-1}u$, it follows

$$ x\mathbf{i} = \sum_{i=1}^n \mathbf{f}_{2i}\overline{\lambda}_{2i-1} + \sum_{i=1}^n \mathbf{f}_{2i-1}\overline{\lambda}_{2i}u = J_u\overline{x}. $$

Therefore, by Lemma 3.1, ${}^LF_\xi(x\mathbf{i}) = {}^LF_\xi(J_u\overline{x}) = {}^LF_{\overline{\xi}J_u}(x)$. □

Viewing V as an L-vector space, $G(k) = GL_n(D)$ is realized as a subgroup in $GL_{2n}(L)$. More precisely, we have

$$ G(k) = \left\{\xi \in GL_{2n}(L) : J_u\overline{\xi}J_u^{-1} = \xi\right\}. $$

Note that the condition $J_u\overline{\xi}J_u^{-1} = \xi$ is the same as $J_u^{-1}\overline{\xi}J_u = \xi$ because of $J_u^{-1} = u^{-1}J_u$. We fixed the good maximal compact subgroup K of $G(\mathbb{A})$ in §2. Since maximal compact subgroups of $GL_{2n}(\mathbb{A}_L)$ are conjugate to each other, there exists an $\eta \in GL_{2n}(\mathbb{A}_L)$ such that $K = \eta^{-1}\,{}^LK\eta \cap G(\mathbb{A})$.

Lemma 3.4. *Being the notation as before, then one has*

$$ H_Q(g^{-1}) = \frac{1}{{}^LH_\eta(e_1)^{2n}} \cdot \frac{{}^LH_{\eta g}(e_1)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}} $$

for $g \in G(\mathbb{A})$.

Proof. This follows from Lemma 3.2 and the same argument as in the proof of Lemma 2.2. □

Therefore, $\widetilde{\gamma}_n(D)$ and $\gamma_n(D)$ are represented as

$$ \widetilde{\gamma}_n(D) = \frac{1}{{}^LH_\eta(e_1)^{2n}} \max_{g\in G(\mathbb{A})} \min_{x\in V-\{0\}} \frac{{}^LH_{\eta g}(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}}, $$
$$ \gamma_n(D) = \frac{1}{{}^LH_\eta(e_1)^{2n}} \max_{g\in G(\mathbb{A})^1} \min_{x\in V-\{0\}} {}^LH_{\eta g}(x)^{2n}. $$

REMARK. If k is a number field, we have

$$\widetilde{\gamma}_n(D) = \gamma_n(D) = \frac{1}{{}^L H_\eta(e_1)^{2n}} \gamma_{n,1}(D_u, H_\eta)^{n[k:\mathbb{Q}]/2},$$

where the right-hand side is a generalized Hermite constant defined in [10, §2]. (This relation holds for any division algebra D of degree d if we take L as a cyclic splitting field of D and ${}^L H$ as a height induced from the standard height on the d-th wedge product of the L-vector space $D^n = L^{dn}$.) In general, η is not contained in $G(\mathbb{A})^1$ and we can not immediately apply [10, Theorem 3] to get an upper bound of $\widetilde{\gamma}_n(D)$.

In order to obtain an upper bound of $\widetilde{\gamma}_n(D)$, we need some arguments of geometry of numbers. In the following, we set $m = 2n$ for simplicity. Let $V_{\mathbb{A}_L} = V \otimes_L \mathbb{A}_L$ be the adele space of V and ω_V the Haar measure on $V_{\mathbb{A}_L}$ normalized so that $\omega_V(V_{\mathbb{A}_L}/V) = 1$. We define the subset $\mathbf{S}$ of $V_{\mathbb{A}_L}$ by

$$\mathbf{S} = \prod_{w \in \mathfrak{W}} S_w, \quad S_w = \left\{x \in V_{L_w} : {}^L F_w(x) \le 1\right\}.$$

Then $\mathbf{S}$ is a compact subset of $V_{\mathbb{A}_L}$. We define the function $\psi : GL_m(\mathbb{A}_L) \to \mathbb{R}_{>0}$ by

$$\psi(\xi) = \frac{\omega_V(\mathbf{S} \cap \xi \mathbf{S})}{\omega_V(\mathbf{S})}.$$

Lemma 3.5. *Let* $\xi_1, \xi_2 \in GL_m(\mathbb{A}_L)$. *If one has*

$$\omega_V(\xi_1 \mathbf{S} \cap \xi_2 \mathbf{S}) > \begin{cases} 2^{m[L:\mathbb{Q}]} & (\mathrm{ch}(L) = 0), \\ 1 & (\mathrm{ch}(L) > 0), \end{cases}$$

then $\xi_1 \mathbf{S} \cap \xi_2 \mathbf{S} \cap V \supsetneqq \{0\}$.

Proof. This follows from a standard argument of geometry of numbers (cf. [8]). For the sake of completeness, we mention a proof. Let Ω be a fundamental domain in $V_{\mathbb{A}_L}$ modulo V. We set

$$\mathbf{S}' = \prod_{w \in \mathfrak{W}_\infty} 2^{-1} S_w \times \prod_{w \in \mathfrak{W}_f} S_w.$$

Then $\omega_V(\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}') > 1$. Since $\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}'$ is compact, the set

$$\{x \in V : (x + \Omega) \cap (\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}') \neq \emptyset\}$$

is finite. We denote the elements of this finite set by $x_1, \ldots, x_r$. Let

$$\Omega_i = ((\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}') - x_i) \cap \Omega$$

for $1 \le i \le r$. Then $\Omega_1 + x_1, \ldots, \Omega_r + x_r$ cover $(\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}')$, so that

$$\sum_{i=1}^{r} \omega_V(\Omega_i) \ge \omega_V(\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}') > 1 = \omega_V(V_{\mathbb{A}_L}/V) = \omega_V(\Omega).$$

Thus $r > 1$ and there are $\Omega_i \ne \Omega_j$ such that $\Omega_i \cap \Omega_j \ne \emptyset$. Let $x \in \Omega_i \cap \Omega_j$. Then $x + x_i, x + x_j \in (\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}')$, and hence

$$0 \ne x_i - x_j \in \{(\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}') + (\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}')\} \cap V \subset (\xi_1 \mathbf{S} \cap \xi_2 \mathbf{S}) \cap V.$$

Here we note that the finite part of $(\xi_1 \mathbf{S}' \cap \xi_2 \mathbf{S}')$ is a module. □

By the definition of ψ, one has

$$\omega_V(\xi_1 \mathbf{S} \cap \xi_2 \mathbf{S}) = |\det \xi_1|_{\mathbb{A}_L} \psi\left(\xi_1^{-1} \xi_2\right) \omega_V(\mathbf{S}) = |\det \xi_2|_{\mathbb{A}_L} \psi\left(\xi_2^{-1} \xi_1\right) \omega_V(\mathbf{S}).$$

We put

$$\kappa_m(L) = \frac{1}{\omega_V(\mathbf{S})} \cdot \begin{cases} 2^{m[L:\mathbb{Q}]} & (\mathrm{ch}(L) = 0). \\ q^m & (\mathrm{ch}(L) > 0). \end{cases}$$

Lemma 3.6. *Let $\xi_1, \xi_2 \in GL_m(\mathbb{A}_L)$. Then*

$$\min_{0 \ne x \in V} {}^L F_{\xi_1}(x) \, {}^L F_{\xi_2}(x) \le |\det \xi_1|_{\mathbb{A}_L}^{2/m} \cdot \left(\frac{\kappa_m(L)}{\psi\left(\xi_1 \xi_2^{-1}\right)} \right)^{2/m}.$$

Proof. Let

$$\kappa_m(L)' = \frac{1}{\omega_V(\mathbf{S})} \cdot \begin{cases} 2^{m[L:\mathbb{Q}]} & (\mathrm{ch}(L) = 0). \\ 1 & (\mathrm{ch}(L) > 0). \end{cases}$$

We take a $\lambda \in \mathbb{A}_L^\times$ such that

$$|\lambda|_{\mathbb{A}_L}^m \left|\det \xi_1^{-1}\right|_{\mathbb{A}_L} \psi\left(\xi_1 \xi_2^{-1}\right) > \kappa_m(L)'.$$

Then, by Lemma 3.5, there is $0 \ne x \in \lambda \xi_1^{-1} \mathbf{S} \cap \lambda \xi_2^{-1} \mathbf{S} \cap V$. Let $x = \lambda \xi_1^{-1} y_1 = \lambda \xi_2^{-1} y_2$, $(y_1, y_2 \in \mathbf{S})$. Then

$$1 \ge {}^L F(y_1) \, {}^L F(y_2) = |\lambda|_{\mathbb{A}_L}^{-2} \cdot {}^L F_{\xi_1}(x) \, {}^L F_{\xi_2}(x).$$

Therefore,

$$\begin{aligned} \min_{0 \ne x \in V} \sqrt{{}^L F_{\xi_1}(x) \, {}^L F_{\xi_2}(x)} &\le \inf\left\{ |\lambda|_{\mathbb{A}_L} : |\lambda|_{\mathbb{A}_L}^m \left|\det \xi_1^{-1}\right|_{\mathbb{A}_L} \psi\left(\xi_1 \xi_2^{-1}\right) > \kappa_m(L)' \right\} \\ &\le |\det \xi_1|_{\mathbb{A}_L}^{1/m} \cdot \left(\frac{\kappa_m(L)}{\psi\left(\xi_1 \xi_2^{-1}\right)} \right)^{1/m}. \end{aligned}$$

□

Next we determine $\eta = (\eta_w) \in GL_{2n}(\mathbb{A}_L)$ such that $K = \eta^{-1\,L}K\eta \cap G(\mathbb{A})$ for the maximal compact subgroups $K \subset G(\mathbb{A})$ and ${}^LK \subset GL_{2n}(\mathbb{A}_L)$. Let $v \in \mathfrak{V}$ and $\mathfrak{W}_v$ be the set of places of L which lie above v. What we need is the form of the coset $(\eta_w)_{w\in\mathfrak{W}_v} G(k_v)$ in $\prod_{w\in\mathfrak{W}_v} GL_{2n}(L_w)$.

(i) The case that $v \in \mathfrak{V}''$ and v splits in L. Let $\mathfrak{W}_v = \{w, w'\}$. Then $GL_{2n}(L_w) = GL_{2n}(L_{w'}) = GL_{2n}(k_v)$ and the Galois automorphism becomes $\overline{(g, g')} = (g', g)$ for $(g, g') \in GL_{2n}(L_w) \times GL_{2n}(L_{w'})$, and hence

$$G(k_v) = \left\{\left(g,\, J_u g J_u^{-1}\right) \in G(L_w) \times G(L_{w'}) \colon g \in GL_{2n}(k_v)\right\}.$$

Since $K_v = (\eta_w, \eta_{w'})^{-1}({}^LK_w \times {}^LK_{w'})(\eta_w, \eta_{w'}) \cap G(k_v)$ and ${}^LK_w = {}^LK_{w'}$ by Lemma 3.1, we must have $J_u \eta_w^{-1}\,{}^LK_w \eta_w J_u^{-1} = \eta_{w'}^{-1}\,{}^LK_{w'}\eta_{w'}$, so that we can take $\eta_{w'}$ as $\eta_w J_u^{-1}$. Therefore, $(\eta_w, \eta_{w'})G(k_v) = \left(1,\, J_u^{-1}\right)G(k_v)$.

(ii) The case that $v \in \mathfrak{V}''$ and v remains prime in L. Let $\mathfrak{W}_v = \{w\}$. Then

$$G(k_v) = \left\{g \in GL_{2n}(L_w) \colon J_u \overline{g} J_u^{-1} = g\right\}.$$

Let

$$G'(k_v) = \{g \in GL_{2n}(L_w) \colon \overline{g} = g\} = GL_{2n}(k_v).$$

Since $L_w = k_v(\theta)$ is a quadratic extension of k_w and $D_v \cong M_2(k_v)$, there exists $\delta_w \in L_w^\times$ such that $u = \delta_w\overline{\delta}_w$. Then we define the $2n$ by $2n$ matrix $T_w \in GL_{2n}(L_w)$ by

$$T_w = \begin{pmatrix} \delta_w & \delta_w\overline{\theta} & & & 0 \\ 1 & \theta & & & \\ & & \ddots & & \\ & & & \delta_w & \delta_w\overline{\theta} \\ 0 & & & 1 & \theta \end{pmatrix}.$$

The inner automorphism $\mathrm{int}(T_w) \colon g \mapsto T_w g T_w^{-1}$ gives an isomorphism from $G'(k_v)$ onto $G(k_v)$. Therefore, $T_w\,{}^LK_w T_w^{-1} \cap G(k_v)$ is a maximal compact subgroup of $G(k_v)$ and there exists $h_v \in G(k_v)$ such that $h_v^{-1}\left(T_w\,{}^LK_w T_w^{-1} \cap G(k_v)\right)h_v^{-1} = K_v$. Hence we have $\eta_w G(k_v) = T_w^{-1}G(k_v)$. This implies that η_w satisfies

$$\overline{\eta}_w J_u \eta_w^{-1} = \delta_w I_{2n}.$$

(iii) The case of $v \in \mathfrak{V}'_f$. In this case, v remains prime in L. Let $\mathfrak{W}_v = \{w\}$. The maximal compact subgroup ${}^LK_w$ is the stabilizer of the $\mathfrak{o}_{L_w}$-lattice

$$\Lambda_w := \mathbf{f}_1\mathfrak{o}_{L_w} + \mathbf{f}_2\mathfrak{o}_{L_w} + \cdots + \mathbf{f}_{2n-1}\mathfrak{o}_{L_w} + \mathbf{f}_{2n}\mathfrak{o}_{L_w}.$$

Since both $\mathfrak{o}_{L_w} + \mathbf{i}\mathfrak{o}_{L_w}$ and $\mathfrak{O}_v$ are $\mathfrak{o}_{L_w}$-lattices of rank 2 in $D \otimes_L L_w$, there exits

$T'_w \in GL_2(L_w)$ such that $T'_w(\mathfrak{o}_{L_w} + \mathbf{i}\mathfrak{o}_{L_w}) = \mathfrak{O}_v$. We define the $2n$ by $2n$ matrix $T_w \in GL_{2n}(L_w)$ by

$$T_w = \begin{pmatrix} T'_w & & 0 \\ & \ddots & \\ 0 & & T'_w \end{pmatrix}.$$

Then $T_w \Lambda_w = \Lambda_{1,v} := e_1\mathfrak{O}_v + \cdots + e_n\mathfrak{O}_v$. Therefore, $T_w {}^L K_w T_w^{-1} \cap G(k_v)$ coincides with the stabilizer of $\Lambda_{1,v}$ in $G(k_v)$, and hence we have $\eta_w G(k_v) = T_w^{-1} G(k_v)$.

(iv) The case of $v \in \mathfrak{V}'_\infty$. Then v remains prime in L and $G(k_v) \cong GL_n(\mathbb{H})$, where $\mathbb{H}$ denotes the Hamilton quaternion algebra, and $GL_{2n}(L_w) = GL_{2n}(\mathbb{C})$ for $\mathfrak{W}_v = \{w\}$. We recall that K_v preserves the norm

$$F_v(e_1x_1 + \cdots + e_nx_n) = \left(\sum_{i=1}^{n} |\mathrm{Nr}_{D/k}(x_i)|_{k_v}\right)^{1/2}, \quad (x_1, \ldots, x_n \in D_v).$$

For $x_i = \lambda_i + \mathbf{i}\mu_i$, $\lambda_i, \mu_i \in L_w$, one has a relation

$$\begin{aligned} F_v\left(\sum_{i=1}^{n} e_i(\lambda_i + \mathbf{i}\mu_i)\right) &= \left(\sum_{i=1}^{n} \lambda_i\overline{\lambda}_i - u\mu_i\overline{\mu}_i\right)^{1/2} \\ &= {}^L F_{I_u}\left(\sum_{i=1}^{n} (\mathbf{f}_{2i-1}\lambda_i + \mathbf{f}_{2i}\mu_i)\right)^{1/2}, \end{aligned}$$

where

$$I_u = \begin{pmatrix} 1 & & & & 0 \\ & \sqrt{-u} & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \sqrt{-u} \end{pmatrix} \in GL_{2n}(\mathbb{C}).$$

Note that $-u > 0$ in $k_v = \mathbb{R}$ because $D_v = \mathbb{H}$. Therefore, we have $\eta_w G(k_v) = I_u G(k_v)$.

Lemma 3.7. *Let $\eta \in GL_{2n}(\mathbb{A}_L)$ be an element such that $K = \eta^{-1}\,{}^L K \eta \cap G(\mathbb{A})$ and $h \in G(\mathbb{A})$ be an arbitrary element. Then, for $\xi_1 = \eta h$ and $\xi_2 = \overline{\eta}\,\overline{h} J_u$, one has*

$$\begin{aligned} \frac{1}{\psi\left(\xi_1\xi_2^{-1}\right)} = \frac{1}{\psi\left(\xi_2\xi_1^{-1}\right)} = \frac{1}{\psi\left(\overline{\eta} J_u \eta^{-1}\right)} &= \prod_{v \in \mathfrak{V}} \max\left(1, |u|_{k_v}^{-2n}\right) \\ &= \prod_{v \in \mathfrak{V}} \max\left(|u|_{k_v}^{n}, |u|_{k_v}^{-n}\right). \end{aligned}$$

Proof. By the definition of ψ and $|\det \xi_1|_{\mathbb{A}_L} = |\det \xi_2|_{\mathbb{A}_L} = |\det \eta h|_{\mathbb{A}_L}$, we have $\psi\left(\xi_1\xi_2^{-1}\right) = \psi\left(\xi_2\xi_1^{-1}\right)$. From $J_u h J_u^{-1} = \overline{h}$ for $h \in G(\mathbb{A}_k)$, it follows

$$\psi\left(\xi_2\xi_1^{-1}\right) = \psi\left(\overline{\eta}\,\overline{h} J_u h^{-1}\eta^{-1}\right) = \psi\left(\overline{\eta} J_u \eta^{-1}\right).$$

Especially, $\psi\left(\xi_2\xi_1^{-1}\right)$ is independent of h. This allows us to consider only η modulo $G(\mathbb{A})$. For $w \in \mathfrak{W}$, let ω_w be a Haar measure on V_{L_w} and $\overline{\eta}_w$ the w-component of $\overline{\eta}$. We evaluate $\omega_w(S_w)/\omega_w\left(S_w \cap \overline{\eta}_w J_u \eta_w^{-1} S_w\right)$.

(i) The case that $\mathfrak{W}_v = \{w, w'\}$ and $v \in \mathfrak{V}''$. In this case, $(\eta_w, \eta_{w'}) = \left(1, J_u^{-1}\right)$ modulo $G(k_v)$. From $\overline{\eta}_w = \eta_{w'}$ and $\overline{\eta}_{w'} = \eta_w$, it follows $\overline{\eta}_w J_u \eta_w^{-1} = J_u^{-1} J_u = 1$ and $\overline{\eta}_{w'} J_u \eta_{w'}^{-1} = J_u^2 = u I_{2n}$. Therefore,

$$\begin{aligned}
&\frac{\omega_w(S_w)}{\omega_w(S_w \cap \overline{\eta}_w J_u \eta_w^{-1} S_w)} \cdot \frac{\omega_{w'}(S_{w'})}{\omega_{w'}\left(S_{w'} \cap \overline{\eta}_{w'} J_u \eta_{w'}^{-1} S_{w'}\right)} \\
&= \frac{\omega_{w'}(S_{w'})}{\omega_{w'}(S_{w'} \cap S_{w'} u)} = \max\left(1, |u|_{L_{w'}}^{-2n}\right) = \max\left(1, |u|_{k_v}^{-2n}\right).
\end{aligned}$$

(ii) The case that $\mathfrak{W}_v = \{w\}$ and $v \in \mathfrak{V}''$. In this case, we have a relation $\overline{\eta}_w J_u \eta_w^{-1} = \delta_w I_{2n}$, where $\delta_w \in L_w^\times$ and $N_{L_w/k_v}(\delta_w) = u$. Therefore,

$$\frac{\omega_w(S_w)}{\omega_w(S_w \cap S_w \delta_w)} = \max\left(1, |\delta_w|_{L_w}^{-2n}\right) = \max\left(1, |u|_{k_v}^{-2n}\right).$$

(iii) The case that $w \in \mathfrak{W}_v$ and $v \in \mathfrak{V}'_f$. In this case, $S_w = \Lambda_w$, $\eta_w = T_w^{-1}$ modulo $G(k_v)$ and $T_w \Lambda_w = \Lambda_{1,v}$. Note that

$$\overline{T}_w \Lambda_w = \overline{T}_w \overline{\Lambda}_w = \overline{\Lambda}_{1,v}.$$

Since $J_u \overline{x} = x\mathbf{i}$ for $x \in V_{L_w}$, we obtain

$$J_u^{-1} \overline{T}_w \Lambda_w = u^{-1} J_u \overline{\Lambda}_{1,v} = \Lambda_{1,v} \cdot \mathbf{i} u^{-1} = \Lambda_{1,v} \cdot \mathbf{i}^{-1}.$$

Therefore,

$$\begin{aligned}
\frac{\omega_w(S_w)}{\omega_w\left(S_w \cap \overline{T}_w^{-1} J_u T_w S_w\right)} &= \frac{\omega_w\left(J_u^{-1}\overline{T}_w S_w\right)}{\omega_w\left(J_u^{-1}\overline{T}_w S_w \cap T_w S_w\right)} = \frac{\omega_w\left(\Lambda_{1,v}\mathbf{i}^{-1}\right)}{\omega_w\left(\Lambda_{1,v}\mathbf{i}^{-1} \cap \Lambda_{1,v}\right)} \\
&= \frac{\omega_w(\Lambda_{1,v})}{\omega_w(\Lambda_{1,v} \cap \Lambda_{1,v}\mathbf{i})} = \begin{cases} 1 & (\mathfrak{O}_v \subset \mathfrak{O}_v \mathbf{i}) \\ [\mathfrak{O}_v : \mathfrak{O}_v \mathbf{i}]^n & (\mathfrak{O}_v \mathbf{i} \subset \mathfrak{O}_v) \end{cases} \\
&= \max\left(1, |\mathrm{Nr}_{D/k}(\mathbf{i})|_{k_v}^{-2n}\right) = \max\left(1, |u|_{k_v}^{-2n}\right).
\end{aligned}$$

(iv) The case that $w \in \mathfrak{W}_v$ and $v \in \mathfrak{V}'_\infty$. In this case, $-u > 0$ in k_v, $\eta_w = I_u$ modulo $G(k_v)$ and $\overline{I}_u = I_u$, so that

$$\overline{I}_u J_u I_u^{-1} = \sqrt{-u}\,J_1, \quad \text{where} \quad J_1 = \begin{pmatrix} 0 & 1 & & & & & & 0 \\ 1 & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & 1 \\ 0 & & & & & & 1 & 0 \end{pmatrix} \in GL_{2n}(k).$$

Therefore,

$$\frac{\omega_w(S_w)}{\omega_w\big(S_w \cap \overline{I}_u J_u I_u^{-1} S_w\big)} = \frac{\omega_w(S_w)}{\omega_w(S_w \cap S_w\sqrt{-u})} = \max\big(1, \big|\sqrt{-u}\big|_{L_w}^{-2n}\big) = \max\big(1, |u|_{k_v}^{-2n}\big).$$

Summing up, we obtain the assertion. □

Theorem 3.8. *Let k be a global field, L/k a separable quadratic extension and $D = (L/k, u)$ a quaternion skew field over k. Then we have*

$$\gamma_n(D) \le \widetilde{\gamma}_n(D) \le \left(\prod_{v\in\mathfrak{V}} \max\big(|u|_{k_v}, |u|_{k_v}^{-1}\big)\right)^{n/2} \cdot \kappa_{2n}(L)^{1/2}.$$

Proof. The inequality $\gamma_n(D) \le \widetilde{\gamma}_n(D)$ is trivial by definition in §1. There is an $h \in G(k) = GL_n(D)$ such that h is a permutation of $\{e_1, \ldots, e_n\}$ and

$${}^L H_{\eta h}(e_n) \le {}^L H_{\eta h}(e_{n-1}) \le \cdots \le {}^L H_{\eta h}(e_1).$$

From Hadamard's inequality, it follows

$$|\det \eta|_{\mathbb{A}_L}^{1/4} = |\det \eta h|_{\mathbb{A}_L}^{1/4} \le {}^L H_{\eta h}(e_1) \cdots {}^L H_{\eta h}(e_n) \le {}^L H_{\eta h}(e_1)^n,$$

and hence

$$\frac{|\det \eta|_{\mathbb{A}_L}^{1/2}}{{}^L H_{\eta h}(e_1)^{2n}} \le 1.$$

Since h is a permutation matrix and hence $h \in K$, we can replace η with ηh and g with $h^{-1}g$ in the formula following Lemma 3.4. Then we have

$$\widetilde{\gamma}_n(D) = \frac{1}{{}^L H_{\eta h}(e_1)^{2n}} \cdot \max_{g\in G(\mathbb{A})} \min_{0\ne x\in V} \frac{{}^L H_{\eta g}(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}}$$

$$\leq \frac{1}{|\det \eta|_{\mathbb{A}_L}^{1/2}} \cdot \max_{g\in G(\mathbb{A})} \min_{0\neq x\in V} \frac{{}^L H_{\eta g}(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}}.$$

By Lemma 3.3,

$${}^L H_{\eta g}(x)^{2n} \leq \left({}^L F_{\eta g}(x)\, {}^L F_{\overline{\eta g} J_u}(x)\right)^{n/2}.$$

Applying Lemma 3.6 to $\xi_1 = \eta g$ and $\xi_2 = \overline{\eta g} J_u = \overline{\eta} J_u g$, one has

$$\min_{0\neq x\in V} {}^L F_{\eta g}(x)\, {}^L F_{\overline{\eta g} J_u}(x) \leq |\det \eta g|_{\mathbb{A}_L}^{1/n} \frac{\kappa_{2n}(L)^{1/n}}{\psi(\overline{\eta} J_u \eta^{-1})^{1/n}}.$$

Therefore, by Lemma 3.7,

$$\frac{1}{|\det \eta|_{\mathbb{A}_L}^{1/2}} \cdot \min_{0\neq x\in V} \frac{{}^L H_{\eta g}(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}}$$
$$\leq \left(\prod_{v\in\mathfrak{V}} \max\left(|u|_{k_v}, |u|_{k_v}^{-1}\right)\right)^{n/2} \cdot \kappa_{2n}(L)^{1/2}$$

holds for all $g \in G(\mathbb{A})$. □

The explicit value of $\kappa_{2n}(L)$ is given as follows:

$$\kappa_{2n}(L) = \begin{cases} \dfrac{|D_L|^n 4^{n[L:\mathbb{Q}]}}{(\pi^n/(\Gamma(1+n)))^{r_1}((2\pi)^{2n}/(\Gamma(1+2n)))^{r_2}} & (L \text{ is a number field}), \\ q_L^{2ng_L} & (L \text{ is a function field}). \end{cases}$$

Here if L is a number field, D_L denotes the absolute discriminant of L, r_1 (resp. r_2) the number of real (resp. imaginary) places of L, and if L is a function field, g_L denotes the genus of L and q_L the cardinality of the constant field of L.

REMARK. Fundamental Hermite constants satisfy a Rankin type inequality ([11, Theorem 4]). This especially deduces the following Mordell's inequality for $\gamma_n(D)$:

$$\gamma_n(D)^{1/n} \leq \widetilde{\gamma}_{n-1}(D)^{1/(n-2)}.$$

If k is a number field, this is written as

$$\left(\gamma_n(D)^{1/n}\right)^{1/(n-1)} \leq \left(\gamma_{n-1}(D)^{1/(n-1)}\right)^{1/(n-2)}.$$

See [4, Theorem 2.3.1] for the original form of Mordell's inequality.

REMARK. Lower bounds of fundamental Hermite constants were also given in [11]. A lower bound of $\gamma_n(D)$ was explicitly computed in [6].

4. The case of a number field

In the rest of this paper, we assume that k is a number field, D a quaternion skew field over k. The aim of this section is to translate the adelic definition of the constant $\gamma_n(D)$ given in §2 into a global setting. We will describe $\gamma_n(D)$ by using the notion of quaternionic Humbert forms over D. This description will be used to develop the Voronoï theory for the quaternionic Hermite invariant in §6.

In §2, we fixed a maximal order $\mathfrak{O}$ of D and the maximal compact subgroup K of $G(\mathbb{A})$ whose finite component K_v, $v \in \mathfrak{V}_f$, is the stabilizer of the free $\mathfrak{O}_v$-lattice $e_1\mathfrak{O}_v+\cdots+e_n\mathfrak{O}_v$ in V_v. Let $\mathcal{L}_{\mathfrak{O}}(V)$ be the set of $\mathfrak{O}$-lattices Λ in V such that $\Lambda\otimes_{\mathfrak{o}_k} k = V$, and let $\mathcal{L}_{\mathfrak{O}}(V)/\cong$ be the set of $G(k)$-equivalent classes of elements in $\mathcal{L}_{\mathfrak{O}}(V)$.

We define the reduced norm of D over $\mathbb{Q}$ as $\mathrm{Nr}_{D/\mathbb{Q}} = N_{k/\mathbb{Q}} \circ \mathrm{Nr}_{D/k}$ (it applies to elements of D and $\mathfrak{O}$-ideals as well).

First, we recall some facts of the ideal theory of simple algebras. Let $\mathfrak{R}$ be a maximal order of $M_n(D)$ and $\mathfrak{R}_v$ be the completion of $\mathfrak{R}$ at $v \in \mathfrak{V}_f$. For $g = (g_v) \in G(\mathbb{A})$,

$$g\mathfrak{R} = \bigcap_{v\in\mathfrak{V}_f} (M_n(D) \cap g_v\mathfrak{R}_v)$$

yields a right $\mathfrak{R}$-ideal in $M_n(D)$. We define the subgroup $G(\mathbb{A})_{\mathfrak{R}}$ by

$$G(\mathbb{A})_{\mathfrak{R}} = \{g \in G(\mathbb{A})\colon g\mathfrak{R} = \mathfrak{R}\}.$$

Then the double coset $G(\mathbb{A})_{\mathfrak{R}}g^{-1}G(k)$ of $g \in G(\mathbb{A})$ corresponds to the right $\mathfrak{R}$-ideal class of $g\mathfrak{R}$, and $G(\mathbb{A})_{\mathfrak{R}}\backslash G(\mathbb{A})/G(k)$ is identified with the set of right $\mathfrak{R}$-ideal classes of $M_n(D)$. It is known that the cardinal number $\sharp(G(\mathbb{A})_{\mathfrak{R}}\backslash G(\mathbb{A})/G(k))$ is finite and is independent of the choice of a maximal order of $M_n(D)$. Thus we denote $\sharp(G(\mathbb{A})_{\mathfrak{R}}\backslash G(\mathbb{A})/G(k))$ by $h_D^{(n)}$. The class number of left $\mathfrak{R}$-ideal classes is also equal to $h_D^{(n)}$. We let $\mathfrak{R} = M_n(\mathfrak{O})$. For $\Lambda \in \mathcal{L}_{\mathfrak{O}}(V)$, the set

$$\mathfrak{A}_\Lambda = \{A \in M_n(D)\colon A(e_1\mathfrak{O} + \cdots + e_n\mathfrak{O}) \subset \Lambda\}.$$

is a right $M_n(\mathfrak{O})$-ideal of $M_n(D)$. The correspondence $\Lambda \mapsto \mathfrak{A}_\Lambda$ gives a bijection from $\mathcal{L}_{\mathfrak{O}}(V)/\cong$ to the set of right $M_n(\mathfrak{O})$-ideal classes (cf. [2, Théorème 7]). As a consequence, $\sharp(\mathcal{L}_{\mathfrak{O}}(V)/\cong)$ is equal to $h_D^{(n)}$, and hence $\sharp(\mathcal{L}_{\mathfrak{O}}(V)/\cong)$ is independent of the choice of a maximal order of D.

We denote by $\mathcal{I}_{\mathfrak{O}}$ the set of all right $\mathfrak{O}$-ideals in D, by $\mathcal{I}_{\mathfrak{O}}/\cong$ the set of all right $\mathfrak{O}$-ideal classes and by h_D the class number $\sharp(\mathcal{I}_{\mathfrak{O}}/\cong)$. For $\mathfrak{A} \in I_{\mathfrak{O}}$, the ideal class of $\mathfrak{A}$ is denoted by $[\mathfrak{A}]$. We put

$$\Lambda(\mathfrak{A}) = e_1\mathfrak{O} + \cdots + e_{n-1}\mathfrak{O} + e_n\mathfrak{A},$$

which is an $\mathfrak{O}$-lattice in V. By [2, Théorème 3], it is known that the correspondance $\mathfrak{A} \mapsto \Lambda(\mathfrak{A})$ give a surjection from $\mathcal{I}_{\mathfrak{O}}/\cong$ to $\mathcal{L}_{\mathfrak{O}}(V)/\cong$, and hence $h_D^{(n)} \le h_D$. We fix, once and for all, a complete system $\{\mathfrak{A}_1, \dots, \mathfrak{A}_{h_D}\}$ of representatives of $\mathcal{I}_D/\cong$ such that $\mathfrak{A}_1 = \mathfrak{O}$ and $\{\Lambda(\mathfrak{A}_i) \colon 1 \le i \le h_D^{(n)}\}$ forms a complete system of representatives of $\mathcal{L}_{\mathfrak{O}}(V)/\cong$. We write Λ_i for $\Lambda(\mathfrak{A}_i)$. The adele group $G(\mathbb{A})$ acts transitively on $\mathcal{L}_{\mathfrak{O}}(V)$ by

$$g\Lambda = \bigcap_{v \in \mathfrak{V}_f} (V \cap g_v \Lambda_v)$$

for $g = (g_v) \in G(\mathbb{A})$ and $\Lambda \in \mathcal{L}_{\mathfrak{O}}(V)$, where Λ_v denotes the completion of Λ in V_v for $v \in \mathfrak{V}_f$. Let $G(\mathbb{A})_{\Lambda_1}$ be the stabilizer of $\Lambda_1 = e_1\mathfrak{O} + \cdots + e_n\mathfrak{O}$ in $G(\mathbb{A})$, namely

$$G(\mathbb{A})_{\Lambda_1} = G(k_\infty)K_f, \quad \text{where} \quad G(k_\infty) = \prod_{v \in \mathfrak{V}_\infty} G(k_v), \quad K_f = \prod_{v \in \mathfrak{V}_f} K_v.$$

The map $g \mapsto g^{-1}\Lambda_1$ on $G(\mathbb{A})$ gives rise to bijections from $G(\mathbb{A})_{\Lambda_1}\backslash G(\mathbb{A})$ to $\mathcal{L}_{\mathfrak{O}}(V)$ and $G(\mathbb{A})_{\Lambda_1}\backslash G(\mathbb{A})/G(k)$ to $\mathcal{L}_{\mathfrak{O}}(V)/\cong$. For each i, we fix $g_i \in G(\mathbb{A}_f)$ such that $g_i^{-1}\Lambda_1 = \Lambda_i$. Then $G(\mathbb{A})$ is decomposed into a disjoint union of double cosets $G(\mathbb{A})_{\Lambda_1} g_i G(k)$, i.e.,

$$G(\mathbb{A}) = \bigsqcup_{1 \le i \le h_D^{(n)}} G(\mathbb{A})_{\Lambda_1} g_i G(k).$$

With the notation of §2, we define the constant $\gamma_n(D)_i$ by

$$\begin{aligned}\gamma_n(D)_i &= \max_{g \in G(\mathbb{A})_{\Lambda_1} g_i G(k)} \min_{x \in V - \{0\}} \frac{H_g(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}} \\ &= \max_{g \in G(k_\infty)} \min_{x \in V - \{0\}} \frac{H_g^\infty(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}} \cdot \frac{H_{g_i}^f(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f}},\end{aligned}$$

where

$$H_g^\infty(x) = \prod_{v \in \mathfrak{V}_\infty} H^v(g_v x), \quad H_{g_i}^f(x) = \prod_{v \in \mathfrak{V}_f} H^v(g_{i,v} x).$$

Note that $H_g^\infty(x) H_{g_i}^f(x)$ is invariant by multiples $x \mapsto xa$ ($a \in D^\times$) by Lemma 2.1 and the product formula. Since $V = \{xa \colon x \in \Lambda_i,\ a \in D\}$ by [2, Théorème in Appendice I], the minimum of the defining equation of $\gamma_n(D)_i$ is attained at a point in Λ_i. Therefore,

$$\gamma_n(D)_i = \max_{g \in G(k_\infty)} \min_{x \in \Lambda_i - \{0\}} \frac{H_g^\infty(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}} \cdot \frac{H_{g_i}^f(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f}}$$

and

$$\gamma_n(D) = \widetilde{\gamma}_n(D) = \max_{1 \le i \le h_D^{(n)}} \gamma_n(D)_i.$$

REMARK. If $h_D = 1$, then we have

$$\gamma_n(D) = \gamma_n(D)_1 = \max_{g \in G(k_\infty)} \min_{x \in \Lambda_1 - \{0\}} \frac{H_g^\infty(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}} = \max_{g \in G(k_\infty)} \min_{\delta \in GL_n(\mathfrak{O})} \frac{H_g^\infty(\delta e_1)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}}.$$

Indeed, for $x = e_1 x_1 + \cdots + e_n x_n \in \Lambda_1 - \{0\}$, there exists $y \in \mathfrak{O}$ such that $\mathfrak{O}x_1 + \cdots + \mathfrak{O}x_n = \mathfrak{O}y$ because of $h_D = 1$. Each x_i is written as $z_i y$, $z_i \in \mathfrak{O}$. Then $z = e_1 z_1 + \cdots + e_n z_n$ is primitive in the sense that $\mathfrak{O}z_1 + \cdots + \mathfrak{O}z_n = \mathfrak{O}$. From the primitivity and [2, Théorème 1], it follows that the set $\{a \in D \colon za \in \Lambda_1\}$ is equal to $\mathfrak{O}$ and $z\mathfrak{O}$ is a direct summand of Λ_1. This implies that there exits $\delta \in GL_n(\mathfrak{O})$ such that $\delta e_1 = z$. Then, by Lemma 2.1 and the product formula,

$$H_g^\infty(x) H_{g_1}^f(x) = |\mathrm{Nr}_{D/k}(y)|_{\mathbb{A}}^{1/2} H_g^\infty(\delta e_1) H_{g_1}^f(\delta e_1) = H_g^\infty(\delta e_1) H_{g_1}^f(\delta e_1).$$

From $GL_n(\mathfrak{O}) \subset K_f$, it follows that $H_{g_1}^f(\delta e_1) = H_{1_n}^f(e_1) = 1$.

In the following, we show that each $\gamma_n(D)_i$, $1 \le i \le h_D^{(n)}$, is independent of the choice of a maximal order of D and a family of isomorphisms $\iota_v \colon D_v \to M_2(k_v)$ ($v \in \mathfrak{V}''$) which was fixed in §2 to define local heights H^v. For a given subset U of D and $h = (h_v) \in D_{\mathbb{A}}^\times$, define the subset U^h of D by

$$U^h = \bigcap_{v \in \mathfrak{V}_f} \left(D \cap h_v^{-1} U_v h_v \right),$$

where U_v denotes the closure of U in D_v. We take another maximal order $\mathfrak{O}'$ of D and a family of isomorphisms $\iota'_v \colon D_v \to M_2(k_v)$ ($v \in \mathfrak{V}''$) such that $\iota'_v(\mathfrak{O}'_v) = M_2(\mathfrak{o}_{k_v})$ if $v \in \mathfrak{V}''_f$. By Skolem-Noether's theorem, there exists $h'_v \in D_v^\times$ such that $\iota'_v = \iota_v \circ \mathrm{int}(h'_v)$ for each $v \in \mathfrak{V}''$. Then $(h'_v)^{-1} \mathfrak{O}_v h'_v$ is equal to $\mathfrak{O}'_v$ for $v \in \mathfrak{V}''_f$. Therefore we can take $h = (h_v) \in D_{\mathbb{A}}^\times$ such that $\mathfrak{O}^h = \mathfrak{O}'$ and $h_v = h'_v$ for all $v \in \mathfrak{V}''$. If $\mathfrak{A} \subset \mathfrak{O}$ is a right integral $\mathfrak{O}$-ideal, then $\mathfrak{A}^h$ gives a right integral $\mathfrak{O}'$-ideal. Define $\widehat{h} \in G(\mathbb{A})$ by

$$\widehat{h} = h I_n = \begin{pmatrix} h & & 0 \\ & \ddots & \\ 0 & & h \end{pmatrix}.$$

Then the family

$$\Lambda'_i := \bigcap_{v \in \mathfrak{V}_f} \left(V \cap \widehat{h}_v^{-1} \Lambda_{i,v} h_v \right), \quad 1 \le i \le h_D^{(n)},$$

of $\mathfrak{O}'$-lattices forms a complete system of representatives of $\mathcal{L}_{\mathfrak{O}'}(V)/\cong$. We put $\Lambda = \Lambda_1$ (resp. $\Lambda' = \Lambda'_1$) and denote by $G(\mathbb{A})_\Lambda$ (resp. $G(\mathbb{A})_{\Lambda'}$) the stabilizer of Λ (resp. Λ')

in $G(\mathbb{A})$. It is obvious that $G(\mathbb{A})_{\Lambda'} = \widehat{h}^{-1}G(\mathbb{A})_{\Lambda}\widehat{h}$. If we take $g_i' = \widehat{h}^{-1}g_i\widehat{h} \in G(\mathbb{A}_f)$, then $(g_i')^{-1}\Lambda' = \Lambda_i'$. Furthermore, we define the local height ${}^hH^v$ on V_v for $v \in \mathfrak{V}$ and the global height ${}^hH$ on V as follows:

$$ {}^hH^v(x) := H^v\big(\widehat{h}_v x h_v^{-1}\big), \quad {}^hH(x) := \prod_{v\in\mathfrak{V}} {}^hH^v(x). $$

We show that ${}^hH$ is the height corresponding to $\mathfrak{O}'$. For $v \in \mathfrak{V}''$, put

$$ \epsilon_v := (\iota_v')^{-1}\left(\begin{pmatrix}1&0\\0&0\end{pmatrix}\right), \quad \epsilon_v' := (\iota_v')^{-1}\left(\begin{pmatrix}0&0\\0&1\end{pmatrix}\right), \quad J_v' := (\iota_v')^{-1}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right), $$
$$ f_{2i-1}^{v\prime} := e_i\epsilon_v, \quad f_{2i}^{v\prime} := e_i\epsilon_v' J_v'. $$

Then we have the following relations:

$$ \epsilon_v = h_v^{-1}e_vh_v, \quad \epsilon_v' = h_v^{-1}e_v'h_v, \quad J_v' = h_v^{-1}J_vh_v, \quad f_i^{v\prime} = \widehat{h}_v^{-1}f_i^vh_v, \quad (1 \le i \le 2n). $$

We define the norm $\widehat{F}_v'$ on $(V_v\epsilon_v) \wedge (V_v\epsilon_v)$ for $v \in \mathfrak{V}''$ as in §2 with respect to the k_v-basis $f_1^{v\prime}, \dots, f_{2n}^{v\prime}$ of $V_v\epsilon_v$.

Lemma 4.1. *One has*

$$ {}^hH^v(x) = \begin{cases} H^v(x) & (v \in \mathfrak{V}') \\ \widehat{F}_v'\big(x\epsilon_v \wedge x\epsilon_v' J_v'\big)^{1/2} & (v \in \mathfrak{V}'') \end{cases} $$

for $x \in V_v$.

Proof. If $v \in \mathfrak{V}'$, this follows from

$$ {}^hH^v(e_1x_1 + \cdots + e_nx_n) = H^v\big(e_1h_vx_1h_v^{-1} + \cdots + e_nh_vx_nh_v^{-1}\big), \quad (x_1, \dots, x_n \in D_v) $$

and $\left|\mathrm{Nr}_{D/k}\big(h_vx_ih_v^{-1}\big)\right|_{k_v} = |\mathrm{Nr}_{D/k}(x_i)|_{k_v}$. Thus we let $v \in \mathfrak{V}_f''$. Note that

$$ \widehat{F}_v\left(\widehat{h}_v\left(\sum_{1\le i<j\le 2n}(f_i^{v\prime} \wedge f_j^{v\prime})\lambda_{ij}\right)h_v^{-1}\right) = \widehat{F}_v\left(\sum_{1\le i<j\le 2n}\left(\big(\widehat{h}_v f_i^{v\prime}h_v^{-1}\big) \wedge \big(\widehat{h}_v f_j^{v\prime}h_v^{-1}\big)\right)\lambda_{ij}\right) $$
$$ = \widehat{F}_v\left(\sum_{1\le i<j\le 2n}(f_i^v \wedge f_j^v)\lambda_{ij}\right) = \sup_{1\le i<j\le 2n}(|\lambda_{ij}|_{k_v}). $$

This means that

$$ \widehat{F}_v'(x \wedge y) = \widehat{F}_v\big(\widehat{h}_v(x \wedge y)h_v^{-1}\big) $$

for any $x \wedge y \in (V_v\epsilon_v) \wedge (V_v\epsilon_v)$. Then

$$\begin{aligned} {}^hH^v(x) &= H^v\big(\widehat{h}_v x h_v^{-1}\big) = \widehat{F}_v\Big(\big(\widehat{h}_v x h_v^{-1} e_v\big) \wedge \big(\widehat{h}_v x h_v^{-1} e_v' J_v\big)\Big)^{1/2} \\ &= \widehat{F}_v\Big(\big(\widehat{h}_v x \epsilon_v h_v^{-1}\big) \wedge \big(\widehat{h}_v x \epsilon_v' J_v' h_v^{-1}\big)\Big)^{1/2} = \widehat{F}_v\Big(\widehat{h}_v\big(x\epsilon_v \wedge x\epsilon_v' J_v'\big)h_v^{-1}\Big)^{1/2} \\ &= \widehat{F}_v'\big(x\epsilon_v \wedge x\epsilon_v' J_v'\big)^{1/2}. \end{aligned}$$

□

This lemma shows that ${}^hH$ is the height with respect to $\mathfrak{O}'$. For $g = (g_v) \in G(\mathbb{A})$, define the twisted height ${}^hH_g$ on V by

$$ {}^hH_g(x) = \prod_{v \in \mathfrak{V}} {}^hH^v(g_v x_v). $$

We set

$$\begin{aligned} \gamma_n(D)' &= \max_{g \in G(\mathbb{A})} \min_{x \in V - \{0\}} \frac{{}^hH_g(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}}, \\ \gamma_n(D)_i' &= \max_{g \in G(\mathbb{A})_{\Lambda'} g_i' G(k)} \min_{x \in V - \{0\}} \frac{{}^hH_g(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{\mathbb{A}}} \\ &= \max_{g \in G(k_\infty)} \min_{x \in \Lambda_i' - \{0\}} \frac{{}^hH_g^\infty(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}} \cdot \frac{{}^hH_{g_i'}^f(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i')|_{\mathbb{A}_f}} \end{aligned}$$

for $1 \le i \le h_D^{(n)}$. Let h_∞ (resp. $\widehat{h}_\infty$) be the infinite component of h (resp. $\widehat{h}$) and h_f (resp. $\widehat{h}_f$) be the finite component of h (resp. $\widehat{h}$). Since $g_i' = \widehat{h}^{-1} g_i \widehat{h} = \widehat{h}_f^{-1} g_i \widehat{h}_f$, we have

$$\begin{aligned} \gamma_n(D)_i' &= \max_{g \in G(k_\infty)} \min_{x \in \Lambda_i' - \{0\}} \frac{\prod_{v \in \mathfrak{V}_\infty} H^v\big(\widehat{h}_v g_v x h_v^{-1}\big)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}} \cdot \frac{\prod_{v \in \mathfrak{V}_f} H^v\big(\widehat{h}_v \widehat{h}_v^{-1} g_{i,v} \widehat{h}_v x h_v^{-1}\big)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f}} \\ &= \max_{g \in G(k_\infty)} \min_{x \in \Lambda_i' - \{0\}} \frac{|\mathrm{Nr}_{D/k}(h_\infty)|_{k_\infty}^{-n} \prod_{v \in \mathfrak{V}_\infty} H^v(g_v x)^{2n}}{\big|\mathrm{Nr}_{M_n(D)/k}\big(\widehat{h}_\infty^{-1} g\big)\big|_{k_\infty}} \cdot \frac{\prod_{v \in \mathfrak{V}_f} H^v\big(g_{i,v} \widehat{h}_v x h_v^{-1}\big)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f}} \\ &= \max_{g \in G(k_\infty)} \min_{x \in \Lambda_i' - \{0\}} \frac{H_g^\infty(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}} \cdot \frac{H_{g_i}^f\big(\widehat{h}_f x h_f^{-1}\big)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f}}, \end{aligned}$$

where we write $H_{g_i}^f\big(\widehat{h}_f x h_f^{-1}\big)$ for $\prod_{v \in \mathfrak{V}_f} H^v\big(g_{i,v} \widehat{h}_v x h_v^{-1}\big)$.

Proposition 4.2. *$\gamma_n(D)_i' = \gamma_n(D)_i$ for $i = 1, \dots, h_D^{(n)}$.*

Proof. We prove $\gamma_n(D)_i \le \gamma_n(D)_i'$. Fix a $g \in G(k_\infty)$ and take an $x_0 \in \Lambda_i' - \{0\}$ such that

$$ H_g^\infty(x_0) H_{g_i}^f\big(\widehat{h}_f x_0 h_f^{-1}\big) = \min_{x \in \Lambda_i' - \{0\}} H_g^\infty(x) H_{g_i}^f\big(\widehat{h}_f x h_f^{-1}\big). $$

From $\widehat{h}_v \Lambda'_{i,v} h_v^{-1} = \Lambda_{i,v}$ for all $v \in \mathfrak{V}_f$, it follows $\widehat{h}_f x_0 h_f^{-1} \in \prod_{v \in \mathfrak{V}_f} \Lambda_{i,v}$. We put $c_v = H^v\big(g_{i,v}\widehat{h}_v x_0 h_v^{-1}\big)$ for $v \in \mathfrak{V}_f$. Clearly, $c_v > 0$ and, for almost all v, $c_v = 1$. We define the open subset U_v in $\Lambda_{i,v}$ by

$$U_v = \{y_v \in \Lambda_{i,v} \colon H^v(g_{i,v} y_v) \leq c_v\}.$$

Since $U_v = \Lambda_{i,v}$ for almost all v, the product $U = \prod_{v \in \mathfrak{V}_f} U_v$ gives an open subset of $\prod_{v \in \mathfrak{V}_f} \Lambda_{i,v}$. From the density of Λ_i in $\prod_{v \in \mathfrak{V}_f} \Lambda_{i,v}$, it follows $\Lambda_i \cap (U - \{0\}) \neq \emptyset$, so that we can take a nonzero $y_0 \in \Lambda_i \cap U$. By the definition of U, y_0 satisfies

$$H^f_{g_i}(y_0) \leq H^f_{g_i}\big(\widehat{h}_f x_0 h_f^{-1}\big).$$

Since the group $SL_n(D) = \{g \in GL_n(D) \colon \mathrm{Nr}_{M_n(D)/k}(g) = 1\}$ acts on $V - \{0\}$ transitively, there exists $\xi \in SL_n(D)$ such that $\xi y_0 = x_0$. Let ξ_∞ be the projection of ξ to $G(k_\infty)$. Then

$$H^\infty_{g\xi_\infty}(y_0) H^f_{g_i}(y_0) \leq H^\infty_g(x_0) H^f_{g_i}\big(\widehat{h}_f x_0 h_f^{-1}\big) = \min_{x \in \Lambda'_i - \{0\}} H^\infty_g(x) H^f_{g_i}\big(\widehat{h}_f x h_f^{-1}\big).$$

Therefore

$$\min_{y \in \Lambda_i - \{0\}} H^\infty_{g\xi_\infty}(y) H^f_{g_i}(y) \leq \min_{x \in \Lambda'_i - \{0\}} H^\infty_g(x) H^f_{g_i}\big(\widehat{h}_f x h_f^{-1}\big).$$

From $|\mathrm{Nr}_{M_n(D)/k}(g\xi_\infty)|_{k_\infty} = |\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}$, it follows that

$$\begin{aligned} &\min_{y \in \Lambda_i - \{0\}} \frac{H^\infty_{g\xi_\infty}(y)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g\xi_\infty)|_{k_\infty}} \cdot \frac{H^f_{g_i}(y)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f}} \\ &\leq \min_{x \in \Lambda'_i - \{0\}} \frac{H^\infty_g(x)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty}} \cdot \frac{H^f_{g_i}\big(\widehat{h}_f x h_f^{-1}\big)^{2n}}{|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f}}. \end{aligned}$$

Taking the maximums of both sides over $g \in G(k_\infty)$, we obtain $\gamma_n(D)_i \leq \gamma_n(D)'_i$. If we change the roles of $\mathfrak{O}$ and $\mathfrak{O}'$ each other, then we get $\gamma_n(D)_i = \gamma_n(D)'_i$. □

Now we define the notion of quaternionic Humbert forms over a quaternion skew field. To that end, we introduce some notation. We denote by $\mathfrak{V}_{\infty,1}$ (resp. $\mathfrak{V}_{\infty,2}$), the set of real (resp. complex) places of k, and by r_1, r_2 the corresponding cardinalities, so that $r_1 + 2r_2 = [k : \mathbb{Q}]$. The ramification index at $v \in \mathfrak{V}_\infty$ over $\mathbb{Q}$ is denoted by d_v, and is 1 or 2 according to $k_v \simeq \mathbb{R}$ or $\mathbb{C}$. The set of real places of k which ramify in D (resp. which split), is denoted by $\mathfrak{V}'_{\infty,1}$ (resp. $\mathfrak{V}''_{\infty,1}$), with cardinality r'_1 and r''_1 respectively. Finally, the index of D_v is denoted by m_v ($m_v = 2$ if $v \in \mathfrak{V}'_{\infty,1}$, 1 if $v \in \mathfrak{V}''_{\infty,1} \cup \mathfrak{V}_{\infty,2}$). We fix, at any $v \in \mathfrak{V}_\infty$, an isomorphism ι_v from D_v onto $\mathbb{H}$ or $M_2(k_v)$, depending on whether v is ramified or not.

DEFINITION 4.3. An n-ary quaternionic Humbert form over D is a (r_1+r_2)-tuple $S = (S_v)_{v\in\mathfrak{V}_\infty}$, where:

– if $v \in \mathfrak{V}'_{\infty,1}$, S_v is an n-ary positive definite Hermitian form on $D_v^n \simeq \mathbb{H}^n$.

– if $v \in \mathfrak{V}''_{\infty,1}$ (resp. $v \in \mathfrak{V}_{\infty,2}$), S_v is a $2n$-ary positive definite symmetric (resp. Hermitian) form on $\mathbb{R}^{2n}$ (resp. $\mathbb{C}^{2n}$).

We denote by $P_{n,D}$ the set of n-ary quaternionic Humbert forms over D. One can view $P_{n,D}$ as a cone in the space $\mathcal{H} = \prod_{v\in\mathfrak{V}_\infty} \mathcal{H}_{n,v}$, where $\mathcal{H}_{n,v}$ stands for the space $\mathcal{H}_n(\mathbb{H})$ of n-ary Hermitian forms over $\mathbb{H}$ if $v \in \mathfrak{V}'_{\infty,1}$, the space $\mathcal{S}_{2n}(\mathbb{R})$ of $2n$-ary symmetric forms over $\mathbb{R}$ if $v \in \mathfrak{V}''_{\infty,1}$ and the space $\mathcal{H}_{2n}(\mathbb{C})$ of $2n$-ary Hermitian forms over $\mathbb{C}$ if $v \in \mathfrak{V}_{\infty,2}$. The group $G(k_\infty)$ acts on $P_{n,D}$ by $S \cdot g = S[g] = gS\overline{g}'$. In particular, we get a natural diagonal action of $k_\infty^\times$ on $P_{n,D}$:

$$\lambda \cdot S = \left(\lambda_v \overline{\lambda_v} S_v\right), \quad \text{for} \quad \lambda = (\lambda_v)_{v\in\mathfrak{V}_\infty} \in k_\infty^\times \quad \text{and} \quad S = (S_v)_{v\in\mathfrak{V}_\infty} \in P_{n,D}.$$

We want to endow $P_{n,D}$ with a structure of Riemannian symmetric space. To that end, we associate to any $S \in P_{n,D}$ a scalar product $\langle\ ,\ \rangle_S$ on $\mathcal{H}$, defined by:

$$\langle X, Y\rangle_S = \sum_{v\in\mathfrak{V}_\infty} \frac{d_v}{m_v} \operatorname{Tr}_v\left(S_v^{-1} X_v S_v^{-1} Y_v\right),$$

in which Tr_v stands for the reduced trace of $M_n(D_v)/k_v$ (more precisely, if v is split, one identifies $M_n(M_2(k_v))$ with $M_{2n}(k_v)$ and Tr_v is just the ordinary trace, while for v ramified, $M_n(D_v) = M_n(\mathbb{H})$ and $\operatorname{Tr}_v = \operatorname{Tr}_{\mathbb{H}/\mathbb{R}} \circ \operatorname{Tr}$, i.e. $\operatorname{Tr}_v A = \operatorname{Tr} A + \operatorname{Tr} \overline{A}$).

This scalar product is $G(k_\infty)$ invariant, in the sense that

$$\langle X \cdot g, Y \cdot g\rangle_{S\cdot g} = \langle X, Y\rangle_S, \quad S \in P_{n,D}, g \in G(k_\infty), \quad (X, Y) \in \mathcal{H}^2. \tag{2}$$

We define the determinant of a form $S \in P_{n,D}$ as follows:

(i) if $v \in \mathfrak{V}'_\infty$, then $S_v = g_v\overline{g_v}'$ for a suitable $g_v \in GL_n(\mathbb{H})$, and we set $\det S_v = \operatorname{Nr}_{M_m(\mathbb{H})/\mathbb{R}}(g_v)$. Alternatively, one can also write S_v as $S_v = h_v \operatorname{diag}(a_1, \ldots, a_n)\overline{h_v}'$, where h_v is an upper triangular unipotent matrix and $a_i > 0$, and set $\det S_v = \prod_{i=1}^n a_i$.

(ii) if $v \in \mathfrak{V}''_\infty$, then $\det S_v$ is the usual determinant of S_v. The determinant of S is then

$$\operatorname{Det} S = \prod_{v\in\mathfrak{V}_\infty} (\det S_v)^{m_v d_v/2}.$$

For any vector u in D^n, we denote, for simplicity, by u_v its image in $\iota_v(D^n)$. If v is ramified, then $S_v[u_v] = u_v S_v \overline{u_v}'$ just stands for the value of the positive definite Hermitian form S_v at u_v. If v is split, we identify $\iota_v(D^n) = M_2(k_v)^n$ with $M_{2n,2}(k_v)$. We choose this identification, rather than $M_{2,2n}(k_v)$, since then the right action of $M_2(k_v)$ on $M_2(k_v)^n$ coincides with the natural right action of $M_2(k_v)$ on $M_{2n,2}(k_v)$, whereas there is no *natural* right action of $M_2(k_v)$ on $M_{2,2n}(k_v)$. Then the image u_v of a vector u in

D^n may be identified with a matrix U_v in $M_{2n,2}(k_v)$, and the value of S_v at u_v is then defined as

$$S_v[u_v] = \det S_v[U_v] = \det U_v' S_v \overline{U}_v$$

(note that the transpose is on the left-hand side, because of the identification $M_2(k_v)^n \simeq M_{2n,2}(k_v)$). Finally, for $S \in P_{n,D}$ and $u \in D^n$, we define the value of S at u as:

$$S[u] = \prod_{v \in \mathfrak{V}_\infty} S_v[u_v]^{m_v d_v/2}.$$

The verification of the following lemma is straightforward.

Lemma 4.4. *For any $\lambda \in k_\infty^\times$, $S \in P_{n,D}$ and $u \in D^n$, one has*:
(i) $\mathrm{Det}(\lambda \cdot S) = |\lambda|_{\mathbb{A}}^{2n} \mathrm{Det}\, S$.
(ii) $(\lambda \cdot S)[u] = |\lambda|_{\mathbb{A}}^{2} S[u]$.
For any $\alpha \in D$, $S \in P_{n,D}$ and $u \in D^n$, one has:
(iii) $S[\alpha u] = \mathrm{Nr}_{D/\mathbb{Q}}(\alpha)^2 S[u]$.

We want to express the constants $\gamma_n(D)$ and $\gamma_n(D)_i$ in terms of quaternionic Humbert forms. In the following, we often identify the vector space V with D^n. To $u = e_1u_1 + \cdots + e_{n-1}u_{n-1} + e_nu_n \in \Lambda_i$, one associates a left $\mathfrak{O}$-ideal $\mathfrak{A}_u$ defined as

$$\mathfrak{A}_u = \mathfrak{O}u_1 + \cdots + \mathfrak{O}u_{n-1} + \mathfrak{A}_i^{-1}u_n.$$

This is an integral left ideal, since $u_j \in \mathfrak{O}$ for $1 \le j \le n-1$, and $u_n \in \mathfrak{A}_i$. A vector $u \in \Lambda_i$ is said to be *primitive* if its associated left $\mathfrak{O}$-ideal $\mathfrak{A}_u$ satisfies the minimality condition, i.e., $\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}) \ge \mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)$ for any integral left $\mathfrak{O}$-ideal $\mathfrak{A}$ in the same class as $\mathfrak{A}_u$.

The minimum of a form $S \in P_{n,D}$ with respect to Λ_i is defined as:

$$m_i(S) = \min_{0 \neq u \in \Lambda_i} \frac{S[u]}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)},$$

and its Hermite invariant with respect to Λ_i as

$$\mu_i(S) = \frac{1}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_i)} \frac{m_i(S)^n}{\mathrm{Det}\, S}.$$

From the previous lemma, we see that the μ_i are invariant under the natural action of $k_\infty^\times$ on $P_{n,D}$. This allows us to restrict μ_i to the set $P_{n,D}^1$ of quaternionic Humbert forms $S = (S_v)$ satisfying $\det S_v = 1$ for any $v \in \mathfrak{V}_\infty$. The μ_i are related to the constants $\gamma_n(D)_i$ through the following proposition:

Proposition 4.5. *For $i = 1, \ldots, h_D^{(n)}$*, $\max_{S \in P_{n,D}} \mu_i(S) = \max_{S \in P_{n,D}^1} \mu_i(S) = \gamma_n(D)_i$.

Proof. First we note that the group $G(k_\infty)$ acts transitively on $P_{n,D}$. Then, if $S = I[g] = g\overline{g}'$, an elementary calculation shows that, for $u \in \Lambda_i$

$$\begin{aligned} &H_g^\infty(u)^2 = S[u] \\ &H_{g_i}^f(u)^2 = \mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)^{-1} \\ &|\mathrm{Nr}_{M_n(D)/k}(g)|_{k_\infty} = \mathrm{Det}\, S \\ &|\mathrm{Nr}_{M_n(D)/k}(g_i)|_{\mathbb{A}_f} = \mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_i), \end{aligned}$$

whence the conclusion. □

REMARK. If the class number of D is 1, which will be the case in the examples below, then we denote m_1, μ_1 and $\gamma_n(D)_1$ by m, μ and $\gamma_n(D)$ respectively.

EXAMPLE. Here we assume $k = \mathbb{Q}$ and $D_\infty \simeq \mathbb{H}$. Let $\mathfrak{O}$ be a maximal order of D. We define

$$\delta_{\mathfrak{O}} = \max_{x \in D} \min_{y \in \mathfrak{O}} \mathrm{Nr}_{D/\mathbb{Q}}(x - y),$$

and we say that $\mathfrak{O}$ is (right)-euclidean if $\delta_{\mathfrak{O}} < 1$. If this is the case, then the class number of $\mathfrak{O}$ is one, and the type number of D as well. Consequently, the value of $\delta_{\mathfrak{O}}$ is independent of $\mathfrak{O}$, and we denote it by δ_D. For such a quaternion skew field D, the methods of Newman [7], Chapter 11, carries over and give the exact value of $\gamma_2(D)$ as well as an upper bound for $\gamma_n(D)$. According to [9], p.156, there are exactly 3 such euclidean quaternion fields over $\mathbb{Q}$, namely, with the standard notation, $D_2 = (-1,-1)_{\mathbb{Q}}$, $D_3 = (-1,-3)_{\mathbb{Q}}$ and $D_5 = (-2,-5)_{\mathbb{Q}}$, (recall that $(a,b)_k$ stands for the quaternion algebra over k generated by i and j with $i^2 = a$, $j^2 = b$ and $ij = -ji$). A maximal order of D_m, $m = 2, 3, 5$, is described as follows:

- $\underline{m = 2}$: $\mathfrak{O} = \mathbb{Z}[1, i, j, (1 + i + j + ij)/2]$.
- $\underline{m = 3}$: $\mathfrak{O} = \mathbb{Z}[1, i, (i + j)/2, (1 + ij)/2]$.
- $\underline{m = 5}$: $\mathfrak{O} = \mathbb{Z}[1, (1 + i + j)/2, j, (2 + i + ij)/4]$.

Their norm constants δ_{D_m} are given by

$$\delta_{D_m} = \frac{m-1}{m}, \quad m = 2, 3, 5.$$

From this we deduce

Proposition 4.6. *For $m = 2, 3, 5$, one has*
(i) $\gamma_n(D_m) \le m^{n(n-1)/2}$,
(ii) $\gamma_2(D_m) = m$.

Proof. (i) The proof follows the same lines as that of Hermite inequality, as given for instance in [4, Theorem 2.2.1. p.39].

(ii) From the first part of the proposition, we know that $\gamma_2(D_m) \le m$ for $m = 2, 3, 5$, so we just have to find, in each case, a binary quaternionic Humbert form, i.e., a binary Hermitian form S over $\mathbb{H}$, achieving this bound.

- $\underline{m = 2}$: We claim that the form $S_2 = \begin{pmatrix} 1 & (1+i)/2 \\ (1-i)/2 & 1 \end{pmatrix}$ satisfies $\mu(S_2) = 2$. Its determinant is $1/2$, so it remains to check that its minimum $m(S)$ is 1. For any $u = (x, y) \in \mathfrak{O}^2$, one has $S[u] = x\overline{x} + y\overline{y} + \mathrm{Tr}(((1+i)/2)\overline{x}y)$, and $S[u] \in \mathbb{Z}$, since $(1+i)/2$ belongs to the codifferent of $\mathfrak{O}$. Consequently, one has $S[u] \ge 1$ for any $0 \ne u = (x, y) \in \mathfrak{O}^2$, with equality for instance for $u = (1, 0)$.
- $\underline{m = 3}$: One shows similarly that the form $S_3 = \begin{pmatrix} 1 & (1+i)/j \\ (-1+i)/j & 1 \end{pmatrix}$ satisfies $\mu(S_3) = 3$.
- $\underline{m = 5}$: Finally, the form $S_5 = \begin{pmatrix} 1 & (2/5)j \\ -(2/5)j & 1 \end{pmatrix}$ satisfies $\mu(S_5) = 5$. □

5. Minimal vectors

To any quaternionic Humbert form S, we want to attach a set of *minimal vectors* with respect to μ_i (or m_i). Namely, we want to consider the set of nonzero vectors $u \in \Lambda_i$ such that $S[u]/\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)$ is minimal. First we take a complete system $\{\mathfrak{B}_1, \dots, \mathfrak{B}_{h_D}\}$ of representatives of left $\mathfrak{O}$-ideal classes of D as follows:
(B1): $\mathfrak{B}_i \subset \mathfrak{O}$ and $[\mathfrak{B}_i] = \left[\mathfrak{A}_i^{-1}\right]$, where $\{\mathfrak{A}_1, \dots, \mathfrak{A}_{h_D}\}$ is the set of representatives of right ideal classes $\mathcal{I}_D/\cong$ we fixed in §4.
(B2): If $\mathfrak{B} \subset \mathfrak{O}$ is a left $\mathfrak{O}$-ideal and $[\mathfrak{B}] = [\mathfrak{B}_i]$, then $\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{B}) \ge \mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{B}_i)$.
Then one can write $m_i(S)$ as

$$m_i(S) = \min_{1 \le j \le h_D} m_{i,j}(S),$$

where

$$m_{i,j}(S) = \min_{0 \ne u \in \Lambda_i, [\mathfrak{A}_u] = [\mathfrak{B}_j]} \frac{S[u]}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)},$$

so that we can split the minimal vectors according to the class of their associated ideal. So doing, we get infinitely many minimal vectors, since for any $u \in \Lambda_i$ and any $\lambda \in k^\times$, one has

(3) $$\frac{S[u]}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)} = \frac{S[\lambda u]}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_{\lambda u})}.$$

This is overcome by the following lemma.

Lemma 5.1. *For $1 \le i \le h_D^{(n)}$ and $1 \le j \le h_D$, one has*:
(i) $m_{i,j}(S) = (1/\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{B}_j)) \min_{0 \ne u \in \Lambda_i, \mathfrak{A}_u = \mathfrak{B}_j} S[u] = (1/\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{B}_j)) \min_{0 \ne u \in \Lambda_i, [\mathfrak{A}_u] = [\mathfrak{B}_j]} S[u]$.
(ii) *There are finitely many nonzero vectors u in Λ_i, up to multiplication by units, such that $\mathfrak{A}_u = \mathfrak{B}_j$ and $S[u]/\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u) = m_{i,j}(S)$.*

Proof. The first assertion is clear, because of (3) and the minimality conditions on $\mathfrak{B}_j$. As for the second one it will follow from classical properties of height functions. For $1 \leq j \leq h_D$, let $\Lambda'_{i,j}$ stand for the set of primitive vectors $u \in \Lambda_i$ with $\mathfrak{A}_u = \mathfrak{B}_j$. With the notation of §3, we have injections $\Lambda'_{i,j}/\mathfrak{O}^\times \hookrightarrow \mathbf{P}_D V \hookrightarrow \mathbf{P}\widehat{V}_L$. If ${}^L H$ denotes the height function on $\mathbf{P}\widehat{V}_L$ defined in §3, we know, by standard properties of height functions on projective spaces that, for any $T > 0$, the set

$$\left\{x \in \mathbf{P}\widehat{V}_L : {}^L H(x) \leq T\right\}$$

is finite. Let $g \in G(k_\infty)$ be such that $S = g\overline{g}'$. Because of the relation between ${}^L H$ and H_{gg_i} (Lemma 2.2 and Lemma 3.4), we can conclude that the set

$$\left\{u \in \Lambda'_{i,j}/\mathfrak{O}^\times : H_{gg_i}(u) \leq T\right\}$$

is finite. But for $u \in \Lambda'_{i,j}$, the finite part $H^f_{g_i}(u)$ of $H_{gg_i}(u)$ is constant, so that the set

$$\left\{u \in \Lambda'_{i,j}/\mathfrak{O}^\times : S[u]^{1/2} = H^\infty_g(u) \leq T\right\}$$

is itself finite, which gives the desired result. □

In other words, one can restrict minimal vectors to *primitive* minimal vectors, and the set of primitive minimal vectors up to multiplication by units is finite. From now on, we fix a finite set $M_i(S)$ of representatives, modulo units, of primitive minimal vectors.

6. Voronoï theory

We prove in this section, using a general method developed by C. Bavard [1], that Voronoï theory holds for the quaternionic Hermite invariants μ_i just defined. According to the classical terminology, we call *μ_i-extreme* a form S that achieves a local maximum of μ_i, viewed as a function on $P_{n,D}$, or $P^1_{n,D}$. We want to characterize μ_i-extreme forms via suitable notions of *perfection* and *eutaxy*. To that end we need to rephrase the definitions of the μ_i in terms of length functions on a certain variety, check that the so-called ‘condition (C)’ (see [1], 2.2) is satisfied, and then apply Lemma 2.2 of [1] to conclude. As mentioned before, we can restrict μ_i to the subvariety $P^1_{n,D}$.

The tangent space $T_S P^1_{n,D}$ of $P^1_{n,D}$ at S is identified with

$$\left\{ M = (M_v)_{v\in\mathfrak{V}_\infty} \in \prod_{v\in\mathfrak{V}_\infty} \mathcal{H}_{n,v} : \ \mathrm{Tr}_v\left(S_v^{-1} M_v\right) = 0 \quad \text{for all} \quad v \in \mathfrak{V}_\infty \right\},$$

and therefore has dimension $r_1' n(2n-1) + r_1'' n(2n+1) + 4r_2 n^2 - (r_1 + r_2)$. It is endowed with the scalar product $\langle\ ,\ \rangle_S$ defined above.

To $u \in D^n$, we associate a length function l_u on $P^1_{n,D}$ defined by

$$l_u(S) = \frac{S[u]^2}{\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)^2},$$

so that

$$\mu_i(S) = \min_{0 \neq u \in \Lambda_i} l_u(S)^{1/2}.$$

An easy computation gives the gradient $X_S(u)$ at S of l_u, with respect to $\langle\ ,\ \rangle_S$, namely

$$X_S(u) = l_u(S)\left(S_v x_S(u)_v S_v - \frac{1}{n} S_v\right)_{v \in \mathfrak{V}_\infty},$$

where

$$x_S(u)_v = \iota_v\big(\overline{u}'_v S_v[u_v]^{-1} u_v\big) = \begin{cases} \dfrac{\overline{u}'_v u_v}{S_v[u_v]} & (v \in \mathfrak{V}'_{\infty,1}), \\ \overline{U}'_v\big(U_v S_v \overline{U}'_v\big)^{-1} U_v & (v \in \mathfrak{V}''_{\infty,1} \cup \mathfrak{V}_{\infty,2}). \end{cases}$$

We set $x_S(u) = (x_S(u)_v)_{v \in \mathfrak{V}_\infty}$. Note that $x_S(u)$, and $X_S(u)$ as well, depends on u only modulo units, i.e., $x_S(\epsilon u) = x_S(u)$ for $\epsilon \in \mathfrak{O}^\times$.

From this, we can deduce a definition for *μ_i-perfection* and *μ_i-eutaxy*. According to the general theory developed in [1], it is natural indeed, to say that a form S is *μ_i-perfect* if the gradients $X_S(u)$, $u \in M_i(S)$, generate the tangent space $T_S P^1_{n,D}$, and *μ_i-eutactic* if 0 belongs to the open convex hull of these gradients. From the above remark, this does not depend on the choice of a set $M_i(S)$ of representatives of minimal vectors modulo units. The following proposition gives a simpler formulation of these properties.

Proposition 6.1. (i) *A quaternionic Humbert form S is μ_i-perfect if and only if*

$$\mathrm{Conv}\{x_S(u)_v, u \in M_i(S)\} = \{M = (M_v)_{v \in \mathfrak{V}_\infty} \in \mathcal{H} \colon \mathrm{Tr}_v(S_v M_v) = 2 \quad (\forall v \in \mathfrak{V}_\infty)\},$$

where Conv *stands for the convex hull. In other words, S is μ_i-perfect if and only if*

$$\dim \mathrm{Span}\{x_S(u), u \in M_i(S)\} = r'_1 n(2n-1) + r''_1 n(2n+1) + 4r_2 n^2 - (r_1 + r_2) + 1.$$

(ii) *A quaternionic Humbert form S is μ_i-eutactic if and only if the form $S^{-1} = (S_v^{-1})_{v \in \mathfrak{V}_\infty}$ belongs to the open convex hull of the vectors $(x_S(u)_v)_{v \in \mathfrak{V}_\infty}$, $u \in M_i(S)$.*

Proof. (i) Let $p_{S^\perp}$ stand for the orthogonal projection on the orthogonal complement of S (orthogonality is with respect to $\langle\ ,\ \rangle_S$). One has $p_{S^\perp}(S x_S(u) S) = (1/l_u(S)) \times X_S(u)$, whence (i). Assertion (ii) is straightforward. □

Lemma 6.2. *The length functions l_u satisfy condition* (C).

Applying Lemma 2.2 of [1] we obtain the foreseen characterization of extreme forms:

Theorem 6.3. *A quaternionic Humbert form $S = (S_v)$ is μ_i-extreme if and only if it is μ_i-perfect and μ_i-eutactic.*

Proof of the Lemma. The proof is absolutely similar to the proof of Proposition 2.8 in [1]. In our context, condition (C) means that: for any $S \in P^1_{n,D}$, and any finite set M of vectors in $\Lambda_i \setminus \{0\}$, if there exists a nonzero vector X in $T_S P^1_{n,D}$ which is orthogonal to the $X_S(u)$, $u \in M$, then there exists a $\mathcal{C}^1$ curve $c\colon [0,\epsilon[\;\to\; P^1_{n,D}$ such that
(C1): $c(0) = S$, $c'(0) = X$.
(C2): $\forall u \in M$, $\forall t \in [0,\epsilon[$, $l_u(c(t)) > l_u(S)$.

From the $SL_n(D \otimes_k k_\infty)$-invariance of $\langle\ ,\ \rangle_S$, it is enough to check condition (C) at $S = I$. In that case, we denote the scalar product $\langle\ ,\ \rangle_I$ simply by $\langle\ ,\ \rangle$. The condition that $X = (X_v)_{v \in \mathfrak{V}_\infty}$ belongs to $T_I P^1_{n,D}$ reads

$$\forall v \in \mathfrak{V}_\infty, \quad \operatorname{Tr}_v X_v = 0,$$

and the orthogonality condition is equivalent to

$$\forall u \in M, \quad \langle x_I(u), X\rangle = 0.$$

We want to find $Y \in T_I P^1_{n,D}$ such that the curve $c(t) = \exp(tX + (t^2/2)Y)$ satisfies conditions (C1) and (C2) above (the exponential is to be understood componentwise, namely $c(t) = (\exp(tX_v + (t^2/2)Y_v))_{v \in \mathfrak{V}_\infty}$). Setting $f_u(t) = l_u(c(t))$ one has

$$f_u'(0) = \langle x_I(u), X\rangle = 0$$

and

$$f_u''(0) = \langle x_I(u), Y\rangle + \langle x_I(u), X^2\rangle - \langle x_I(u)X, x_I(u)X\rangle.$$

As in the proof of Proposition 2.8 in [1], it's easy to see that $\langle x_I(u), X^2\rangle - \langle x_I(u)X, x_I(u)X\rangle$ is positive, unless

$$x_I(u)X = X x_I(u) \tag{4}$$

$$\text{i.e.} \quad \forall v \in \mathfrak{V}_\infty,\ \ x_I(u)_v X_v = X_v x_I(u)_v. \tag{5}$$

If this commutation relation is not satisfied, then, for small enough $Y \in T_I P^1_{n,D}$, we can conclude that the second derivative $f_u''(0)$ is positive, whence $f_u(t) > f_u(0)$, for small enough t. For those u satisfying (4) to the contrary, one has

$$f_u^{(3)}(0) = 0$$

and

$$f_u^{(4)}(0) = 3\big(\langle x_I(u), Y\rangle^2 + \langle x_I(u), Y^2\rangle - \langle x_I(u)Y, x_I(u)Y\rangle\big).$$

Arguing as in the proof of Proposition 2.8 of [1], one shows that there exists $Y \in T_I P^1_{n,D}$ such that $\langle x_I(u), Y^2\rangle - \langle x_I(u)Y, x_I(u)Y\rangle > 0$, whence $f_u^{(4)}(0) > 0$. Moreover, this Y can be chosen arbitrarily small so that, again, for small enough t, $f_u(t) > f_u(0)$. □

Proposition 6.4. *Any μ_i-perfect form $S \in P^1_{n,D}$ is algebraic, i.e. the entries of each S_v, $v \in \mathfrak{V}_\infty$, belong to $\overline{\mathbb{Q}}$.*

Proof. Let S be a perfect form. Let us consider the algebraic variety $\mathcal{V}(S) = \{T \in \mathcal{H} \colon \forall u \in M_i(S),\ T[u] = 1\}$. This is an algebraic subvariety of $\mathcal{H}$, defined over $\mathbb{Q}$. The μ_i-perfect forms belonging to $\mathcal{V}(S)$ are isolated real points ot this variety, thus they are finitely many, and they are defined over $\overline{\mathbb{Q}}$. □

Corollary 6.5. *For $i = 1, \dots, h_D^{(n)}$, $\gamma_n(D)_i$ is algebraic.*

Proof. There exists one μ_i-extreme, hence μ_i-perfect, form S such that $\gamma_n(D)_i = \mu_i(S)$. The conclusion follows since $\mu_i(S)$ is a rational expression in S, and S is algebraic. □

We end this section by showing that there are only finitely many $\mu_i(S)$-perfect forms in a given dimension. To that end, we introduce the notion of $\mu_i(S)$-perfect sets of vectors in Λ_i.

DEFINITION 6.6. A set $\{u_1, \dots, u_t\}$ of vectors in Λ_i is $\mu_i(S)$-perfect if it is the set of minimal vectors of a $\mu_i(S)$-perfect quaternionic Humbert form.

Two sets $\{u_1, \dots, u_t\}$ and $\{u_1', \dots, u_t'\}$ of vectors in Λ_i are equivalent if there exists $g \in GL(\Lambda_i)$, and units $\epsilon_1, \dots, \epsilon_t$ in $\mathfrak{O}^\times$ such that $u_j' = \epsilon_j g u_j$, $1 \le j \le t$. The main result of this subsection is the following:

Theorem 6.7. *Modulo the actions of $GL(\Lambda_i)$ and $\mathfrak{O}^\times$, the set of $\mu_i(S)$-perfect sets in Λ_i is finite.*

From this we easily derive the following corollary

Corollary 6.8. *Modulo the actions of $GL(\Lambda_i)$ and $\mathfrak{O}^\times$, the set of $\mu_i(S)$-perfect forms is finite.*

Proof. Let M be a $\mu_i(S)$-perfect set. We see as before, that the set of $\mu_i(S)$-perfect forms having M as their set of minimal vectors is contained in the set of isolated real points of an algebraic variety, so they are finitely many. $\square$

The proof of Theorem 6.7 relies on the following sequel of lemmas.

Lemma 6.9. *There exists a positive constant $C = C(k)$ such that for any $S \in P_{n,D}$ and any $u \in D^n$,*

$$\inf_{\{\epsilon \in \mathfrak{o}_k^\times\}} \sup_{\{v \in \mathfrak{V}_\infty\}} \frac{S_v[\epsilon_v u_v]}{S[\epsilon u]^{2/(m_v d_v (r+s))}} \leq C.$$

Proof. Let $k_\infty^1 := \left\{\lambda = (\lambda_v)_{v \in \mathfrak{V}_\infty} \colon \prod |\lambda_v| = 1\right\}$. For fixed $S \in P_{n,D}$ and $u \in D^n$, we define an element of k_∞^1 by setting

$$\lambda_v := \frac{S_v[u_v]^{m_v/2}}{S[u]^{1/(d_v(r+s))}}.$$

From Dirichlet unit theorem, the quotient $k_\infty^1 / \mathfrak{o}_k^{\times 2}$ is compact, so there exists a constant $C = C(k)$, depending only on k, such that any element in k_∞^1 admits a representative $\lambda' = (\lambda'_v)_{v \in \mathfrak{V}_\infty}$ modulo multiplication by elements of $\mathfrak{o}_k^{\times 2}$, satisfying $|\lambda'_v| \leq C$. Applying this to the above defined element λ, we can find a unit ϵ such that

$$\frac{S_v[\epsilon_v u_v]^{m_v/2}}{S[\epsilon u]^{1/(d_v(r+s))}} = \epsilon_v^2 \frac{S_v[u_v]^{m_v/2}}{S[u]^{1/(d_v(r+s))}} \leq C \quad \text{for any infinite place} \quad v,$$

which gives the desired conclusion. $\square$

Lemma 6.10. *There exists a positive constant $C' = C'(D)$ such that for any $S \in P_{n,D}$ and any $u \in D^n$,*

$$\inf_{\{\epsilon \in \mathfrak{O}^\times\}} \sup_{\{v \in \mathfrak{V}''_{\infty,1} \cup \mathfrak{V}_{\infty,2}\}} \frac{\operatorname{Tr} S_v[\epsilon_v U_v]}{\det S_v[U_v]^{1/2}} \leq C'.$$

Proof. First, by homogeneity, we can restrict to $S \in P_{n,D}^1$ and $u \in S^{n-1}(D) := \left\{u \in D^n \colon \sum_{i=1}^n \mathrm{Nr}_{D/k}(u_i) = 1\right\}$. If D does not satisfy the Eichler condition, i.e. both $\mathfrak{V}''_{\infty,1}$ and $\mathfrak{V}_{\infty,2}$ are empty, then the assertion is obvious. Otherwise, one knows from [5], Theorem 8.12, that the image of $\mathfrak{O}^1 := \{\epsilon \in \mathfrak{O}^\times \colon \mathrm{Nr}_{D/k}(\epsilon) = 1\}$ in

$\prod_{v\in\mathfrak{V}''_{\infty,1}\cup\mathfrak{V}_{\infty,2}} SL_2(k_v)$ is co-compact, from which the assertion of the lemma is easily derived. □

We can now proceed to the proof of Theorem 6.7 itself. In what follows, a vector $u \in D^n$ satisfying the conditions of Lemma 6.9 and 6.10, i.e.

$$\frac{S_v[u_v]}{S[u]^{2/(m_v d_v (r+s))}} \le C \quad \text{for any} \quad v \in \mathfrak{V}_\infty \tag{6}$$

and

$$\frac{\operatorname{Tr} S_v[U_v]}{\det S_v[U_v]^{1/2}} \le C' \quad \text{for any} \quad v \in \mathfrak{V}''_{\infty,1} \cup \mathfrak{V}_{\infty,2}, \tag{7}$$

will be said to be *normalized with respect to S*, or simply *normalized* for short.

Let $M = \{u_1, \dots, u_t\}$ be a $\mu_i(S)$-perfect set in Λ_i. Using Lemma 6.9, we can assume that the u_j, $1 \le j \le t$, are normalized with respect to S (this amounts to multiply them by suitable units, if necessary). The D-subspace spanned by M is D^n, otherwise the dimension of the subspace spanned by the $x_S(u)$ would be strictly less than $r_1' n(2n-1) + r_1'' n(2n+1) + 4r_2 n^2 - (r_1 + r_2) + 1$, contradicting the $\mu_i(S)$-perfection of M.

So one can extract from M a D-basis $u_1, \dots, u_n$ of D^n. The $\mathfrak{O}$ sublattice of Λ_i spanned by this basis is denoted by Λ. Let u be any non zero vector in Λ_i. Due to the arithmetic-geometric mean inequality, one has

$$\begin{aligned}
(S[u])^{1/r+s} &\le \frac{1}{r+s}\left(\sum_{v\in\mathfrak{V}_\infty} S_v[u_v]^{m_v d_v/2}\right) \\
&\le \frac{1}{r+s}\left(\sum_{v\in\mathfrak{V}'_{\infty,1}} S_v[u_v] + \sum_{v\in\mathfrak{V}''_{\infty,1}} \det S_v[U_v]^{1/2} + \sum_{v\in\mathfrak{V}_{\infty,2}} \det S_v[U_v]\right) \\
&\le \frac{1}{r+s}\left(\sum_{v\in\mathfrak{V}'_{\infty,1}} S_v[u_v] + \sum_{v\in\mathfrak{V}''_{\infty,1}} \frac{1}{2}\operatorname{Tr} S_v[U_v] + \sum_{v\in\mathfrak{V}_{\infty,2}} \frac{1}{4}(\operatorname{Tr} S_v[U_v])^2\right).
\end{aligned}$$

One can write u as $\sum_{j=1}^n \alpha_j u_j$, $\alpha_j \in D$. Set $\mathfrak{n}_u = \operatorname{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_u)$ (resp. $\mathfrak{n}_{u_i} = \operatorname{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_{u_{u_i}})$, $1 \le i \le n$). Because of Lemma 5.1, we can assume that $\mathfrak{A}_u$ is one of the $\mathfrak{B}_i$, so that $\mathfrak{n}_u \ge 1$ ($\mathfrak{B}_i \subset \mathfrak{O}$). Applying repeatedly the triangle inequality and inequalities (6) and (7) one gets:

$$\begin{aligned}
S_v[u_v] &\le \left(\sum_i |\alpha_i|_v S_v[u_{i,v}]^{1/2}\right)^2 \\
&\le C\left(\sum_i |\alpha_i|_v S[u_i]^{1/(2(r+s))}\right)^2
\end{aligned}$$

$$\leq C\left(\sum_i |\alpha_i|_v \mathfrak{n}_i^{1/(2(r+s))}\right)^2 m_i(S)^{1/r+s} \quad \text{for} \quad v \in \mathfrak{V}'_{\infty,1},$$

$$\operatorname{Tr} S_v[U_v] \leq 2\left(\sum_i |\alpha_i|_v^{1/d_v} (\operatorname{Tr} S_v[U_{i,v}])^{1/2}\right)^2$$

$$\leq 2CC'\left(\sum_i |\alpha_i|_v^{1/d_v} S[u_i]^{1/(2m_v d_v(r+s))}\right)^2$$

$$\leq 2CC'\left(\sum_i |\alpha_i|_v^{1/d_v} \mathfrak{n}_i^{1/(2m_v d_v(r+s))}\right)^2 m_i(S)^{1/(m_v d_v(r+s))} \quad \text{for} \quad v \in \mathfrak{V}''_{\infty,1} \cup \mathfrak{V}_{\infty,2}.$$

Combined with the previous inequality, this yields

$$(\mathfrak{n}_u m_i(S))^{1/r+s} \leq (S[u])^{1/r+s}$$
$$\leq \frac{1}{r+s}\left(\sum_{v\in\mathfrak{V}'_{\infty,1}} C\left(\sum_i |\alpha_i|_v \mathfrak{n}_i^{1/(2(r+s))}\right)^2 + \sum_{v\in\mathfrak{V}''_{\infty,1}} CC'\left(\sum_i |\alpha_i|_v \mathfrak{n}_i^{1/(2(r+s))}\right)^2 + \sum_{v\in\mathfrak{V}_{\infty,2}} (CC')^2\left(\sum_i |\alpha_i|_v^{1/2} \mathfrak{n}_i^{1/(4(r+s))}\right)^4\right) m_i(S)^{1/(r+s)},$$

whence

$$1 \leq \mathfrak{n}_u^{1/r+s} \leq \frac{1}{r+s}\left(\sum_{v\in\mathfrak{V}'_{\infty,1}} C\left(\sum_i |\alpha_i|_v \mathfrak{n}_i^{1/(2(r+s))}\right)^2 + \sum_{v\in\mathfrak{V}''_{\infty,1}} CC'\left(\sum_i |\alpha_i|_v \mathfrak{n}_i^{1/(2(r+s))}\right)^2 + \sum_{v\in\mathfrak{V}_{\infty,2}} (CC')^2\left(\sum_i |\alpha_i|_v^{1/2} \mathfrak{n}_i^{1/(4(r+s))}\right)^4\right).$$

In particular, the convex body

$$\frac{1}{r+s}\left(\sum_{v\in\mathfrak{V}'_{\infty,1}} C\left(\sum_i |\alpha_i|_v \mathfrak{n}_i^{1/(2(r+s))}\right)^2 + \sum_{v\in\mathfrak{V}''_{\infty,1}} CC'\left(\sum_i |\alpha_i|_v \mathfrak{n}_i^{1/(2(r+s))}\right)^2 + \sum_{v\in\mathfrak{V}_{\infty,2}} (CC')^2\left(\sum_i |\alpha_i|_v^{1/2} \mathfrak{n}_i^{1/(4(r+s))}\right)^4\right) < 1 \tag{8}$$

in $\mathbb{R} \otimes_{\mathbb{Q}} D^n$, contains no nonzero point in Λ_i.

According to Minkowski convex body theorem, this implies that its volume is bounded from above by $2^{4[k:\mathbb{Q}]}\Delta_i$, where Δ_i stands for the discriminant of Λ_i, viewed as a lattice in $\mathbb{R}\otimes_{\mathbb{Q}} D^n \simeq \mathbb{R}^{4[k:\mathbb{Q}]}$. On the other hand, an easy computation shows that this volume can be expressed as

$$[\Lambda_i : \Lambda]V, \tag{9}$$

where V is a constant depending only on k and n. Consequently, we see that $[\Lambda_i : \Lambda]$ is bounded from above by a constant, so that there are finitely many possible Λ's, whence finitely many bases $\{u_1, \ldots, u_n\}$ of D^n up to $GL(\Lambda_i)$ satisfying (6) and (7) and consisting on minimal vectors of a Humbert form.

It remains to prove that each of these bases is contained in finitely many weakly perfect sets. Without loss of generality, we can assume that $\det S_v = 1$ for any $v \in \mathfrak{V}_\infty$ (this amounts to scale the components of S by suitable positive factors, which does not affect the set of minimal vectors). Let $\{u_1, \ldots, u_n\}$ be a n-tuple of linearly independent normalized minimal vectors of a n-ary Humbert form S. If u is any minimal vector of S, we can assume, from Lemma 5.1, that $\mathfrak{A}_u$ is one of the $\mathfrak{B}_i$, so that, in particular, $\mathfrak{n}_u$ is bounded by a constant depending only on $\mathfrak{O}$. Once this is achieved we can assume moreover, that u is normalized with respect to S (this amounts to scale u by a suitable unit, which does not affect $\mathfrak{A}_u$). If we write u as

$$u = \sum_{j=1}^{n} u_j \alpha_j, \quad \alpha_j \in D,$$

we will show that there are finitely many possibilities for the α_j, which will complete the proof. To that end, we only need to bound the images $\alpha_{j,v}$ of α_j in $D \otimes_k k_v$, v in $\mathfrak{V}_\infty$.

(i) If $v \in \mathfrak{V}'_{\infty,1}$, we consider, for each $j = 1, \ldots, n$, the matrix $P_j \in M_n(D)$ the columns of which are the u'_k, but for the j-th which is defined to be u'. Then, the determinant of the hermitian form $S_v[P_{j,v}]$ is given by

$$\det S_v[P_{j,v}] = \mathrm{Nr}_{\mathbb{H}/\mathbb{R}}(\alpha_{j,v}) \det S_v = \mathrm{Nr}_{\mathbb{H}/\mathbb{R}}(\alpha_{j,v}).$$

On the other hand, bounding the determinant of $S_v[P_{j,v}]$ by the product of its diagonal entries (Hadamard inequality), we get

$$
\begin{aligned}
\mathrm{Nr}_{\mathbb{H}/\mathbb{R}}(\alpha_{j,v}) = \det S_v[P_{j,v}] &\le S_v[u_v]\prod_{k\neq j}^{n} S_v[u_{k,v}] \\
&\le C^n\left(S[u]\prod_{k\neq j} S[u_k]\right)^{1/(r+s)} \qquad \text{because of (6)} \\
&= C^n\left(\mathfrak{n}_u\prod_{k\neq j}\mathfrak{n}_{u_k}\right)^{1/(r+s)} \qquad m_i(S)^{n/(r+s)} \\
&\le C^n\left(\left(\mathfrak{n}_u\prod_{k\neq j}\mathfrak{n}_{u_k}\right)\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_i)\gamma_n(D)_i\right)^{1/(r+s)}.
\end{aligned}
\tag{10}
$$

From our assumption on u, we know that $\mathfrak{n}_u$ is bounded, so (10) gives a bound on $\mathrm{Nr}_{\mathbb{H}/\mathbb{R}}(\alpha_{j,v})$.

(ii) If $v \in \mathfrak{V}''_{\infty,1} \cup \mathfrak{V}_{\infty,2}$, each $\alpha_{j,v}$ identifies with a 2 by 2 matrix $\left(\begin{smallmatrix}\lambda_j & \nu_j\\ \mu_j & \eta_j\end{smallmatrix}\right) \in M_2(k_v)$, and what we want to show is that $|\lambda_j|_v$, $|\nu_j|_v$, $|\mu_j|_v$ and $|\eta_j|_v$ are bounded. We show it for $|\lambda_j|_v$ (the other cases are similar). We denote by U (resp. U_j) the image of u (resp. u_j) in $M_{2n,2}(k_v)$, so that the equality $u = \sum_{j=1}^n u_j\alpha_j$ reads $U = \sum_{j=1}^n U_j \left(\begin{smallmatrix}\lambda_j & \nu_j\\ \mu_j & \eta_j\end{smallmatrix}\right)$ or, transposing,

$$
U' = \sum_{j=1}^{n}\begin{pmatrix}\lambda_j & \mu_j\\ \nu_j & \eta_j\end{pmatrix} U'_j. \tag{11}
$$

Let X, Y (resp. X_j, Y_j) in D^n be the first and second rows of U' (resp. U'_j). Multiplying (11) on the left by $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, we get

$$
X = \sum_{j=1}^{n}\lambda_j X_j + \mu_j Y_j. \tag{12}
$$

We now consider the matrix $P_{j,v} \in M_{2n}(k_v)$, the rows of which are defined as follows: for $1 \le k \le n$, the $2k$-th row is Y_k, the $(2k-1)$-th row is X_k if $k \neq j$ and the $(2j-1)$-th row is X. As before, we see that the determinant of the positive definite Hermitian form $S_v[P_{j,v}]$ is

$$
\det S_v[P_{j,v}] = |\lambda_j|_v^{2/d_v} \det S_v = |\lambda_j|_v^{2/d_v}.
$$

On the other hand, applying the Hadamard inequality, we get

$$
\begin{aligned}
\det S_v[P_{j,v}] &\le S_v[X]S_v[Y_j]\prod_{k\neq j}(S_v[X_k]S_v[Y_k]) \\
&\le C'^n \det S_v[U_j]\prod_{k\neq j}\det S_v[U_k] \quad \text{because of (7)} \\
&\le C'^n C^n \left(S[u]\prod_{k\neq j}^{n} S[u_k]\right)^{2/(m_v d_v(r+s))} \quad \text{because of (6)} \\
&= C'^n C^n \left(\mathfrak{n}_u \prod_{k\neq j}\mathfrak{n}_{u_k}\right)^{2/(m_v d_v(r+s))} \quad m_i(S)^{2n/(m_v d_v(r+s))} \\
&\le C'^n C^n \left(\left(\mathfrak{n}_u \prod_{k\neq j}\mathfrak{n}_{u_k}\right)\mathrm{Nr}_{D/\mathbb{Q}}(\mathfrak{A}_i)\gamma_n(D)_i\right)^{2/(m_v d_v(r+s))}.
\end{aligned}
\tag{13}
$$

Again, the assumption on u ensures that $\mathfrak{n}_u$ is bounded, so (13) gives a bound on $|\lambda_j|_v$.

In conclusion, (10) and (13), together with the assumption that u is in Λ_i, leaves finitely many possibilities for the α_j, whence we conclude that there are finitely many weakly perfect sets. □

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