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A NON QUASI-INVARIANCE OF THE BROWNIAN MOTION ON LOOP GROUPS

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1. Introduction

In this paper, we will prove a non quasi-invariance of the Brownian motion on loop groups.

In [9], Fang proved an integration by parts formula for a natural gradient on path space over loop groups. His gradient is constructed on the parallel translation operator which was first introduced by Driver [7].

On the other hand, on path spaces over finite dimensional Lie groups, there is a natural constraction of the gradient based on the group translations. In this case, the integration by parts formula is computed via the quasi-invariance under the group translations of the reference measure.

And, there are many works on the quasi-invariance on path groups and loop groups over finite dimensional Lie groups: See, for example, Albeverio-Høeph-Krohn [3], Shigekawa [15], Malliavin-Malliavin [12].

On the contrary, our result shows that there is no extension of these results to the case of the path group over loop groups. If a smooth path acts on the law of the Brownian motion, the shifted measure is singular to the original measure except the case of the constant path.

The proof of this non quasi-invariance relies on two recent results.

One is the two parameter stochastic calculus on Lie groups which is developed in Driver-Srimurthy [8], Srimurthy [17]. This plays an essential role in the non quasi-invariance of the Brownian motion on path groups (Section 3). For two parameter stochastic calculus on manifolds, see also Norris [13].

The other one is the equivalence between the heat kernel measure and the pinned measure which is shown in Driver-Srimurthy [8] and Aida-Driver [1]. This theorem enables us to reduce the result in the path group case to the loop group case.

The organization of this paper is as follows. In Section 2, we fix some notations and give a proof for the key lemma (Lemma 2.2). In Section 3 and Section 4, we will prove the non quasi-invariance of the Brownian motion on path groups and loop groups, respectively.

2. Preliminaries

The aim of this section is to fix some notations and collect several results which we will need in the next section.

Let K be a compact semi-simple simply connected Lie group, and $e \in K$ be the identity, d be the dimension of K. We denote by $\mathfrak{k} = T_eK$ the Lie algebra of K. On $\mathfrak{k}$, we fix an Ad_K-invariant inner product which is denoted by $\langle \cdot, \cdot \rangle$.

For a topological space X with a base point x, we denote by $\mathcal{P}_x(X)$ the space of based continuous paths over X. Then, $\mathcal{P}_e(K)$ denotes the group of based continuous paths on K. We will use the same symbol e to denote the identity of $\mathcal{P}_e(K)$.

We set, for $t, \tau \in [0, 1]$, $G(t, \tau) = t \wedge \tau$. Let $\{\beta(t, s)\}_{(t,s)\in[0,1]^2}$ be $\mathfrak{k}$ valued Brownian sheet, i.e., β is a $\mathfrak{k}$ valued centered Gaussian process such that

$$E[\langle A, \beta(t, s)\rangle\langle B, \beta(\tau, \sigma)\rangle] = \langle A, B\rangle G(t, \tau)G(s, \sigma)$$

for all $t, \tau, s, \sigma \in [0, 1]$, and $A, B \in \mathfrak{k}$.

Let $\{A_\alpha\}_{\alpha=1}^d$ be an orthonormal basis of $\mathfrak{k}$. For $C \in \mathfrak{k}$, set $C^\alpha = \langle A_\alpha, C\rangle$. We denote by $\tilde{A}_\alpha$ the left invariant vector field corresponding to A_α.

Let $\Sigma(t, s)$ denote the solution to the following Stratonovich stochastic differential equation in t with s as a parameter:

$$d_t\Sigma(t, s) = \sum_{\alpha=1}^{d} \tilde{A}_\alpha(\Sigma(t, s)) \circ d_t\beta^\alpha(t, s) \quad \text{with} \quad \Sigma(0, s) = e. \tag{2.1}$$

By Malliavin [11] and Driver [7, Theorem 3.8], we may choose a version of $\Sigma(t, s)$ which is jointly continuous in (t, s). And then, the law of $\{\Sigma(t, s)\}_{(t,s)\in[0,1]^2}$ is a probability measure on $C([0, 1]^2 \to K)$ which is supported on $\mathcal{P}_e(\mathcal{P}_e(K))$. We denote by ν this measure.

It is shown in Driver-Srimurthy [8, Theorem 2.15] that, for a fixed $t \in [0, 1]$, $s \mapsto \Sigma(t, s)$ is a Brownian motion on K with variance t. From this fact, we obtain another Brownian motion $s \mapsto x(t, s)$ on $\mathfrak{k}$ with variance t by

$$x(t, s) = \int_0^s \omega(\circ d_\sigma\Sigma(t, \sigma)) \tag{2.2}$$

where ω denote the left invariant Maurer-Cartan form on K. More precisely, ω is the $\mathfrak{k}$ valued one form which is determined by $\omega(\tilde{A}) = A$ $(A \in \mathfrak{k})$. It is equivalent that $\{\Sigma(t, s)\}$ satisfies the following stochastic differential equation with parameter t:

$$d_s\Sigma(t, s) = \sum_{\alpha=1}^{d} \tilde{A}_\alpha(\Sigma(t, s)) \circ d_s x^\alpha(t, s) \quad \text{with} \quad \Sigma(t, 0) = e. \tag{2.3}$$

REMARK. By the same proof of Srimurthy [17, Theorem 4.1], we shall obtain a Brownian sheet from $\Sigma(t, s)$ by using the right invariant Maurer-Cartan form instead

of ω in (2.2). But since we do not need this fact, we will use the left invariant one, to avoid confusion.

By (2.3), we obtain the following lemma:

Lemma 2.1. *$Ad_{\Sigma(t,s)^{-1}}$ satisfies the following matrix stochastic differential equation with parameter t:*

$$d_s Ad_{\Sigma(t,s)^{-1}} = -\sum_{\alpha=1}^{d} ad_{A_\alpha} Ad_{\Sigma(t,s)^{-1}} \circ d_s x^\alpha(t,s). \tag{2.4}$$

Proof. This is a consequance of Itô calculus. See [16, Proposition 2.1]. □

We will state our key lemma, which will be used in the next section. Let $H^1(\mathfrak{k})$ be the space of H^1-paths on $\mathfrak{k}$. More precisely, we set

$$H^1(\mathfrak{k}) = \left\{ h \in C([0,1] \to \mathfrak{k});\ \begin{array}{l} h \text{ is an absolute continuous function such that} \\ h(0) = 0 \text{ and } \int_0^1 |(d/ds)h(s)|_{\mathfrak{k}}^2 \, ds < \infty \end{array} \right\}.$$

Lemma 2.2. *Let $h \in H^1(\mathfrak{k})$ and set $l(t,s) = Ad_{\Sigma(t,s)^{-1}}(h(s))$. Then, for any fixed $t \in [0,1]$, $\{l(t,s)\}_{s\in[0,1]}$ is a semi-martingale and its quadratic variation process $\langle l(t,s)\rangle$ is given by*

$$\langle l(t,s)\rangle = -t \int_0^s K(h(\sigma), h(\sigma))\, d\sigma \tag{2.5}$$

where K denotes

$$K(X,Y) = \mathrm{tr}(ad_X \circ ad_Y),$$

the Killing form of $\mathfrak{k}$.

Proof. From (2.4), we deduce that $l(t,s) = Ad_{\Sigma(t,s)^{-1}}(h(s))$ satisfies

$$d_s l(t,s) = -\sum_{\alpha=1}^{d} ad_{A_\alpha}(l(t,s)) \circ d_s x^\alpha(t,s) + Ad_{\Sigma(t,s)^{-1}}\left(\frac{d}{ds}h(s)\right) ds.$$

And then, we have the quadratic variation of $l(t,s)$ as follows:

$$\langle l(t,s)\rangle = t \sum_{\alpha=1}^{d} \int_0^s |ad_{A_\alpha} Ad_{\Sigma(t,\sigma)^{-1}}(h(\sigma))|_{\mathfrak{k}}^2 \, d\sigma.$$

By noting that $\{A_\alpha\}_{\alpha=1}^d$ is an orthonormal basis of $\mathfrak{k}$ and Ad-invariance of the Killing form, we have obtained (2.5). This proves the lemma. □

Since we will use the Hellinger integral to show the non quasi-invariance, we review some properties of it. Let ρ be the Hellinger integral:

$$\rho(\nu^1, \nu^2) = \int \sqrt{\frac{d\nu^1}{d\nu^3}} \sqrt{\frac{d\nu^2}{d\nu^3}}\, d\nu^3$$

where, ν^1, ν^2, ν^3 are probability measures on Ω in relation that ν^1, $\nu^2 \ll \nu^3$. It is well-known that this definition is independent of the choice of such ν^3. See, e.g. [18, Section 1.4].

For a probability measure ν on $(\Omega, \mathcal{F})$, we denote by ν_n the restriction of ν to the σ-algebra $\mathcal{F}_n$, and by $\nu(X|\mathcal{F}_n)$ the conditional expectation of X with respect to $\mathcal{F}_n$.

Proposition 2.1. *The following properties holds for* ρ .

(2.6) 1. $\rho(\nu^1, \nu^2) = \lim_{n\to\infty} \rho(\nu_n^1, \nu_n^2)$.

(2.7) 2. $\rho(\nu^1, \nu^2) = 0$ *is equivalent to* $\nu^1 \perp \nu^2$.

Proof. We set

$$\alpha_n^p(x) := \frac{d\nu_n^p}{d\nu_n^3}(x), \quad \alpha^p(x) := \frac{d\nu^p}{d\nu^3}(x) \quad (p = 1,\ 2)$$

for short. First, as for (2.6), we note that

$$\begin{aligned}\rho(\nu_n^1, \nu_n^2) &= \int \sqrt{\alpha_n^1(x)} \sqrt{\alpha_n^2(x)}\, d\nu_n^3 \\ &= \int \sqrt{\alpha_n^1(x)} \sqrt{\alpha_n^2(x)}\, d\nu^3,\end{aligned}$$

where we regard $\alpha_n^p(x)$ as a function on $(\Omega, \mathcal{F}_n)$ in the first line and as on $(\Omega, \mathcal{F})$ in the second line. Since $\alpha_n^p = \nu^3(\alpha^p|\mathcal{F}_n)$, α_n^p converges to α^p in $L^1(\Omega, \nu^3)$. (see, e.g., [10, Proposition 2.2.4 and Theorem 2.6.6].) We have obtained (2.6).

As for (2.7), we refer to [18, Lemma 1.4.1]. □

3. Non quasi-invariance: over path groups

The purpose of this section is to show a non quasi-invariance of the measure ν under the group transformations. For the proof, we need the approximation from finite dimensional subgroup which was first introduced by Driver-Lorentz [6]. First, we review Driver-Lorentz's approximation quickly.

For a partition

(3.1) $$\mathcal{P} = \{0 < s_1 < \cdots < s_n < 1\}$$

of $[0,1]$, we set

$$\mathfrak{k}^{\mathcal{P}} = \overbrace{\mathfrak{k} \times \cdots \times \mathfrak{k}}^{n}, \quad K^{\mathcal{P}} = \overbrace{K \times \cdots \times K}^{n}.$$

We define $\langle \vec{A}, \vec{B} \rangle_{\mathcal{P}}$ for $\vec{A} = (A_1, \ldots, A_n)$, $\vec{B} = (B_1, \ldots, B_n) \in \mathfrak{k}^{\mathcal{P}}$ by

$$\langle \vec{A}, \vec{B} \rangle_{\mathcal{P}} = \sum_{i,j=1}^{n} Q^{\mathcal{P}}_{i,j} \langle A_i, B_j \rangle,$$

where $(Q^{\mathcal{P}}_{i,j})$ denotes the inverse matrix of $(G(s_i, s_j))_{s_i, s_j \in \mathcal{P}}$. Let $H^1(\mathfrak{k}^{\mathcal{P}})$ denote the space of based H^1-paths on $\mathfrak{k}^{\mathcal{P}}$. The map $i\colon \mathfrak{k}^{\mathcal{P}} \to H^1(\mathfrak{k}^{\mathcal{P}})$ defined by

$$i((A_1, \ldots, A_n))(t) = \sum_{i=1}^{n} G(s_i, t) A_i$$

is an isometric embedding of $\mathfrak{k}^{\mathcal{P}}$ into $H^1(\mathfrak{k}^{\mathcal{P}})$.

We now state our theorem. For $k \in \mathcal{P}_e(\mathcal{P}_e(K))$, we denote by ν^k the image measure of ν by the map $L_k\colon \mathcal{P}_e(\mathcal{P}_e(K)) \to \mathcal{P}_e(\mathcal{P}_e(K))$. In other words, ν^k is the measure which is characterized by

$$\int f(\Sigma)\, d\nu^k = \int f(k\Sigma)\, d\nu$$

for all bounded Borel function f on $\mathcal{P}_e(\mathcal{P}_e(K))$.

And, we introduce the notion of H^1-paths on $\mathcal{P}_e(K)$ as follows.

$$H^1(\mathcal{P}_e(K)) = \left\{ k \in \mathcal{P}_e(\mathcal{P}_e(K)); \begin{array}{l} \text{For each } s \in [0,1], \text{ the map } t \mapsto k(t,s) \text{ is an} \\ \text{absolute continuous function, and for a.a. } t, \\ \text{the map } s \mapsto (\partial_t k)(t,s) k(t,s)^{-1} \text{ is in } H^1(\mathfrak{k}), \\ \text{and } \int_0^1 |(\partial_t k)(t,s) k(t,s)^{-1}|^2_{H^1(\mathfrak{k})}\, dt < \infty. \end{array} \right\},$$

where we set $(\partial_t k)(t,s) = (\partial/\partial t) k(t,s)$ and $(\partial_t k)(t,s) k(t,s)^{-1} = R_{k(t,s)^{-1}*}((\partial_t k)(t,s))$, for ease of reading.

Theorem 3.1. *Let $k \in H^1(\mathcal{P}_e(K))$ be a non-constant path. Then, ν^k and ν are mutually singular.*

To show Theorem 3.1, we need some notations. We set

$$\mathcal{P}_n = \left\{ \frac{1}{2^n} < \cdots < \frac{2^n - 1}{2^n} \right\}. \tag{3.2}$$

And then, we set

$$\mathcal{F}_n = \sigma\Big(\Sigma(t,s); t \in [0,1], s \in \mathcal{P}_n \Big)$$

and denote by ν_n^k and ν_n the restrictions of ν^k and ν to $\mathcal{F}_n$ respectively.

For k in Theorem 3.1, we define $h \in L^2([0,1] \to H^1(\mathfrak{k}))$ by

$$h(t,s) = (\partial_t k)(t,s)k(t,s)^{-1}, \tag{3.3}$$

and then define $l \in H^1(\mathcal{P}_e(K))$ by the following ordinary differential equation with parameter s:

$$\frac{d}{dt}l(t,s) = \frac{1}{2}h(t,s)l(t,s) \quad \text{with} \quad l(0,s) = e. \tag{3.4}$$

For $\gamma \in \mathcal{P}_e(\mathcal{P}_e(K))$ (resp. $\gamma \in \mathcal{P}_e(\mathcal{P}_e(\mathfrak{k}))$), we use $\gamma_{n,t}$ to denote the following path in $K^{\mathcal{P}_n}$ (resp. in $\mathfrak{k}^{\mathcal{P}_n}$):

$$\gamma_{n,t} = \left(\gamma\left(t, \frac{1}{2^n}\right), \ldots, \gamma\left(t, \frac{2^n - 1}{2^n}\right)\right).$$

The following proposition is well-known ([3],[15]), but for its importance, we will give a proof for this case.

Proposition 3.1. *Let $k \in H^1(\mathcal{P}_e(K))$. Then, ν_n^k and ν_n are equivalent and the Radon-Nykodim derivative is given by*

$$\frac{d\nu_n^k}{d\nu_n}(\Sigma) = \exp\left(\int_0^1 \langle Ad_{\Sigma_{n,t}^{-1}}(h_{n,t}), d\beta_{n,t}\rangle_{\mathcal{P}_n} - \frac{1}{2}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(h_{n,t})|^2_{\mathcal{P}_n}\, dt\right), \tag{3.5}$$

where h is the path on $\mathcal{P}_0(\mathfrak{k})$ given in (3.3)*, and Ad denotes the Adjoint representation of $G^{\mathcal{P}_n}$.*

Proof. First, noting that $G^{\mathcal{P}_n}$ is compact, we set $M_n = \sup_{g \in G^{\mathcal{P}_n}} \|Ad_g\|$. Here, $\|\cdot\|$ denotes the operator norm with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{P}_n}$.

By Itô formula, we have that $k_{n,t}^{-1}\Sigma_{n,t}$ satisfies

$$d(k_{n,t}^{-1}\Sigma_{n,t}) = \sum_{\alpha=1}^{d} \tilde{A}_\alpha(k_{n,t}^{-1}\Sigma_{n,t}) \circ (d\beta_{n,t}^\alpha - Ad_{\Sigma_{n,t}^{-1}}(h_{n,t})^\alpha\, dt). \tag{3.6}$$

As we note above, $G^{\mathcal{P}_n}$ is compact and the Novikov condition is satisfied as follows:

$$\begin{aligned} E\left[\exp\left\{\frac{1}{2}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(h_{n,t})|^2_{\mathcal{P}_n}\, dt\right\}\right] &\le \exp\left(\frac{M_n^2}{2}\int_0^1 |h_{n,t}|^2_{\mathcal{P}_n}\, dt\right) \\ &\le \exp\left(\frac{M_n^2}{2}\int_0^1 |h(t,\cdot)|^2_{H^1(\mathfrak{k})}\, dt\right) < \infty. \end{aligned}$$

By [14, Chapter VIII, Proposition (1.15)], (3.5) holds. □

Lemma 3.1. *The Hellinger integral of ν_n and ν_n^k is given by*

$$\rho(\nu_n^k, \nu_n) = E\Big[\exp\Big(-\frac{1}{8}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{\mathcal{P}_n}\, dt\Big)\Big]. \tag{3.7}$$

Proof. First, by (3.4) and (3.5), we have

$$\frac{d\nu_n^l}{d\nu_n} = \exp\left(\frac{1}{2}\int_0^1 \langle Ad_{\Sigma_{n,t}^{-1}}(h_{n,t}), d\beta_{n,t}\rangle_{\mathcal{P}_n} - \frac{1}{8}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(h_{n,t})|^2_{\mathcal{P}_n}\, dt\right).$$

Then, by using the expression $\rho(\nu_n^k, \nu_n) = \int \sqrt{d\nu_n^k/d\nu_n}\, d\nu_n$, we obtain (3.7) as follows:

$$\begin{aligned}
\rho(\nu_n^k, \nu_n) &= \int \exp\left(\frac{1}{2}\int_0^1 \langle Ad_{\Sigma_{n,t}^{-1}}(h_{n,t}), d\beta_{n,t}\rangle_{\mathcal{P}_n} - \frac{1}{4}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(h_{n,t})|^2_{\mathcal{P}_n}\, dt\right) d\nu_n \\
&= \int \exp\left(-\frac{1}{8}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(h_{n,t})|^2_{\mathcal{P}_n}\, dt\right) \frac{d\nu_n^l}{d\nu_n}\, d\nu_n \\
&= \int \exp\Big(-\frac{1}{8}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(h_{n,t})|^2_{\mathcal{P}_n}\, dt\Big)\, d\nu_n^l \\
&= \int \exp\Big(-\frac{1}{8}\int_0^1 |Ad_{((l\Sigma)_{n,t})^{-1}}(h_{n,t})|^2_{\mathcal{P}_n}\, dt\Big)\, d\nu_n \\
&= E\Big[\exp\Big(-\frac{1}{8}\int_0^1 |Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{\mathcal{P}_n}\, dt\Big)\Big].
\end{aligned}$$

The proof is completed. □

Proof of Theorem 3.1. First, set $T = \{t \in [0,1]; \forall s \in [0,1], h(t,s) = 0\}$. Then, by Lemma 3.1 and Jensen's inequality, we have an estimation of $\rho(\nu_n^k, \nu_n)$ as follows:

$$\begin{aligned}
\rho(\nu_n^k, \nu_n) &= E\Big[\exp\Big(-\frac{1}{8}\int_{T^c} |Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{\mathcal{P}_n}\, dt\Big)\Big] \\
&\le \frac{1}{|T^c|}\int_{T^c} E\Big[\exp\Big(-\frac{|T^c|}{8}|Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{\mathcal{P}_n}\Big)\Big]\, dt,
\end{aligned} \tag{3.8}$$

where $|T^c|$ denotes the Lebesgue measure of T^c. By Lemma 2.2 and since K is semi-simple, $s \mapsto Ad_{\Sigma(t,s)^{-1}}(Ad_{l(t,s)^{-1}}(h(t,s)))$ is a semi-martingale of positive quadratic variation. In particular, the path $s \mapsto Ad_{\Sigma(t,s)^{-1}}(Ad_{l(t,s)^{-1}}(h(t,s)))$ is not in $H^1(\mathfrak{k})$ almost surely. Since $|\cdot|_{\mathcal{P}_n}$ is increasing with respect to n, and approximates the H^1-norm, we have

$$\lim_{n\to\infty} \exp\Big(-\frac{|T^c|}{8}|Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{\mathcal{P}_n}\Big) = 0 \quad \text{a.s.} \tag{3.9}$$

And then by (2.6), we have

$$\begin{aligned}\rho(\nu^k, \nu) &\leq \lim_{n\to\infty} \frac{1}{|T^c|} \int_{T^c} E\Big[\exp\Big(-\frac{|T^c|}{8} |Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{\mathcal{P}_n}\Big)\Big]\, dt \\ &= 0.\end{aligned}$$

The proof is completed. □

4. Non quasi-invariance: over loop groups

In this section, we will deduce the non quasi-invariance on loop groups from Theorem 3.1. To state the theorem precisely, we fix some notations.

We keep some notations in previous sections. We set, for $s, \sigma \in [0,1]$, $G_0(s,\sigma) = s \wedge \sigma - s\sigma$. Let $\{\chi(t,s)\}_{(t,s)\in[0,1]^2}$ be $\mathfrak{k}$ valued Brownian bridge sheet, i.e., χ is a $\mathfrak{k}$ valued centered Gaussian process such that

$$E[\langle A, \chi(t,s)\rangle\langle B, \chi(\tau,\sigma)\rangle] = \langle A, B\rangle G(t,\tau)G_0(s,\sigma).$$

Let $\Sigma^0(t,s)$ denote the solution to the following Stratonovich stochastic differential equation in t with s as a parameter:

$$d_t\Sigma^0(t,s) = \sum_{\alpha=1}^{d} \tilde{A}_\alpha(\Sigma^0(t,s)) \circ d_t\chi^\alpha(t,s) \quad \text{with} \quad \Sigma^0(0,s) = e. \tag{4.1}$$

Following the notation in path group case, we set

$\mathcal{L}_x(X)$ = the based loop space over X with base point x,
ν_0 = the law of $\Sigma^0(t,s)$, and $\nu_0^k = \nu_0 \circ L_k^{-1}$,
$\mathcal{F}_{0,n} = \sigma\big(\Sigma^0(t,s); t \in [0,1], s \in \mathcal{P}_n\big)$,
$\nu_{0,n}$, $\nu_{0,n}^k$ = the restriction of ν_0, ν_0^k to $\mathcal{F}_{0,n}$ repectively,
$H^1(\mathcal{L}_e(K)) = H^1(\mathcal{P}_e(K)) \cap \mathcal{P}_e(\mathcal{L}_e(K))$,
$Q_0^{\mathcal{P}_n}$ = the inverse matrix of $(G_0(s_i,s_j))_{s_i,s_j\in\mathcal{P}_n}$,
$\langle \vec{A}, \vec{B}\rangle_{0,\mathcal{P}_n} = \sum_{i,j=1}^n Q_{0,i,j}^{\mathcal{P}_n}\langle A_i, B_j\rangle$.

In this case, the matrix $Q_0^{\mathcal{P}_n}$ is given as follows:

$$Q_{0,i,j}^{\mathcal{P}_n} = \begin{cases} 2^{n+1} & (i = j), \\ -2^n & (|i-j| = 1), \\ 0 & (\text{otherwise}). \end{cases} \tag{4.2}$$

Our theorem in the loop group case is the following.

Theorem 4.1. *Let $k \in H^1(\mathcal{L}_e(K))$ be a non-constant path. Then, ν_0^k and ν_0 are mutually singular.*

The way of the proof is the same as in Theorem 3.1. First, we prepare a lemma which is corresponding to Lemma 3.1. For $k \in H^1(\mathcal{L}_e(K))$, we define h and l as in (3.3) and (3.4) respectively.

Lemma 4.1. *The Hellinger integral of $\nu_{0,n}$ and $\nu^k_{0,n}$ is given by*

$$\rho(\nu^k_{0,n}, \nu_{0,n}) = E\left[\exp\left(-\frac{1}{8}\int_0^1 |Ad_{(\Sigma^0_{n,t})^{-1}}(Ad_{l^{-1}_{n,t}}(h_{n,t}))|^2_{0,\mathcal{P}_n}\, dt\right)\right]. \tag{4.3}$$

We omit the proof, since it is the same as in Lemma 3.1.

For the proof of the Theorem 4.1, we use the result on the equivalence between the heat kernel measure and the pinned measure. For a later use of this fact, we fix some notations.

For the partition $\mathcal{P}_n$, we define $\pi_n\colon \mathcal{L}_e(K) \to K^{\mathcal{P}_n}$ by

$$\pi_n(\gamma) = \left(\gamma\left(\frac{1}{2^n}\right), \ldots, \gamma\left(\frac{2^n-1}{2^n}\right)\right). \tag{4.4}$$

Let $p_t(g)$ the heat kernel on K. Let μ_t and μ^0_t be the Wiener measure and the pinned Wiener measure on $\mathcal{P}_e(K)$ with variance t, respectively. More precisely, μ^0_t is the unique measure such that, if f is a bounded function of the form $f(\gamma) = F(\gamma(s_1), \ldots, \gamma(s_n))$ for some partition $\mathcal{P} = \{0 < s_1 < \cdots < s_n < 1\}$,

$$\int f(\gamma)\, d\mu^0_t = \frac{1}{p_{t(1-s_n)}(e)} \int f(\gamma) p_{t(1-s_n)}(\gamma(1-s_n))\, d\mu. \tag{4.5}$$

Let ν^0_t be the heat kernel measure on $\mathcal{L}_e(K)$. In other words, ν^0_t is the law of $\{\Sigma^0(t, \cdot)\}$.

Proof of Theorem 4.1. Set $T = \{t \in [0,1]; \forall s \in [0,1], h(t,s) = 0\}$. Then, as in (3.8), we have

$$\rho(\nu^k_{0,n}, \nu_{0,n}) \le \frac{1}{|T^c|}\int_{T^c}\int_{\mathcal{L}_e(K)} \exp\left(-\frac{|T^c|}{8}|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l^{-1}_{n,t}}(h_{n,t}))|^2_{0,\mathcal{P}_n}\right) d\nu^0_t(\gamma)\, dt.$$

To complete the proof, it suffices to show that, for any $t \in T^c$,

$$\lim_{n\to\infty}\int_{\mathcal{L}_e(K)} \exp\left(-\frac{|T^c|}{8}|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l^{-1}_{n,t}}(h_{n,t}))|^2_{0,\mathcal{P}_n}\right)\nu^0_t(\gamma) = 0. \tag{4.6}$$

Since $t \in T^c$, we may take $\alpha_t \in \bigcup_{n=1}^\infty \mathcal{P}_n$ so that

$$\int_0^{\alpha_t} |h(t,s)|^2_{\mathfrak{k}}\, ds > 0. \tag{4.7}$$

For this number, we define $\tilde{h}(t,s)$ by

$$\tilde{h}(t,s) = \begin{cases} h(t,s) & (s \le \alpha_t), \\ 0 & (s > \alpha_t). \end{cases}$$

From the definition of $\tilde{h}(t,s)$, if we write $\alpha_t = i/2^m$, we can deduce

$$|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{0,\mathcal{P}_n} \ge |Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n} \quad (n \ge m). \tag{4.8}$$

To check (4.8), we set

$$\varphi(t,s) = Ad_{\gamma(s)^{-1}}(Ad_{l(t,s)^{-1}}(h(t,s))), \quad \tilde{\varphi}(t,s) = Ad_{\gamma(s)^{-1}}(Ad_{l(t,s)^{-1}}(\tilde{h}(t,s))).$$

Then, by (4.2), we have (4.8) as follows.

$$\begin{aligned} |\varphi_{n,t}|^2_{0,\mathcal{P}_n} &= 2^n \sum_{k=1}^{2^n} \left| \varphi_{n,t}\Big(\frac{k}{2^n}\Big) - \varphi_{n,t}\Big(\frac{k-1}{2^n}\Big) \right|^2_{\mathfrak{k}} \\ &\ge 2^n \sum_{k=1}^{i2^{n-m}} \left| \varphi_{n,t}\Big(\frac{k}{2^n}\Big) - \varphi_{n,t}\Big(\frac{k-1}{2^n}\Big) \right|^2_{\mathfrak{k}} = |\tilde{\varphi}_{n,t}|^2_{0,\mathcal{P}_n}. \end{aligned} \tag{4.9}$$

The expression of $|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n}$ in (4.9) also shows that

$$|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n} \text{ is } \sigma(\gamma(s); s \le \alpha_t) \text{ measurable.}$$

By [8, Theorem 2.16] and (4.5), ν_t^0 is absolute continuous with respect to μ_t on the σ-field $\sigma(\gamma(s); s \le \alpha_t)$ with bounded density F_t. By using this fact, we have, for $n \ge m$,

$$\begin{aligned} &\int_{\mathcal{L}_e(K)} \exp\Big(-\frac{|T^c|}{8}|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(h_{n,t}))|^2_{0,\mathcal{P}_n}\Big)\, d\nu_t^0(\gamma) \\ &\le \int_{\mathcal{L}_e(K)} \exp\Big(-\frac{|T^c|}{8}|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n}\Big)\, d\nu_t^0(\gamma) \\ &= \int_{\mathcal{P}_e(K)} \exp\Big(-\frac{|T^c|}{8}|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n}\Big) F_t\, d\mu_t(\gamma) \\ &\le R_t \int_{\mathcal{P}_e(K)} \exp\Big(-\frac{|T^c|}{8}|Ad_{\pi_n(\gamma)^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n}\Big)\, d\mu_t(\gamma) \\ &= R_t E\Big[\exp\Big(-\frac{|T^c|}{8}|Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n}\Big)\Big], \end{aligned}$$

where $R_t = \operatorname{esssup}_{\gamma \in \mathcal{P}_e(K)} F_t(\gamma)$. By (4.7), $[0, \alpha_t] \ni s \mapsto Ad_{\Sigma(t,s)^{-1}}(Ad_{l(t,s)^{-1}}(h(t,s)))$ is a semi-martingale of positive quadratic variation, and then we have

$$\lim_{n\to\infty} \exp\Big(-\frac{|T^c|}{8}|Ad_{\Sigma_{n,t}^{-1}}(Ad_{l_{n,t}^{-1}}(\tilde{h}_{n,t}))|^2_{0,\mathcal{P}_n}\Big) = 0 \quad \text{a.s.} \tag{4.10}$$

We have obtained (4.6), and the proof is completed. □

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