Hasegawa, I.
Osaka J. Math.
41 (2004), 891–909

THE LOWER BOUND OF THE w-INDICES OF NON-RIBBON SURFACE-LINKS

ISAO HASEGAWA

(Received April 18, 2003)

1. Preliminaries

Surface braids are defined by Rudolph [13], [14] and Viro [18], corresponding to oriented surface-links. Viro [18] and Kamada [10] proved the Alexander's theorem and Markov's theorem for the oriented surface-links and surface braids. In this paper, we will give the lower bound of *w-index* for non-ribbon oriented surface-links and determine values for some examples.

We review some definitions and notations for surface braids. Refer to [6] for more details. Let D_1^2, D_2^2 be 2-disks, and $pr_i\colon D_1^2 \times D_2^2 \to D_i^2 \quad (i = 1, 2)$ be the projection map to the i-th factor. Let Q_m be a set of m interior points of D_1^2.

DEFINITION (see [6]). A *surface braid of degree m* (or a *surface m-braid*) is a compact oriented 2-manifold S embedded properly and locally flatly in $D_1^2 \times D_2^2$ such that the restriction map $pr_2|_S\colon S \to D_2^2$ is a branched covering map of degree m and $\partial S = Q_m \times \partial D_2^2$. It is a *simple surface braid* if the associated branched covering is simple.

In what follows, we assume that surface braids are simple.

Two surface braids S and S' are *equivalent* if there is a fiber-preserving ambient isotopy of $D_1^2 \times D_2^2$ which carries S into S'. Here, we regard $D_1^2 \times D_2^2$ as a trivial D_1^2-bundle over D_2^2.

Identifying $D_1^2 \times D_2^2$ with a standard four ball in $\mathbf{R}^4$, we obtain a closed surface in $\mathbf{R}^4$ by attaching 2-disks onto the boundary of a surface braid S so that the 2-disks are included in the outside of four ball. We call it a *closure* of S.

Theorem 1 (Viro; Kamada [9] Theorem 1). *Any oriented surface-link in* $\mathbf{R}^4$ *is ambient isotopic to a closure of some simple surface braid of a certain degree.*

Take a bi-parameterization $D_1^2 \simeq I_1 \times I_2$. Let $\pi\colon D_1^2 \times D_2^2 \to I_1 \times D_2^2$ be a projection map. We can assume a surface braid S is generic with respect to π. Then the singularity set $\Sigma(\pi|_S)$ consists of double points, isolated triple points and isolated branch points. For each double-point arc, which connects two of isolated triple or branch

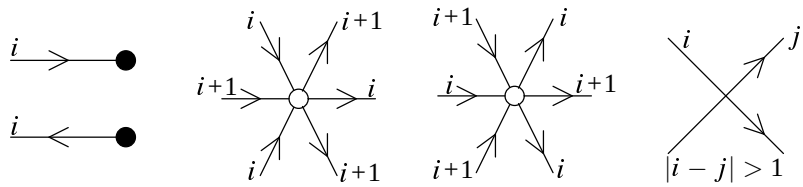

Fig. 1. Vertices and a crossing of a surface braid chart

points in the singularity set, we define a label and an orientation, called *Alexander numbering* (see [2] and [4]). The image of $\Sigma(\pi|_S)$ projected onto D_2^2 is considered to be an immersed graph with labels and orientations. Moreover, these labels and orientations ensure that we can reconstruct the surface braid from the immersed graph.

We assume that a graph may be empty or have closed edges without vertices called *hoops*.

DEFINITION (see [7]). A *surface braid chart of degree m* (or simply, *m-chart*) is an immersed one-, six-valent graph Γ in the interior of D_2^2 whose edges are oriented and labeled, with additional conditions for the immersion around six-valent vertices and the sigularity of the immersion as shown in Fig. 1. An edge is labeled an integer in $\{1, 2, \ldots, m-1\}$.

We call a one-valent (resp. six-) vertex a *black vertex* (resp. a *white vertex*), and a point of the singularity set is called a *crossing*.

REMARK. In [6], an m-chart is treated as an embedded graph. Crossings are also considered to be vertices.

For each m-chart Γ, we construct a surface braid as the above observation shows. We denote it S_Γ and call an *associated surface braid*. Kamada proved the following theorem.

Theorem 2 ([7] Theorem 14). *Let S be a simple surface braid of degree m. Then there exists a surface braid chart Γ of degree m such that S is equivalent to S_Γ.*

Corresponding to the equivalence relation of surface braids, local moves of m-charts, called *CI-move*, *CII-move and CIII-move*, are defined. Each move changes an m-chart Γ into another m-chart Γ' with outside of some 2-disk E unchanged, satisfying one of the following conditions:
(CI) There are no black vertices in $\Gamma \cap E$ and $\Gamma' \cap E$.
(CII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 2, where $|i - j| > 1$.
(CIII) $\Gamma \cap E$ and $\Gamma' \cap E$ are as in Fig. 3, where $|i - j| = 1$.

We say Γ and Γ' are *C-move equivalent* if they are related by a finite sequence of C-moves and ambient isotopies of D_2^2.

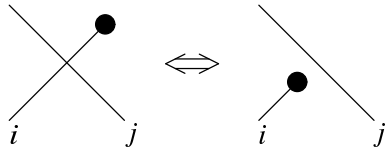

Fig. 2. Chart moves of type II

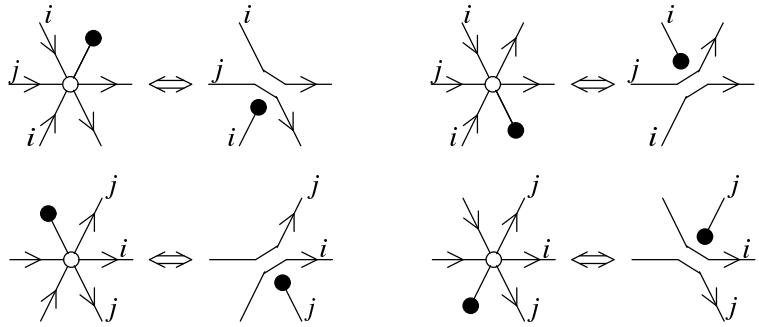

Fig. 3. Chart moves of type III

Theorem 3 ([12] Theorem 1). *Two charts of degree m represent the same, up to equivalence, simple surface braid if and only if they are C-move equivalent.*

At the end of this section, we see some notations and facts about an m-chart Γ.

A *free edge* of an m-chart is an edge both of whose endvertices are black vertex. An m-chart is *trivial* if it is C-move equivalent to the empty graph.

An m-chart is a *ribbon chart* if it is C-move equivalent to the one which has no white vertices.

We denote the number of white vertices by $\mathrm{w}(\Gamma)$, black vertices by $\mathrm{b}(\Gamma)$, crossings by $\mathrm{c}(\Gamma)$. The *w-index* of a surface braid S, denoted by $\mathrm{w}(S)$, is defined as the minimum number of $\mathrm{w}(\Gamma)$ such that S_Γ is equivalent to S. The *w-index* of an oriented surface-link L, denoted by $\mathrm{w}(L)$, is defined as the minimum number of $\mathrm{w}(S)$ such that the closure $\widehat{S}$ is ambient isotopic to L. The w-index is first defined in [8].

A surface-link L in $\mathbf{R}^4$ is a *ribbon surface-link* if L is ambient isotopic to a suface-link which is obtained from a trivial S^2-link by surgery along 1-handles.

Proposition 4 ([7] Lemma 2). *If an m-chart has no black vertices, then it is unknotted.*

Theorem 5 ([7] Proposition 20). *Assume L is an oriented surface-link. Then L is a ribbon surface-link if and only if* $\mathrm{w}(L)$ *is equal to* 0.

In this paper, we will prove following theorems.

Theorem 14. *Assume that an oriented surface-link L is non-ribbon. Then w-index* $\mathrm{w}(L)$ *is more than three.*

Theorem 16. *Assume that an S^2-link L is non-ribbon. Then the w-index* $\mathrm{w}(L)$ *is more than five.*

2. Rails and Genus lemma

Let Γ be an m-chart. For each edge e which is not a hoop of Γ, an initial (resp. terminal) vertex of e with respect to the orientation is denoted by $\mathrm{init}(e)$ (resp. $\mathrm{term}(e)$).

DEFINITION. Let Γ be an m-chart and $k \geq 2$ be an integer. A *rail with ends of length k* in Γ is a finite sequence of distinct edges $(e_1, e_2, \ldots, e_k)$ which satisfies following conditions:

- $\mathrm{init}(e_1)$ and $\mathrm{term}(e_k)$ are black vertices,
- $\mathrm{term}(e_i)$ and $\mathrm{init}(e_{i+1})$ are the same white vertex,
- e_i and e_{i+1} are diagonal with respect to $\mathrm{term}(e_i)$, for $i = 1, 2, \ldots, k-1$.

A *rail without ends of length k* in Γ is a finite sequence of distinct edges $(e_1, e_2, \ldots, e_k)$ which satisfies following conditions:

- $\mathrm{term}(e_i)$ and $\mathrm{init}(e_{i+1})$ are the same white vertex,
- e_i and e_{i+1} are diagonal with respect to $\mathrm{term}(e_i)$,

for $i = 1, 2, \ldots, k$, where indices are considered as mod k.

If $l = (e_1, e_2, \ldots, e_k)$ is a rail without ends, then $l' = (e_2, e_3, \ldots, e_1)$ is also a rail without ends. Thus, we consider that l and l' are the same.

Let Γ be an m-chart and S_Γ be an associated surface braid. Let $\Sigma(\pi|_{S_\Gamma})$ be the singularity set of the projection $\pi|_{S_\Gamma} : S_\Gamma \to I_1 \times D_2^2$. If $x \in \Sigma(\pi|_{S_\Gamma})$ is a double point, the inverse image $\pi^{-1}(x) \cap S_\Gamma$ consists of two points. Then one is higher than the other with respect to I_2, the direction of the projection π. We say that the higher point is *over* and the lower point is *under* and these signatures, 'over' or 'under', is refered to as *height information.*

Along a rail with ends l, we consider an immersion of the interval I into D_2^2. Then this immersion is uniquely lifted into $\Sigma(\pi|_{S_\Gamma})$, for Γ corresponds to the projected image of $\Sigma(\pi|_{S_\Gamma})$. Since consecutive two edges in l are diagonal, the lift is moreover lifted into S_Γ in two ways such that one is the over curve and the other is the under curve. We define a generic immersion ϕ_l of the circle S^1 into S_Γ as the immersion going along the over curve lift and coming back along the under curve lift.

For a rail without ends l, we also define generic immersions $\overline{\psi_l}$ and $\underline{\psi_l}$ in a similar way. We consider an immersion of the circle S^1 into D_2^2 along l. It is uniquely

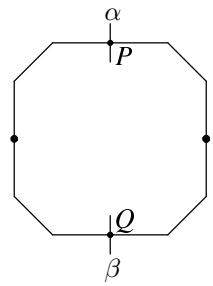

Fig. 4. Pull back of handle rail

lifted into $\Sigma(\pi|_{S_\Gamma})$ and moreover lifted into S_Γ in two ways. We define $\overline{\psi_l}$ as the lift which is higher and $\underline{\psi_l}$ as the other lift.

Let Γ be an m-chart and $l = (e_1, e_2)$ be a rail with ends of length 2. The rail l is said to be a *handle rail* if e_1 is the middle edge of a set of consecutive inward three edges incident to the white vertex term(e_1).

Lemma 6 (Genus Lemma. cf. [16]). *If an m-chart Γ contains a handle rail, the associated surface braid S_Γ contains a surface with positive genus.*

Proof. Consider the inverse image $\pi^{-1}(\Sigma(\pi|_{S_\Gamma})) \cap S_\Gamma$. Let l_0 be a handle rail. Then $\phi_{l_0}(S^1)$ is a simple closed curve in the surface braid S_Γ, and on $\phi_{l_0}(S^1)$ there are two points P and Q of three inverse image of the triple point which corresponds to the white vertex on l_0. By the definition of a handle rail, we see that P and Q are the highest and the lowest points in the three inverse image. Moreover $\phi_{l_0}(S^1)$ crosses arcs α and β, parts of the inverse image $\pi^{-1}(\Sigma(\pi|_{S_\Gamma})) \cap S_\Gamma$, transversely at P and Q and we see also that height information of α is over and that of β is under (see Fig. 4).

We show that the assumption that S_Γ does not include a surface with positive genus contradicts the following fact.

Fact. *Suppose two generic immersions from S^1 to a planar surface cross transversely. Then the number of intersection points is even.*

If we assume the above, α and β in Fig. 4 must be parts of the same generic immersion. But height information of α and β shows that it cannot be $\overline{\psi_l}$ or $\underline{\psi_l}$. Let l be the rail of length k such that $\phi_l(S^1)$ includes α and β, and parametrize $S^1 \subset \mathbf{C}$, the domain of ϕ_l, so that the following conditions are satisfied:

- $\phi_l(\xi^0)$ and $\phi_l(\xi^k)$ correspond to black vertices and
- $\phi_l(\xi^j)$ and $\phi_l(\xi^{2k-j})$ correspond to the same white vertices,

where $\xi = \exp\{(\sqrt{-1}\,\pi)/k\}$ and $j = 1, 2, \ldots, k-1$.

The image $\pi(\alpha)$ is the intersection of the top and the middle sheets around the triple point, thus if $\phi_l(\xi^j) = P$ on α, $\phi_l(\xi^{2k-j}) \neq Q$ because $\phi_l(\xi^{2k-j})$ is on the middle sheet and β is on the bottom sheet. Then there exists $j' \neq j$ such that $\phi_l(\xi^{2k-j'}) = Q$ on β and $\phi_l(\xi^{2k-j}) = \phi_l(\xi^{j'})$, for $\{P, Q, \phi_l(\xi^{2k-j}) = \phi_l(\xi^{j'})\}$ is the inverse image of the triple point corresponding to the white vertex on l_0. However, the restriction of ϕ_l to $J = \{\exp\{(\sqrt{-1}\,\theta\pi)/k\} \mid j' \leq \theta \leq 2k - j\}$ is assumed to be a generic immersion of S^1, which intersect $\phi_{l_0}(S^1)$ exactly at one point, contradicting the Fact. □

REMARK. First part of this proof is proved by Satoh, i.e. for the case $\overline{\psi_l}$ and $\underline{\psi_l}$. The author generalize the statement.

3. Braid monodromy

The m-strand braid group B_m is generated by standard generators $\sigma_1, \sigma_2, \ldots, \sigma_{m-1}$.

We identify the braid group with the fundamental group $\pi_1(F_m(D_1^2), Q_m)$, where $F_m(D_1^2)$ is the *configuration space* of the interior of D_1^2.

Let S be a surface braid of degree m and Σ be a set of branch points in D_2^2. Let y be a fixed point of $D_2^2 \backslash \Sigma$. For any loop $\gamma\colon (I, \partial I) \to (D_2^2 \backslash \Sigma, y)$, we define a loop $\tilde{\gamma}\colon (I, \partial I) \to (F_m(D_1^2), \tilde{y})$ such that

$$\tilde{\gamma}(t) = pr_2^{-1}(\gamma(t)) \cap S,$$

where $\tilde{y}$ is the inverse image of y by $(pr_2|S)^{-1}$. By choosing a path from Q_m to $\tilde{y}$, we obtain an element of the braid group $B_m = \pi_1(F_m(D_1^2), Q_m)$. This braid is called a *braid monodromy* of the loop γ and denoted by $\rho(\gamma)$. If we choose another path, then the resulting braid may be different, but the conjugacy class of them are the same. Moreover, if we fix a path from Q_m to $\tilde{y}$, then the map

$$\rho\colon \pi_1(D_2^2 \backslash \Sigma, y) \to B_m$$

becomes a homomorphism.

Let Γ be an m-chart and S_Γ be an associated surface braid. We consider the subset of $F_m(D_1^2)$ each of whose element consists of m points of $D_1^2 \simeq I_1 \times I_2$ such that I_1 coordinates of m points are distinct from each other. For each $y \in D_2^2 \backslash \Gamma$, the inverse image $\tilde{y} = pr|_2^{-1}(y) \cap S_\Gamma$ is in this subset. Since this subset is contractible, we choose a path from Q_m to $\tilde{y}$ uniquely determined up to homotopy for each $y \in D_2^2 \backslash \Gamma$. Then we can easily read the braid monodromy of a simple closed curve γ of $D_2^2 \backslash \Sigma$ which starts from y and intersects Γ transversely as the following way:

Assign each intersection a letter σ_i^ϵ where i is the label on the edge/loop and ϵ is ± 1 which is determined from the orientation of edge/loop and γ. By reading the letters along γ, we obtain a braid word, denoted by $W_\Gamma(\gamma)$, of the braid monodromy $\rho(\gamma)$ (see [11]).

EXAMPLE. Let Γ be an m-chart. Assume that there is an edge e satisfying init(e) = term(e). Then e separates D_2^2 into two components, a 2-disk and an annulus. One of the two components includes only one edge incident to init(e), and the other includes three edges incident to init(e). Let E be former. Now, we consider the monodromy along the simple closed curve γ which is obtained by perturbing ∂E into Int(E). Assume that the label of e is i and another label of the edges incident to init(e) is j, where $|i - j| = 1$. We get the braid word

$$W_\Gamma(\gamma) = \sigma_j^{\epsilon} \cdot \sigma_{i_1}^{\epsilon_1} \cdot \cdots \cdot \sigma_{i_p}^{\epsilon_p},$$

where $\sigma_{i_q}^{\epsilon_q}$ is obtained from the edge which intersect e. From the condition for the label at crossings, it satisfies $|i - i_q| > 1$.

Let S_m be the symmetric group and $s\colon B_m \to S_m$ be the canonical homomorphism. It is easy to see that $s([W_\Gamma(\gamma)])(j) \neq j$ and it follows that $s([W_\Gamma(\gamma)]) \neq id_{S_m}$. Thus γ is not null-homotopic in $D_2^2 \backslash \Sigma$, for the braid monodromy ρ is assumed to be a map from free homotopy class of simple closed curves to a set of conjugacy classes of B_m. Moreover γ is not homotopic to ∂D_2^2 in $D_2^2 \backslash \Sigma$ because $\partial D_2^2 \cap \Gamma = \emptyset$ and $W_\Gamma(\partial D_2^2)$ = empty. Therefore we conclude:

Lemma 7. *In the above situation, $\Gamma \cap E$ and $\Gamma \cap (D_2^2 \backslash E)$ must have a black vertex.*

Let SA_m be the subset of B_m consisting of conjugates of the standard generators and their inverses. Each element of SA_m is called a *band*.

It is easy to see that the permutation associated with a band is a transposition and the braid obtained by removing two strings which correspond to the transposition is the trivial braid. It is also clear that the closure of a band is the trivial link consisting of $m - 1$ components. Since the square of a band is conjugate to the square of one of standard generators and their inverses, it is easy to see that the permutation associated with the square of a band is the identity id_{S_m} and the braid obtained by removing from the square one of two strings which correspond to the transposition of the band is the trivial braid of $m - 1$ strings.

Let $\gamma\colon (I, \partial I) \to (D_2^2 \backslash \Sigma)$ be a simple closed curve. If γ encloses only one black vertex, γ is homotopic to a loop which goes along a path from y to the black vertex and turns around the black vertex and comes back along the same path (We call a loop of this form a *lasso*). Hence by reading the braid monodromy with Γ we see that $\rho(\gamma)$ becomes a band. Since $\Gamma \cap \partial D_2^2 = \phi$, the monodromy $\rho(\partial D_2^2)$ is identity id_{B_m}. Thus it is easy to see that if γ encloses all but one black vertex, the braid monodromy $\rho(\gamma)$ also becomes a band.

Let Γ be an m-chart and γ be a lasso which goes along a path α from the initial point of γ to a black vertex. Suppose that a braid word $w^{-1}\sigma_i^{\epsilon} w$ represents the monodromy $\rho(\gamma)$, which may be different from W_Γ. S. Kamada showed that there exists a

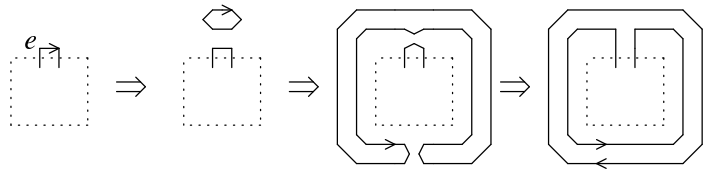

Fig. 5. Turning edge over

C-move which changes Γ into another m-chart Γ' with outside of a neighborhood of α unchanged such that the C-move does not move black vertices and $W_{\Gamma'}(\gamma)$ is equal to $w^{-1}\sigma_i^{\epsilon}w$ as a word.

Using this technique, the following lemma is shown easily.

Lemma 8 (cf. [6] Lemma 29.5). *A simple surface braid which has exact two branch points is ribbon.*

Proof. Suppose an m-chart Γ has two black vertices. We choose two lassos γ_1 and γ_2 as they are disjoint except their initial point. Then it holds that $\rho(\gamma_1)\rho(\gamma_2) = id_{B_m}$. Thus we choose a representative of $\rho(\gamma_1)$ of the form $w^{-1}\sigma_i^{\epsilon}w$ and change Γ into Γ' such that $W_{\Gamma'}(\gamma_1) = w^{-1}\sigma_i^{\epsilon}w$ and $W_{\Gamma'}(\gamma_2) = w^{-1}\sigma_i^{-\epsilon}w$. We can find a C-move which generate a free edge in Γ' and Γ' becomes a ribbon chart. □

4. Surface braid charts as graphs

An m-chart Γ is an immersed graph into the interior of D_2^2. But we can consider an m-chart as an immersed graph into S^2, for there exists a sequence of C-moves such that it turns an outermost edge over to the otherside (see Fig. 5). The remaining outermost hoop is not affect whether Γ is a ribbon chart or not, thus it will be ignored in this paper.

REMARK. There exists the notion of "conjugation" for surface braids. In view of m-charts, it is to add or delete an outermost hoop to an m-chart Γ. It does not change the ambient isotopy class of a closure of a surface braid.

DEFINITION. In an m-chart Γ, a pair of edge (e_1, e_2) is called a *canceling pair* if it satisfies the following conditions:

1. the endvertices of e_1 and the endvertices of e_2 are the common two white vertices $\{W_1, W_2\}$
2. e_1 is left to e_2 at W_1 and e_2 is left to e_1 at W_2
3. the edge which is left to e_1 at W_1 and the edge which is left to e_2 at W_2 are oriented to the same. (both are oriented inward or outward)

If (e_1, e_2) is a canceling pair, $e_1 \cup e_2$ separates D_2^2 into two components, a 2-disk and

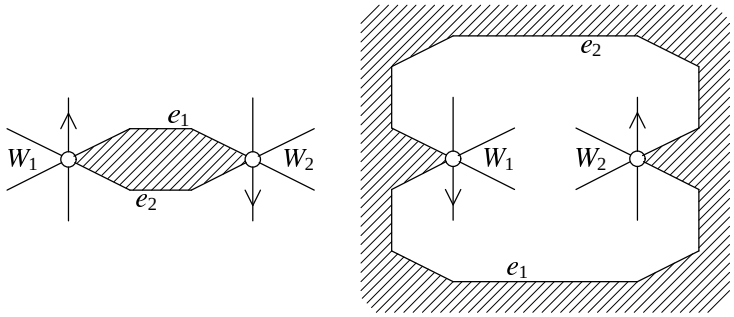

Fig. 6. Canceling pair

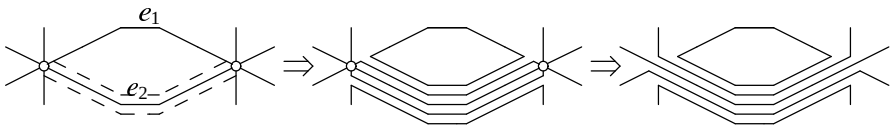

Fig. 7. Cancel white vertices

an annulus. A *canceling disk* is the one of two components which includes no edge incident to W_1. Here, regarding D_2^2 as in S^2, an annulus which includes ∂D_2^2 is called to be a 'disk' (see Fig. 6).

Lemma 9. *If an m-chart* Γ *has a canceling pair* (e_1, e_2) *whose canceling disk* E *includes at most one white vertex, then there exists an m-chart* Γ' *such that*

- Γ *is C-move equivalent to* Γ', *and*
- $\mathrm{w}(\Gamma') = \mathrm{w}(\Gamma) - 2$

Proof. We show by three steps that two white vertices, endvertices of a canceling pair, are cancelled by C-move while a new white vertex is not generated. Assume that i, $i+1$ are the label of the edges e_1, e_2 respectively.

If each label of edges which intersect the edge e_2 transversely is not equal to $i-1$, we cancel init(e_2) and term(e_2) by CI-move as in Fig. 7.

If there exists no white vertex in E, there exist three cases of arcs in E which intersect e_2 transversely:

(i) Arcs one of whose endvertices is on e_1 and the other is on e_2
(ii) Arcs one of whose endvertices is black and the other is on e_2
(iii) Arcs both of whose endvertices is on e_2

The case (i) is ignored. Arcs of the case (ii) are able to be pulled out by CII-move. Let X be the set of arcs of the case (iii). We define a partial order of X as follows:

For each arc α in X, there exists a unique 2-disk D_α surrounded by $\alpha \cup e_2$.

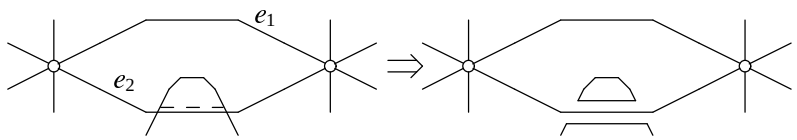

Fig. 8. Delete crossings with arcs in X

Thus we define a partial order $\alpha > \alpha'$ is defined by $D_\alpha \subset D_{\alpha'}$.
Then we delete crossings of e_2 and arcs in X according to this order as in Fig. 8. Hence two white vertices are cancelled as in Fig. 7.

Assume that there exists one white vertex in E. Since labels of edges incident to the white vertex differ by 1, these edges cannot intersect one of the canceling pair (e_1, e_2). Applying the previous argument to the edge e_1 or the edge e_2, we cancel two white vertices. □

Corollary 10. *Let Γ be an m-chart with four white vertices. If Γ has two canceling disks, then Γ is C-move equivalent to an m-chart with two white vertices.*

Proof. At least two white vertices are endvertices of canceling pairs. If at least three white vertices are endvertices of canceling pairs, it is easy to see that we can apply Lemma 9. If only two white vertices are endvertices of canceling pairs, the intersection of canceling disks consists of white vertices or edges. Thus we can apply Lemma 9 regardless of the placement of the remaining two white vertices. □

5. Main Theorem

DEFINITION. An m-chart Γ is a *C_{23}-minimal chart* if Γ satisfies a following condition:

there exists no m-chart Γ' which is obtained from Γ by at most one CII-move or one CIII-move and satisfies the inequality $\mathrm{w}(\Gamma') + \mathrm{c}(\Gamma') < \mathrm{w}(\Gamma) + \mathrm{c}(\Gamma)$.

Lemma 11. *For any m-chart Γ, there exists a C_{23}-minimal chart Γ' such that*
- *Γ is C-move equivalent to Γ', and*
- $\mathrm{w}(\Gamma) \geq \mathrm{w}(\Gamma')$.

Proof. If Γ is not C_{23}-minimal, there exists Γ' which is obtained from Γ by one CII-move or CIII-move and satisfies $\mathrm{w}(\Gamma') + \mathrm{c}(\Gamma') < \mathrm{w}(\Gamma) + \mathrm{c}(\Gamma)$. Moreover, CII-move (resp. CIII-) keeps $\mathrm{w}(\Gamma)$ (resp. $\mathrm{c}(\Gamma)$) unchanged. Thus it satisfies $\mathrm{w}(\Gamma) \geq \mathrm{w}(\Gamma')$. We can show the lemma by induction with $\mathrm{w}(\Gamma) + \mathrm{c}(\Gamma)$. □

The edge whose endvertices are black and white is called *a bw-edge*. Then a

white vertex W in C_{23}-minimal charts is one of following three types:
(A) there is no bw-edge incident to W,
(B) there is exact one bw-edge incident to W and it is a middle edge of consecutive three edges which are oriented in the same direction,
(C) there are exact two bw-edge incident to W and they are middle edges.

A white vertex satisfying the condition (A) (resp. (B) or (C)) is called an *A-type* white vertex (resp. *B-type* or *C-type*).

For any m-chart Γ, the number of A-type white vertex is denoted by $w_A(\Gamma)$, B-type is denoted by $w_B(\Gamma)$ and C-type is denoted by $w_C(\Gamma)$. And we denote by $b_0(\Gamma)$ the number of black vertices which is not an end of a free edge.

REMARK. If Γ is C_{23}-minimal, then it holds that $b_0(\Gamma) = w_B(\Gamma) + 2w_C(\Gamma)$.

Lemma 12. *For any m-chart Γ, $b_0(\Gamma)$ is even.*

Proof. $b_0(\Gamma)+6w(\Gamma)$ is equal to a twice of the number of edges except free edges and hoops. □

Lemma 13 (cf. [7]). *Let Γ be an m-chart. If it satisfies $b_0(\Gamma) \leq 2$ then it is a ribbon chart.*

Proof. Note that $b_0(\Gamma)$ is even. The case $b_0(\Gamma) = 0$ follows the argument of Lemma 19 in [7]. The case $b_0(\Gamma) = 2$ is similar. At first, we sweep free edges away from a subgraph of Γ which includes all of white vertices. Then this subgraph includes only two black vertices and is C-move equivalent to an m-chart which includes no white vertex from Lemma 8. Thus Γ is a ribbon chart even if it has free edges. □

Theorem 14. *Assume that an oriented surface-link L is non-ribbon. Then w-index $w(L)$ is more than three.*

Proof. We show that any chart Γ with $w(\Gamma) \leq 3$ is ribbon. By Lemma 11 and 13, we may assume that Γ is C_{23}-minimal and $b_0(\Gamma) > 2$. Since $w_A(\Gamma) + w_B(\Gamma) + w_C(\Gamma) = w(\Gamma) \leq 3$ and $w_B(\Gamma) + 2w_C(\Gamma) = b_0(\Gamma) > 2$ and since $b_0(\Gamma)$ is even, we have the following cases:

$$(w_A(\Gamma), w_B(\Gamma), w_C(\Gamma)) = (0, 0, 2), (1, 0, 2), (0, 2, 1), (0, 0, 3)$$

But for the case $(1, 0, 2)$, $(0, 2, 1)$ and $(0, 0, 3)$, the BW-orientation shows that there is no such chart (see [15]). For the case $(0, 0, 2)$, essentially, there exists only one way to immerse Γ as shown in Fig. 9. It clearly has two canceling disks and becomes ribbon by Corollary 10. Thus the theorem is proved. □

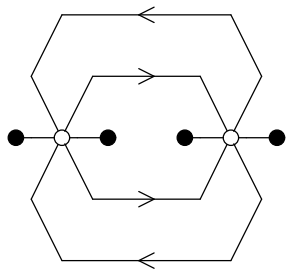

Fig. 9. Immersion for $w_C(\Gamma) = 2$

Corollary 15. *There exists a surface-link whose w-index is four.*

Proof. In [3], a non-ribbon surface-link L which is named *twisted Hopf* 2*-link* is presented. In [1], there is a chart representing L with four white vertices. Thus $w(L)$ is four. □

REMARK. The twisted Hopf 2-link is a pseudo-ribbon surface-link, i.e. it is described by a surface diagram without triple points and branch points.

Theorem 16. *Suppose L is a non-ribbon S^2-link. Then w-index $w(L)$ is more than five.*

Proof. We show that any chart Γ with $w(\Gamma) \leq 5$ is ribbon if the associated surface braid S_Γ is a planar suface. By Lemma 11 and 13, we may assume that Γ is C_{23}-minimal and $b_0(\Gamma) > 2$. Moreover by Lemma 6, we may assume that $w_C(\Gamma) = 0$. Since $w_A(\Gamma) + w_B(\Gamma) = w(\Gamma) \leq 5$ and $w_B(\Gamma) = b_0(\Gamma) > 2$ and since $b_0(\Gamma)$ is even, we have the following cases:

$$(w_A(\Gamma), w_B(\Gamma), w_C(\Gamma)) = (0, 4, 0), (1, 4, 0)$$

But in the case $(1, 4, 0)$, the BW-orientation shows that there is no such chart. Thus, by showing that any chart in the case $(0, 4, 0)$ is a ribbon chart, we can prove the theorem. This will be done in the next section. □

Corollary 17. *The w-index of* 2*-twist spun trefoil is equal to six.*

Proof. In [6], there is a chart representing 2-twist spun trefoil K with six white vertices. 2-twist spun trefoil is not ribbon, thus $w(K)$ is six. □

REMARK. In [17], it is shown that the triple point number of 2-twist spun trefoil is four, i.e. it is represented by a surface diagram with four triple points and can not

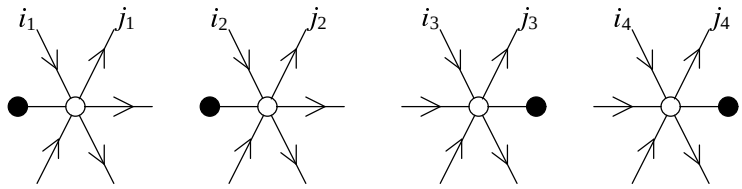

Fig. 10. Labels and orientations of four white vertices.

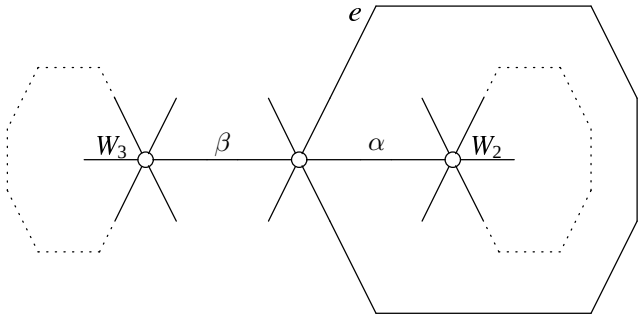

Fig. 11. An edge satisfying init(e) = term(e)

be represented by surface diagram with less than four triple points.

6. The last part of the proof of Theorem 16

Let Γ be a C_{23}-minimal m-chart which has four white vertices labeled and oriented as in Fig. 10. We assume that the associated surface braid S_Γ is a planar surface. Counting the number of edges for labels and orientations, the following equation holds:

$$\{i_1, i_2, i_3, i_3, i_4, i_4\} = \{j_1, j_1, j_2, j_2, j_3, j_4\}, \tag{1}$$

where the equation holds including multiplicity. Note that the equalities $|i_k - j_k| = 1$ hold for $k = 1, 2, 3, 4$.

We show that Γ is a ribbon chart through four steps.

STEP 1. Γ has no edge whose endvertices are the same white vertex.

Proof. Assume that e satisfies init(e) = term(e) and let W_1 be init(e). Then e separates D_2^2 into two components and the component including only one edge incident to init(e) is denoted by E. Let α and β be edges incident to W_1 which are not diagonal to the edge e (see Fig. 11). Since W_1 is B-type white vertex, α and β are not a bw-edge. Let W_2 and W_3 be another endvertices of α and β respectively. W_2 in E and

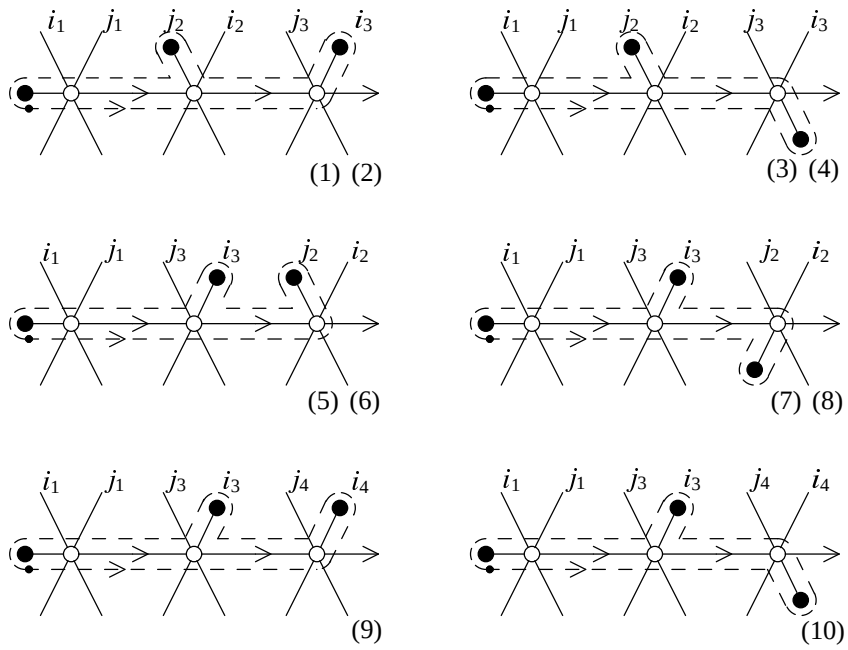

Fig. 12. Rails of length more than three.

W_3 in $D_2^2 \backslash E$.

Now we observe the place of W_4, either in E or out of E. If W_4 is in E then edges which are incident to W_3 and whose label is the same as β must be connected each other. Thus there exists another edge e' which satisfies init(e') = term(e'). But the component E' which is separated by e' including only one edge which is incident to W_3 has no black vertex, for the edge included in E' is not a bw-edge and the other bw-edge are out of E. Thus it contradicts to Lemma 7. It is the same in the case when W_4 is out of E and there is no e satisfying init(e) = term(e). □

STEP 2. Suppose that the rail which starts from the bw-edge incident to W_1 has the length larger than three. Then Γ is a ribbon chart.

Proof. By Step 1, first three white vertices on the rail are distinct each other. We examine two cases of positions of the bw-edge incident to the second white vertex as in Fig. 12. If the bw-edge is nearer to the first white vertex, then the orientation of the bw-edge is from black to white and it holds that $i_1 = i_2$. The equation (1) shows that $i_1 = i_2 = j_3 = j_4$. Hence we assume that the label of the third edge in the rail is i_3 and two cases of positions of the bw-edge incident to the third white vertex must be considered. Moreover we consider whether the labels j_1 and j_2 are the same or not.

If the bw-edge is nearer to the third white vertex, then the orientation is from white to black and we assume that $i_1 = i_3$. The equation (1) shows that $i_1 = i_3 =$

Table 1. Ten cases of the monodromies γ

	monodromy of γ	result
(1)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_q^{-1}\sigma_{q+1}^{-1}\ W\ \sigma_{q+1}\sigma_q\sigma_{q+1}^{-1}\sigma_{q+1}^{-1}\ W^{-1}\ \sigma_{q+1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	σ_q^{-1}
(2)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_{q+2}^{-1}\sigma_{q+1}^{-1}\ W\ \sigma_{q+1}\sigma_{q+2}\sigma_{q+1}^{-1}\sigma_{q+1}^{-1}\ W^{-1}\ \sigma_{q+1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	σ_q^{-1}
(3)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_q^{-1}\sigma_{q+1}^{-1}\ W\ \sigma_{q+1}^{-1}\sigma_{q+1}^{-1}\sigma_q\sigma_{q+1}\ W^{-1}\ \sigma_{q+1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	NOT
(4)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_{q+2}^{-1}\sigma_{q+1}^{-1}\ W\ \sigma_{q+1}^{-1}\sigma_{q+1}^{-1}\sigma_{q+2}\sigma_{q+1}\ W^{-1}\ \sigma_{q+1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	NOT
(5)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_q\sigma_{q+1}\ W\ \sigma_{q+1}^{-1}\sigma_q^{-1}\sigma_{q+1}^{-1}\sigma_q\ W^{-1}\ \sigma_q\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	σ_q^{-1}
(6)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_{q+2}\sigma_{q+1}\ W\ \sigma_{q+1}^{-1}\sigma_{q+2}^{-1}\sigma_{q+1}^{-1}\sigma_{q+2}\ W^{-1}\ \sigma_{q+2}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	σ_q^{-1}
(7)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_q\sigma_{q+1}\ W\ \sigma_q\sigma_{q+1}^{-1}\sigma_q^{-1}\sigma_{q+1}^{-1}\ W^{-1}\ \sigma_q^{-1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	NOT
(8)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_{q+2}\sigma_{q+1}\ W\ \sigma_{q+2}\sigma_{q+1}^{-1}\sigma_{q+2}^{-1}\sigma_{q+1}^{-1}\ W^{-1}\ \sigma_{q+2}^{-1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	NOT
(9)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_q\sigma_{q+1}\ W\ \sigma_{q+1}\sigma_q\sigma_{q+1}^{-1}\sigma_{q+1}^{-1}\ W^{-1}\ \sigma_q^{-1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	NOT
(10)	$\sigma_{q+1}\sigma_q^{-1}\ V\ \sigma_q\sigma_{q+1}\ W\ \sigma_{q+1}^{-1}\sigma_{q+1}^{-1}\sigma_q\sigma_{q+1}\ W^{-1}\ \sigma_q^{-1}\ V^{-1}\ \sigma_q^{-1}\sigma_{q+1}$	b

$j_2 = j_4$, $i_2 = j_3$ and $i_4 = j_1$, for the first equation induces that $\{i_2, i_4, i_4\} = \{j_1, j_1, j_3\}$. Hence the label of the third edge is i_2 or i_4 and four cases of positions of the bw-edge incident to the third white vertex is must be considered. For the case that the label of the third edge is i_2, we consider whether the labels j_1 and j_3 are the same or not.

Now we read the monodromy of the simple closed curve γ for each of ten cases. In the Table 1, V is the word obtained from the crossings on the second edge in the rail and W is the word from the crossings on the third edge. Let b be a braid word $\sigma_{q+1}^{-1}\sigma_q\sigma_{q+1}\sigma_q^{-1}\sigma_{q+1}$.

The monodromy must be a band for each case, for there exists exact one black vertex outside of the simple closed curve γ. But there exist five cases such that the monodromy cannot become a band, the mark 'NOT' in the Table 1. The reason why the monodromy of the case marked 'NOT' cannot be a band is that the closure of the monodromy includes a non-trivial link. This is showed as the following way: For the case (8) in the Table 1. At first, we see that the associated permutation is a transposition of the type $(q+1, r)$, where r is not equal to q, $q+1$, $q+2$. Hence q-th and $q+2$-th strings are the part of the trivial braid obtained by removing $q+1$-th and r-th strings. Second, we see that the permutation $s(W^{-1})$ associated with W^{-1} preserves $q+1$ so that q-th string becomes the part of the trivial braid and moreover W^{-1} must include the positive full twist of two strings corresponding to q-th and $q+2$-th strings. This full twist, in turn, becomes negative full twist of two strings correspoding to $q+2$-th and r-th strings in the word W. Hence the link consisting of $q+1$-th, $q+2$-th and r-th strings is ambient isotopic to the closure of the 3-braid $\sigma_1\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_2^{-1}\sigma_1$. This is a Whitehead link and it contradicts that the monodromy is a band. For the cases (3), (4), (7) and (9), similar arguments show that the monodromy includes a non-trivial

link.

In the other five cases in the Table 1, the monodromies are determined as the following way: For the case (10). At first, we see that the associated transposition is $(q, q+1)$. Then the permutation $s(W^{-1})$ preserves $q+2$, so that the end of $q+1$-th string becomes q. Moreover the permutation $s(V)$ must preserve q, for if otherwise, the monodromy includes a Whitehead link. Second, we see that $q+2$-th string is separated in W so that the braid obtained by removing q-th and $q+1$-th string becomes a trivial braid. Here, the word *separeted* means that W is equivalent to a braid word which does not include σ_{q+1}^{ϵ} nor σ_{q+2}^{ϵ}. Hence the monodromy is equivalent to the word $\sigma_{q+1}\sigma_q^{-1}V\sigma_q\sigma_{q+1}^{-1}\sigma_q\sigma_{q+1}\sigma_q^{-1}V^{-1}\sigma_q^{-1}\sigma_{q+1}$. To see that q-th string is separated in V, we examine the square of the monodromy. Then we get the equation

$$[\sigma_q^{-1}V\sigma_q^2V^{-1}\sigma_q^2V\sigma_q^{-2}V^{-1}\sigma_q^{-1}] = id_{B_m},$$

for the braid obtained by removing $q+1$-th string from the square of the band becomes a trivial braid. This equation implies that $V\sigma_q^2V^{-1}$ and σ_q^2 commute with each other. Recall that the permutation $s(V)$ preserves q and $q+1$. The following lemma is easily proved by the argument similar to the Lemma 3.2 in [5].

Lemma 18 ([5] Lemma 3.2). *Let β be a braid satisfying the following two conditions*:

- *$s(\beta)$ stabilizes $\{q, q+1\}$, and*
- *$\beta\sigma_q^2\beta^{-1}$ commutes with σ_q^2.*

Then β commutes with σ_q.

Outline of proof. Since $\beta\sigma_q^2\beta^{-1}$ commutes with σ_q^2, $\beta\sigma_q^2\beta^{-1}$ has a (q,q)-band. Moreover the assumption that $s(\beta)$ stabilizes $\{q, q+1\}$ shows that β itself has a (q,q)-band, for the interval $[q, q+1]$ is the only arc stable by the action of σ_q^2 among the arcs of D^2 whose ends are $\{q, q+1\}$ and interior is included in $D^2\backslash Q_m$. □

At last we see that the monodromy is equal to

$$\sigma_q\sigma_{q+1}\sigma_q^{-1}\sigma_q^{-1}\sigma_{q+1} = \sigma_{q+1}^{-1}\sigma_q\sigma_{q+1}\sigma_q^{-1}\sigma_{q+1}.$$

For the cases (1), (2), (5) and (6), much easier arguments determine the monodromy, without the observation of the square of a band. Now we can find the C-move for each of five cases that construct a free edge from two of black vertices which are endvertices of bw-edges. Note that each of V and W commutes with σ_q and σ_{q+1}. By sweeping this free edge out, we see that Γ is a ribbon chart from Lemma 13. □

Now we suppose that Γ has two rails with ends of length three as in Fig. 13. We call each of the arcs in the following figures one of whose endvertices is described

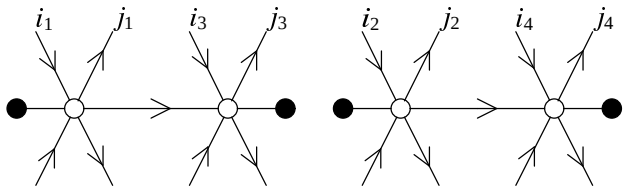

Fig. 13. Two rails of length three.

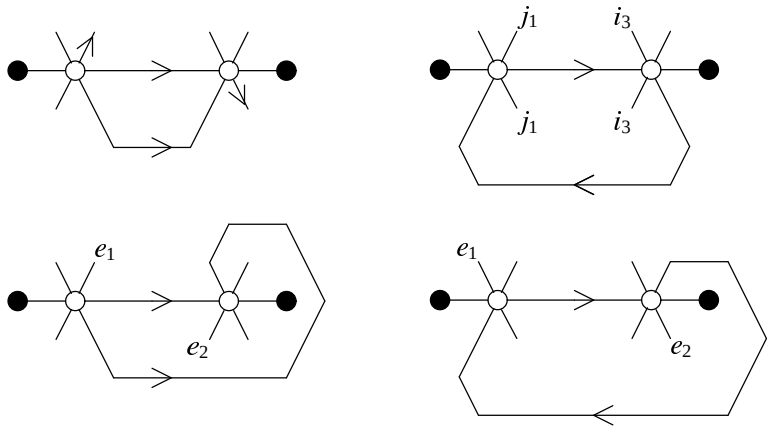

Fig. 14. Two edges incident to W_1 and W_3.

an *open edge*. We note that the affection of each of four white vertices is even when counting the number of open edges for labels and orientations.

STEP 3. If there exists another edge incident to W_1 and W_3, then Γ is a ribbon chart.

Proof. We consider only four cases in Fig. 14. For the case (1), there exists a canceling disk and if there exists at least one white vertex then Γ is C-move equivalent to an m-chart with two white vertices from Lemma 9. Otherwise, there exist no edge adjacent to one of $\{W_1, W_3\}$ and one of $\{W_2, W_4\}$. Hence two open edge marked e are connected with each other and Γ is ribbon. For the case (2), we see that two open edges marked e_1 or two open edges marked e_2 are connected with each other, for the placement of W_2 and W_4 is limited by counting the number of open edges. Hence this case turns to the case (1). On the other hand, since the affection of white vertices for open edges is even, we cannot connect the open edges which have the same labels marked e_1 or e_2. Thus there exist no m-chart which includes the part such as the case (3) or (4). □

STEP 4. The remaining case.

Proof. We assume that there exist no edge incident to W_1 and W_3 except one in Fig. 13 and that there exist no edge incident to W_2 and W_4 except one in Fig. 13. By observing labels and orientations of open edges in Fig. 13, we see that if there exists an edge incident to W_1 and W_2 then all of four open edge are incident to W_1 and W_2. Then the placement of four edges is similar to the chart in Theorem 14 (Fig. 9) and Γ has two canceling disks. Hence Γ is ribbon from Lemma 9. We prove the main theorem. □

ACKNOWLEDGEMENT. The author would like to thank Yukio Matsumoto, Seiichi Kamada and Shin Satoh for helpful suggestions.

References

[1] H. Aiba and T. Nagase: *On the closure of a surface braid, represented by a* 4*-chart with at most two crossings in one house*, preprint.
[2] J.S. Carter, S. Kamada and M. Saito: *Alexander numbering of knotted surface diagrams*, Proc. Amer. Math. Soc. **128** (2000), 3761–3771.
[3] J.S. Carter, S. Kamada, M. Saito and S. Satoh: *Bordism of unoriented surfaces in* 4*-space*, Michigan Math. J. **50** (2002), 575–591.
[4] J.S. Carter and M. Saito: Knotted Surfaces and Thier Diagrams, Math. Surv. Mono. **55**, Amer. Math. Soc., 1998.
[5] R. Fenn, D. Rolfsen and J. Zhu: *Centralisers in the braid group and singular braid monoid*, Enseign. Math. **42** (1996), 75–96.
[6] S. Kamada: Braid and Knot Theory in Dimension Four, Math. Surv. Mono. **95**, Amer. Math. Soc., 2002.
[7] S. Kamada: *Surfaces in* $\mathbf{R}^4$ *of braid index three are ribbon*, J. Knot Theory Ramifications **1** (1992), 137–160.
[8] S. Kamada: 2*-dimensional braids and chart descriptions*, 'Topics in knot theory' (Erzurum, 1992), 277–287, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 399, Kluwer Acad. Publ., Dordrecht, 1993.
[9] S. Kamada: *A characterization of groups of closed orientable surfaces in* 4*-space*, Topology **33** (1994), 113–122.
[10] S. Kamada: *Alexander's and Markov's theorems in dimension four*, Bull. Amer. Math. Soc. (N.S.) **31** (1994), 64–67.
[11] S. Kamada: *On braid monodromies of non-simple braided surfaces*, Math. Proc. Camb. Phil. Soc. **120** (1996), 237–245.
[12] S. Kamada: *An observation of surface braids via chart description*, J. Knot Theory Ramifications **5** (1996), 517–529.
[13] L. Rudolph: *Braided surfaces and Seifert ribbons for closed braids*, Comment. Math. Helv. **58** (1983), 1–37.
[14] L. Rudolph: *Special positions for surfaces bounded by closed braids*, Rev. Mat. Iberoamericana **1** (1985), 93–133.
[15] S. Satoh: *Lifting a generic surface in* 3*-space to an embedded surface in* 4*-space*, Topology Appl. **106** (2000), 103–113.
[16] S. Satoh: *No* 2*-knot has triple point number one, two nor three*, preprint.

[17] S. Satoh and A. Shima: *The* 2*-twist-spun trefoil has the triple point number four*, Trans. Amer. Math. Soc. **356** (2004), 1007–1024.
[18] O.Ya. Viro: Lecture given at Osaka City University, September, 1990

Graduate School of Mathematical Sciences
University of Tokyo
Komaba 3-8-1, Meguro
Tokyo 153-8914, Japan
e-mail: haseisao@ms.u-tokyo.ac.jp