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CROSSING CHANGES FOR PSEUDO-RIBBON SURFACE-KNOTS

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1. Introduction

In classical knot theory, it is well-known that any knot diagram can be deformed into a diagram of a trivial knot by some crossing changes. This fact plays an important role to compute various knot invariants defined by skein relations. We consider a similar problem on surface-knot theory and obtain a partial answer to it.

A *surface-knot* F is a connected closed surface embedded locally flatly in $\mathbb{R}^4$. Throughout this paper, we assume that surface-knots are oriented. Let $\pi\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the natural projection. A *pseudo-ribbon surface diagram* is a projection image $\pi(F)$ whose singularity set consists of only double points and has crossing information with respect to the natural projection. We denote a surface-knot recovered from a diagram D by F_D. We prove the following theorem.

Theorem 1.1 (Theorem 3.3). *Let D be a pseudo-ribbon surface diagram. We can deform D into D' by some crossing changes on D so that $\pi_1(\mathbb{R}^4 - F_{D'})$ becomes isomorphic to $\mathbb{Z}$.*

REMARK 1.2. We can also consider a similar problem on higher dimensional knot theory. We introduce two related consequences on this problem.

Let $p\colon \mathbb{R}^{n+2} = \mathbb{R}^{n+1} \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}$ be the natural projection. An *S^n-knot* K is an n-dimensional sphere embedded locally flatly in $\mathbb{R}^{n+2}$. An S^n-knot K is said to be *trivial* if K bounds an $(n+1)$-disk in $\mathbb{R}^{n+2}$.

E. Ogasa [5] proved that there exists an S^n-knot K ($n \geq 3$) having the following properties.

1. The singularity set of $p(K)$ consists of only double points and is homeomorphic to a disjoint union of $(n-1)$-dimensional tori.
2. The image $p(K)$ is not the projection image of any trivial S^n-knot.

K. Yoshida [9] proved the following result for an S^n-knot K ($n = 2$ or $n \geq 5$). If the singularity set of $p(K)$ consists of only double points and is homeomorphic to a disjoint union of $(n-1)$-dimensional spheres, then $p(K)$ is the projection image of some trivial S^n-knot.

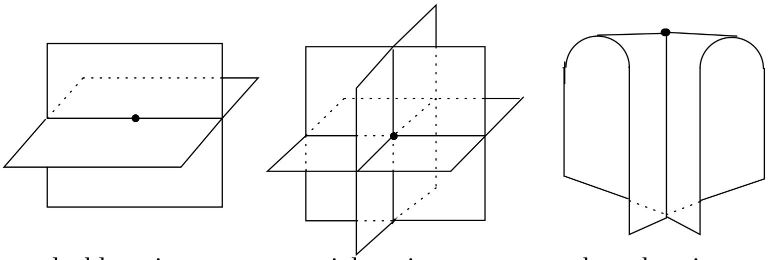

Fig. 1. The singularity set of a projection

This paper is organized as follows. In Section 2, we review some basic notions of surface-knots. In Section 3, we state the main theorem of this paper (Theorem 3.3). In Section 4, we introduce *pseudo-ribbon graph*s and give a way to construct a pseudo-ribbon graph from a pseudo-ribbon surface diagram. In section 5, we prove the key proposition (Proposition 5.5) for the proof of Theorem 3.3. In Section 6, we give the proof of the main theorem (Theorem 3.3).

2. Basic notations of surface-knots

In this section, we review some basic notions of surface-knots from the viewpoint of the diagrammatic theory. See [1] for more details.

Two surface-knots F and F' are said to be *equivalent* if they are related by a (smooth or piecewise-linear) ambient isotopy of $\mathbb{R}^4$. A surface-knot is said to be *trivial* if it is equivalent to the boundary of a handlebody in $\mathbb{R}^4$. F. Hosokawa and A. Kawauchi [2] proved that the boundary of a handlebody is unique up to ambient isotopies of $\mathbb{R}^4$.

Let $\pi\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the projection defined by $\pi(x_1, x_2, x_3, t) = (x_1, x_2, x_3)$. The closure of the self-intersection set of the projection image $\pi(F)$ is called the *singularity set*. The image $\pi(F)$ is said to be *generic* if the singularity set of $\pi(F)$ consists of double points, isolated triple points, and isolated branch points. See Fig. 1. By a slight perturbation if necessary, we may assume that $\pi(F)$ is generic.

The singularity set of the generic projection image $\pi(F)$ is regarded as a disjoint union of graphs with 1-, 6-valent vertices (which correspond to isolated branch points and isolated triple points, respectively) and circles without self-intersections called *hoop*s. An *edge* of $\pi(F)$ is an edge of their graphs.

We find in $\pi(F)$ two sheets intersecting along each edge or hoop, one of which is higher than the other with respect to the t-coordinate. They are called an *over-sheet* and an *under-sheet* along the edge or the hoop, respectively. A *surface diagram* of F is the generic projection image $\pi(F)$ with such crossing information, and is denoted by D_F. A method to indicate crossing information is to split along the edges

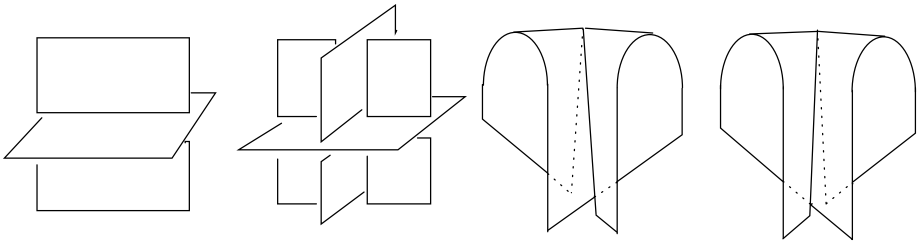

Fig. 2. Crossing information on the singularity set

and hoops on under-sheets. See Fig. 2.

We give every surface diagram an orientation inherited from that of the original surface-knot, and often use a normal direction of the sheets to indicate it. See Fig. 3.

REMARK 2.1. The generic projection image $\pi(F)$ cannot recover the original surface-knot F, but the surface diagram D_F can. We denote a surface-knot recovered from a surface diagram D by F_D.

Theorem 2.2 (Wirtinger presentation of a knot group) ([3], [7]). *Let D_F be a surface diagram of a surface-knot F. We label the connected components obtained from D_F by splitting along the edges and hoops on the under-sheets by $x_0, x_1, \dots, x_s$, where $s+1$ is the number of the components. The knot group $\pi_1(\mathbb{R}^4 - F)$ has the following presentation*

$$\pi_1(\mathbb{R}^4 - F) = \langle\, x_0, \dots, x_s \mid r_1, \dots, r_n \,\rangle.$$

Here each x_i is regarded as a meridian element of the knot group and the knot group is generated $x_0, x_1, \dots, x_s$. Each edge or hoop of the singularity set of the underlying surface of D_F induces a relator. Precisely, if x_{i_2} is the label of the over-sheet and x_{i_1} (*resp. x_{i_3}*) *is of the under-sheet in back* (*resp. front*) *of the over-sheet with respect to the orientation of D_F, then a relator of the knot group is of the form*

$$r_i = x_{i_3}^{-1} x_{i_2}^{-1} x_{i_1} x_{i_2} \quad (1 \leq i \leq n),$$

where n is the number of edges and hoops. See Fig. 3*.*

3. Main theorem

A surface-knot F is said to be a *pseudo-ribbon surface-knot* if there exists a surface-knot F' such that F' is equivalent to F and the singularity set of the generic projection image $\pi(F')$ consists of only hoops. A surface diagram D is said to be a *pseudo-ribbon surface diagram* if the singularity set of D consists of only hoops.

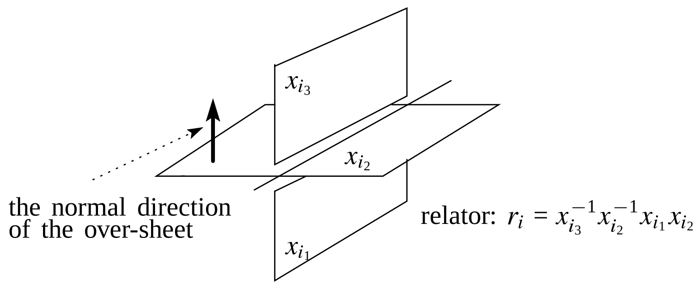

Fig. 3. A relator of the knot group

REMARK 3.1. A *ribbon handlebody* is an immersed image of a handlebody V in $\mathbb{R}^4$ such that the singularity set consists of ribbon singularities, where a ribbon singularity means a singularity with the disjoint union of a properly embedded 2-disk in V and an embedded disk in $\operatorname{int} V$ as the preimage. A *ribbon surface-knot* is a surface-knot bounding a ribbon handlebody in $\mathbb{R}^4$. T. Yajima [8] proved that an S^2-knot K is a ribbon S^2-knot if and only if K is a pseudo-ribbon S^2-knot. On the other hand, a higher genus pseudo-ribbon surface-knot is not necessarily a ribbon surface-knot (cf. A. Kawauchi [4] and A. Shima [6]).

DEFINITION 3.2 (Crossing change). Let D be a pseudo-ribbon surface diagram. A *crossing change* on D is to assign opposite crossing information to some hoop of the singularity set of D.

Theorem 3.3. *Let D be a pseudo-ribbon surface diagram. We can deform D into D' by some crossing changes on D so that $\pi_1(\mathbb{R}^4 - F_{D'})$ becomes isomorphic to $\mathbb{Z}$.*

The proof is given in Section 6. In surface-knot theory, the following conjecture is well-known as the (smooth) *unknotting conjecture* for (orientable) surface-knots.

Conjecture 3.4 (Unknotting conjecture). *For any* (*orientable*) *surface-knot F, F is trivial if and only if $\pi_1(\mathbb{R}^4 - F)$ is isomorphic to $\mathbb{Z}$.*

Combining Theorem 3.3 with Conjecture 3.4, we have the following conjecture.

Conjecture 3.5. *Let D be a pseudo-ribbon surface diagram. We can deform D into D' by some crossing changes on D so that $F_{D'}$ is trivial.*

REMARK 3.6. Any ribbon surface-knot is presented by a *ribbon surface diagram* which is a pseudo-ribbon surface diagram obtained from 2-spheres by attaching

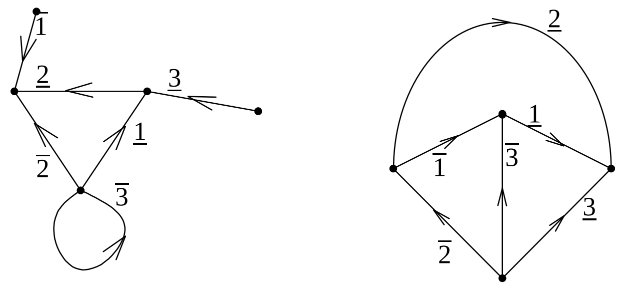

Fig. 4. Examples of pseudo-ribbon graphs

1-handles. By definition, it is easy to see that Conjecture 3.5 is true for ribbon surface diagrams.

4. Pseudo-ribbon graphs

Since it is difficult to treat a pseudo-ribbon surface diagram, we give a way to construct a graph, called a *pseudo-ribbon graph*, from a pseudo-ribbon surface diagram in this section. A pseudo-ribbon graph has knot group information of a pseudo-ribbon surface diagram, and a relationship of a pseudo-ribbon graph to a knot group is mentioned in Section 5.

DEFINITION 4.1 (Pseudo-ribbon graph). Let L be a finite connected graph. We say that L is a *pseudo-ribbon graph* of degree n ($n \in \mathbb{N}$) if the edges of L are oriented and L has just $2n$ edges labeled by $\overline{1}, \underline{1}, \overline{2}, \underline{2}, \ldots, \overline{n}, \underline{n}$. We call the edge labeled by $\overline{i}$ (resp. $\underline{i}$) the $\overline{i}$-edge (resp. $\underline{i}$-edge) for any i ($1 \leq i \leq n$). See Fig. 4.

Let D be a pseudo-ribbon surface diagram, F_D be its recovered pseudo-ribbon surface-knot, $\pi(F_D)$ be its underlying generic projection image, and Γ_D be the singularity set of $\pi(F_D)$. There exists a connected closed oriented surface Σ_D and a locally flat embedding $f\colon \Sigma_D \longrightarrow \mathbb{R}^4$ such that $f(\Sigma_D) = F_D$. The singularity set Γ_D consists of the disjoint union of circles $\gamma_1, \gamma_2, \ldots, \gamma_n$, where n is the number of hoops. For any i ($1 \leq i \leq n$), $f^{-1}(\gamma_i)$ consists of the two disjoint circles, then we assign $c_{\overline{i}}$ (resp. $c_{\underline{i}}$) to the one of them belonging to the over-sheet (resp. under-sheet). The complement $\Sigma_D - f^{-1}(\Gamma_D)$ is separated into some connected components $R_1, R_2, \ldots, R_l$, where l is the number of the connected components. We construct a pseudo-ribbon graph of degree n from D as follows and denote it by L_D. See Example 4.2.

- Vertices of L_D correspond to $R_1, R_2, \ldots, R_l$.
- Edges of L_D correspond to pairs of regions adjacent to $c_{\overline{i}}$ and $c_{\underline{i}}$ for every i ($1 \leq i \leq n$).
- A label of an edge corresponding to $c_{\overline{i}}$ (resp. $c_{\underline{i}}$) is $\overline{i}$ (resp. $\underline{i}$) for every i ($1 \leq i \leq n$).

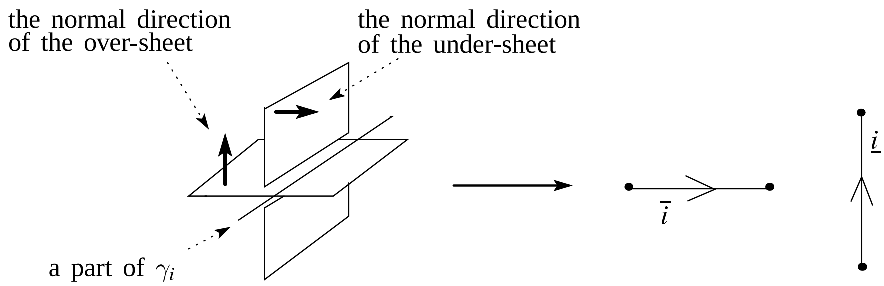

Fig. 5. Constructions of graphs from diagrams

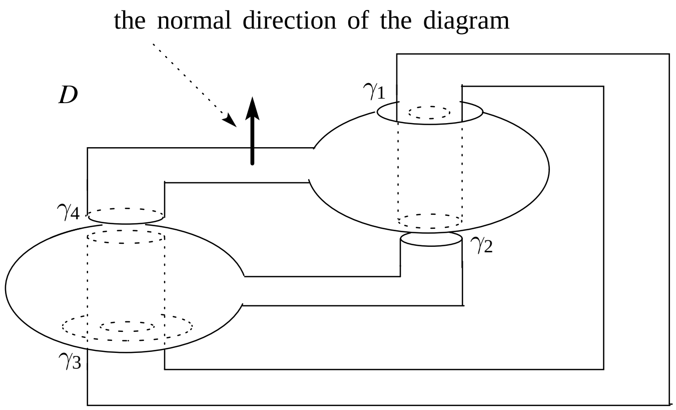

Fig. 6. An example of a pseudo-ribbon surface diagram

- The orientation of the $\overline{i}$-edge (resp. $\underline{i}$-edge) corresponds to the orientation of the under-sheet (resp. over-sheet) for every i $(1 \leq i \leq n)$. See Fig. 5.

EXAMPLE 4.2. We try to construct a pseudo-ribbon graph from the pseudo-ribbon surface diagram D in Fig. 6, then the preimage of D is Σ_D in Fig. 7 and the consequence of the construction is the pseudo-ribbon graph L_D in Fig. 8.

REMARK 4.3. We introduce a way to construct a pseudo-ribbon graph from a pseudo-ribbon surface diagram above. On the other hand, we cannot always construct a pseudo-ribbon surface diagram from a pseudo-ribbon graph.

5. The groups of pseudo-ribbon graphs

In this section, we define the group of a pseudo-ribbon graph and prove the key proposition (Proposition 5.5) for the proof of Theorem 3.3.

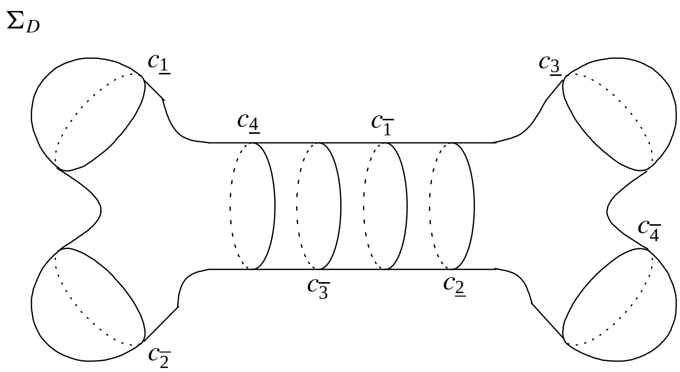

Fig. 7. The preimage of the diagram D in Fig. 6

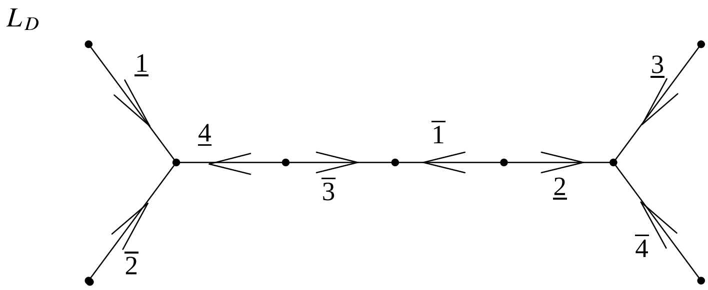

Fig. 8. The pseudo-ribbon graph constructed from D in Fig. 6

DEFINITION 5.1 (Group of a pseudo-ribbon graph). Let L be a pseudo-ribbon graph of degree n ($n \in \mathbb{N}$). If we eliminate the $\underline{i}$-edges ($1 \leq i \leq n$) from L, then the graph is separated into some connected components $L_0, L_1, \ldots, L_s$, where $s + 1$ is the number of connected components. We assign x_i to all the vertices belonging to L_i for every i ($0 \leq i \leq s$). The *group of the pseudo-ribbon graph* L is defined by the following presentation and denoted by $G(L)$.

$$G(L) = \langle\, x_0, \ldots, x_s \mid r_1, \ldots, r_n \,\rangle$$

Here the relator r_i is defined as follows: if two end vertices of the $\overline{i}$-edge are assigned x_{i_2} and an initial (resp. a terminal) vertex of the $\underline{i}$-edge is assigned x_{i_1} (resp. x_{i_3}), then we have the form $r_i = x_{i_3}^{-1} x_{i_2}^{-1} x_{i_1} x_{i_2}$. See Fig. 9.

REMARK 5.2. Let D be a pseudo-ribbon surface diagram. By Theorem 2.2 and Definition 5.1, $G(L_D)$ is isomorphic to $\pi_1(\mathbb{R}^4 - F_D)$.

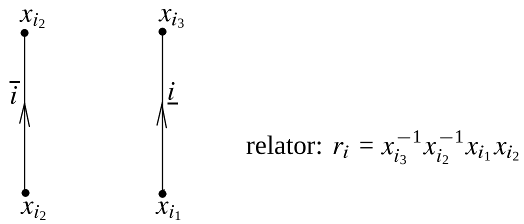

Fig. 9. A relator of the group of a pseudo-ribbon graph

Let C_n be the set $\{\overline{1}, \underline{1}, \overline{2}, \underline{2}, \dots, \overline{n}, \underline{n}\}$, and $\widetilde{C_n}$ the set of pseudo-ribbon graphs of degree n. Now we consider two kinds of maps $\phi_A^n \colon C_n \longrightarrow C_n$ and $\psi_\sigma^n \colon C_n \longrightarrow C_n$. For any subset $A \subset \{1, 2, \dots, n\}$, ϕ_A^n is a bijective map defined by

$$\phi_A^n(\overline{i}) = \begin{cases} \overline{i} & \text{if } i \in A^c \\ \underline{i} & \text{if } i \in A \end{cases}, \quad \phi_A^n(\underline{i}) = \begin{cases} \underline{i} & \text{if } i \in A^c \\ \overline{i} & \text{if } i \in A \end{cases}.$$

For any element $\sigma \in S_n$ (the symmetric group of degree n), ψ_σ^n is a bijective map defined by

$$\psi_\sigma^n(\overline{i}) = \overline{\sigma(i)}, \ \psi_\sigma^n(\underline{i}) = \underline{\sigma(i)} \ (1 \leq i \leq n).$$

The map ϕ_A^n induces a bijective map $\widetilde{\phi_A^n} \colon \widetilde{C_n} \longrightarrow \widetilde{C_n}$. In the same way, ψ_σ^n induces a bijective map $\widetilde{\psi_\sigma^n} \colon \widetilde{C_n} \longrightarrow \widetilde{C_n}$. We note that $G\big(\widetilde{\psi_\sigma^n}(L)\big)$ is isomorphic to $G(L)$ for any pseudo-ribbon graph L and any element $\sigma \in S_n$ by definition.

Lemma 5.3. *Let L be a pseudo-ribbon graph of degree n which is a tree as a 1-dimensional complex. There exists a subset $A_L \subset \{1, 2, \dots, n\}$ such that $G\big(\widetilde{\phi_{A_L}^n}(L)\big)$ is isomorphic to $\mathbb{Z}$.*

Proof.

STEP 1. We assign a sequence of nonnegative integers $(k_1, k_2, \dots, k_l)$ to a vertex, denoted by $v(k_1, k_2, \dots, k_l)$, and an edge, denoted by $e(k_1, k_2, \dots, k_l)$, of L inductively as follows (see Fig. 10):

1. We choose any one vertex of L and assign (0) to this. We assign $(1), (2), \dots, (m)$ to the edges connecting to $v(0)$.
2. For any integer k_1 with $1 \leq k_1 \leq m$, $e(k_1)$ has two end vertices. We assign (k_1) to the vertex which is distinct from $v(0)$. There are some edges connecting to $v(k_1)$. We assign $(k_1, 1), (k_1, 2), \dots, (k_1, m_{k_1})$ to them except for $e(k_1)$.
3. For k_1, k_2 with $1 \leq k_1 \leq m$ and $1 \leq k_2 \leq m_{k_1}$, the edge $e(k_1, k_2)$ has two end vertices. We assign (k_1, k_2) to the vertex which is distinct from $v(k_1)$. There are some edges connecting to $v(k_1, k_2)$. We assign $(k_1, k_2, 1), (k_1, k_2, 2), \dots, (k_1, k_2, m_{k_1,k_2})$ to them except for $e(k_1, k_2)$.

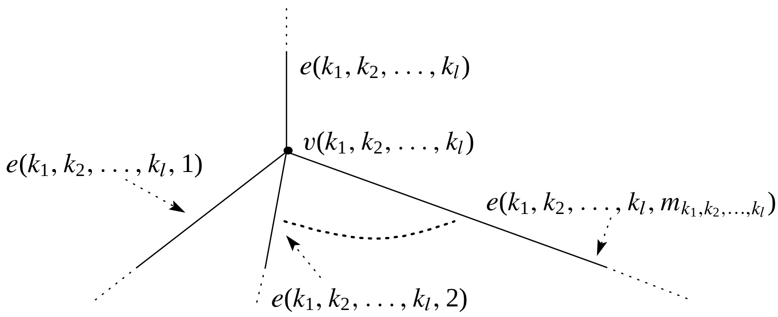

Fig. 10. The way to assign sequences of nonnegative integers

4. In this way, we assign a sequence of nonnegative integers to every vertex and every edge of L inductively.

Since L is a tree, the sequences of nonnegative integers can be assigned to vertices and edges without duplication. We note that the sequences of nonnegative integers assigned edges are independent of labels of them and the way to assign the sequences of nonnegative integers to L is not unique.

STEP 2. We give a total order to the edges with respect to the sequences of nonnegative integers. Precisely, we say that $e(k_1, k_2, \ldots, k_l)$ is *smaller* than $e(k'_1, k'_2, \ldots, k'_{l'})$ if either of the following conditions is satisfied

- $\exists s \leq \min\{l, l'\}$ s.t. $k_1 = k'_1, k_2 = k'_2, \ldots, k_{s-1} = k'_{s-1}, k_s < k'_s$
- $l < l'$, $k_1 = k'_1, k_2 = k'_2, \ldots, k_l = k'_l$

and denote

$$e(k_1, k_2, \ldots, k_l) < e(k'_1, k'_2, \ldots, k'_{l'}).$$

In the same way, we can give a total order to vertices. When $v(k_1, k_2, \ldots, k_l)$ is *smaller* than $v(k'_1, k'_2, \ldots, k'_{l'})$, we denote

$$v(k_1, k_2, \ldots, k_l) < v(k'_1, k'_2, \ldots, k'_{l'}).$$

STEP 3. We define a set of positive integers, A_L, by using the above order of edges as follows:

$$A_L = \left\{ i \in \{1, 2, \ldots, n\} \mid (\underline{i}\text{-edge}) < (\bar{i}\text{-edge}) \right\}.$$

For the pseudo-ribbon graph $\widetilde{\phi^n_{A_L}}(L)$, there exist an element $\sigma \in S_n$ such that

$$(\sigma(1)\text{-edge}) < (\sigma(2)\text{-edge}) < \cdots < (\underline{\sigma(n)}\text{-edge}).$$

Since $\widetilde{\psi^n_{\sigma^{-1}}}$ does not change the groups of pseudo-ribbon graphs, we can consider $\widetilde{\psi^n_{\sigma^{-1}}} \circ \widetilde{\phi^n_{A_L}}(L)$ instead of $\widetilde{\phi^n_{A_L}}(L)$. We note that the order of the $\underline{i}$-edges ($1 \leq i \leq n$) of $\widetilde{\psi^n_{\sigma^{-1}}} \circ \widetilde{\phi^n_{A_L}}(L)$ is

$$(1\text{-edge}) < (2\text{-edge}) < \cdots < (\underline{n}\text{-edge}).$$

Refer to Example 5.4.

STEP 4. We prove that $G\bigl(\widetilde{\psi^n_{\sigma^{-1}}} \circ \widetilde{\phi^n_{A_L}}(L)\bigr)$ is isomorphic to $\mathbb{Z}$ by an actual calculation. Hereafter, as long as there is no confusion, we denote $G\bigl(\widetilde{\psi^n_{\sigma^{-1}}} \circ \widetilde{\phi^n_{A_L}}(L)\bigr)$ by G.

In Definition 5.1, we have assigned the generators to the vertices of L. Further, we may have the following additional conditions:

- All vertices which belong to the component containing $v(0)$ are assigned x_0.
- All vertices which belong to the component containing the greater vertex in the two end vertices of the $\underline{m}$-edge are assigned x_m ($1 \leq m \leq n$).

The generators assigned to the vertices satisfy the following conditions by the order of vertices and edges (see Fig. 12).

- The generator assigned to the greater vertex in the two end vertices of the $\underline{m}$-edge is x_m ($1 \leq m \leq n$).
- There exists a positive integer i_m such that $i_m < m$ and the generators assigned to the two end vertices of the $\bar{m}$-edge are x_{i_m} ($1 \leq m \leq n$).
- There exists a positive integer j_m such that $j_m < m$ and the generator assigned to the smaller vertex in the two end vertices of the $\underline{m}$-edge is x_{j_m} ($1 \leq m \leq n$).

Then the group presentation of G is

$$\begin{aligned}
&\langle\, x_0, x_1, \ldots, x_n \mid x_1^{-1} x_{i_1}^{\epsilon_1} x_{j_1} x_{i_1}^{-\epsilon_1},\ x_2^{-1} x_{i_2}^{\epsilon_2} x_{j_2} x_{i_2}^{-\epsilon_2}, \ldots,\ x_n^{-1} x_{i_n}^{\epsilon_n} x_{j_n} x_{i_n}^{-\epsilon_n} \,\rangle\,,\\
&\qquad\qquad (1 \leq m \leq n,\ i_m,\ j_m < m,\ \epsilon_m \in \{1, -1\})
\end{aligned}$$

and we calculate G as follows:

$$\begin{aligned}
G &\cong \langle\, x_0, x_1, \ldots, x_n \mid x_0 = x_1,\ x_2^{-1} x_{i_2}^{\epsilon_2} x_{j_2} x_{i_2}^{-\epsilon_2}, \ldots,\ x_n^{-1} x_{i_n}^{\epsilon_n} x_{j_n} x_{i_n}^{-\epsilon_n} \,\rangle\\
&\qquad\qquad (2 \leq m \leq n,\ i_m,\ j_m < m,\ \epsilon_m \in \{1, -1\})\\
&\cong \langle\, x_0, x_1, \ldots, x_n \mid x_0 = x_1 = x_2,\ x_3^{-1} x_{i_3}^{\epsilon_3} x_{j_3} x_{i_3}^{-\epsilon_3}, \ldots,\ x_n^{-1} x_{i_n}^{\epsilon_n} x_{j_n} x_{i_n}^{-\epsilon_n} \,\rangle\\
&\qquad\qquad (3 \leq m \leq n,\ i_m,\ j_m < m,\ \epsilon_m \in \{1, -1\})\\
&\cong \qquad \cdots\cdots\cdots\cdots\\
&\cong \langle\, x_0, x_1, \ldots, x_n \mid x_0 = x_1 = x_2 = \cdots = x_n \,\rangle\\
&\cong \mathbb{Z}\,.
\end{aligned}$$

$\sqcap$

EXAMPLE 5.4. We observe the proof of Lemma 5.3 for L in Fig. 11. If we assign the sequences of nonnegative integers to edges and vertices in such a way as shown in

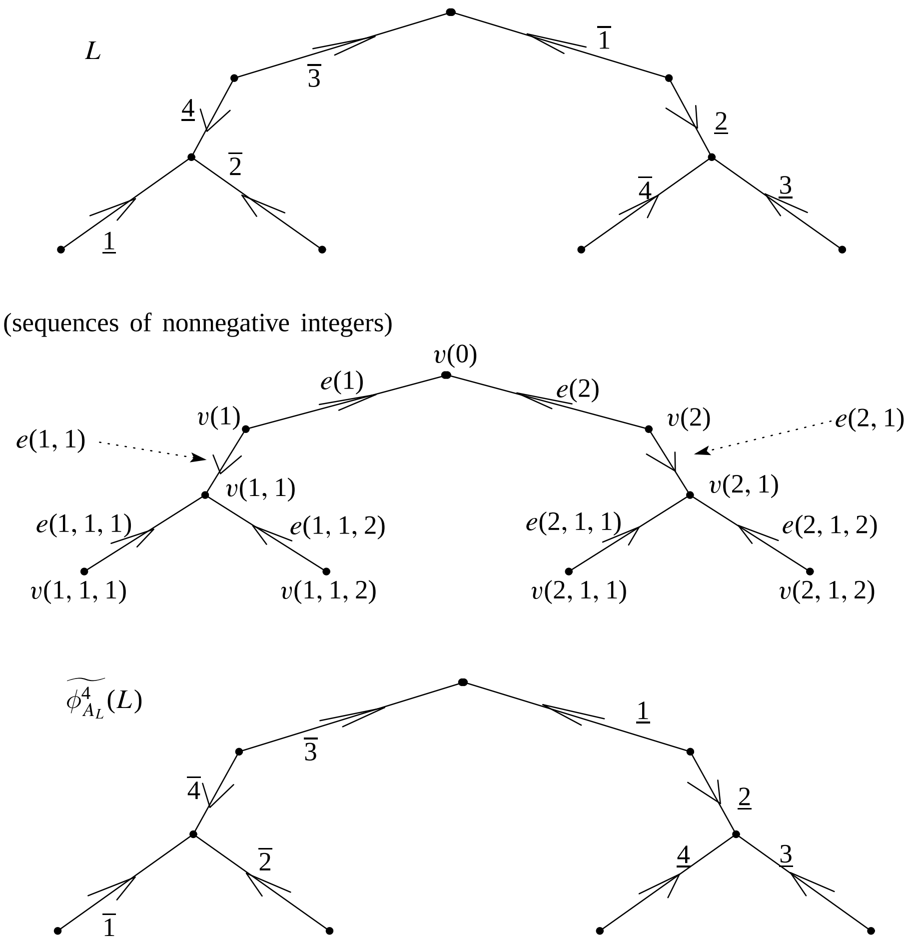

Fig. 11.

the middle part of Fig. 11, then the set of positive integers, A_L, is

$$\{1, 4\} \left(\subset \{1, 2, 3, 4\}\right),$$

the order of $\underline{i}$-edges ($1 \leq i \leq 4$) of $\widetilde{\phi^4_{A_L}}(L)$ is

$$(\underline{1}\text{-edge}) < (\underline{2}\text{-edge}) < (\underline{4}\text{-edge}) < (\underline{3}\text{-edge}),$$

and the element $\sigma \in S_4$ is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}.$$

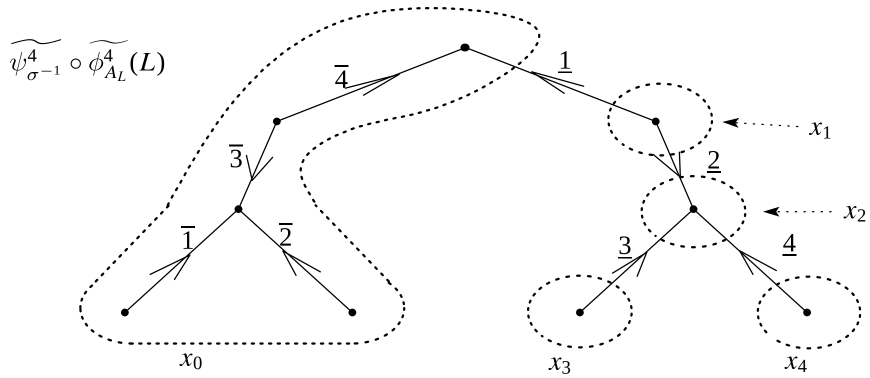

Fig. 12.

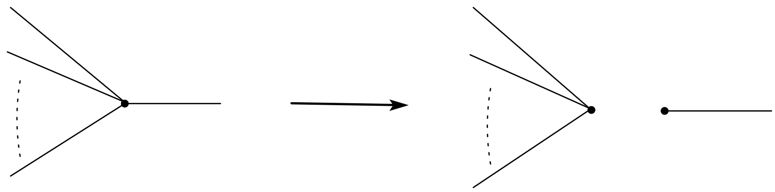

Fig. 13. An operation for pseudo-ribbon graphs

So the group presentation of $G\ \left(= G\left(\widetilde{\psi^4_{\sigma^{-1}}} \circ \widetilde{\phi^4_{A_L}}(L)\right)\right)$ is

$$\langle\ x_0, x_1, x_2, x_3, x_4 \mid x_1 = x_0x_0x_0^{-1},\ \ x_2 = x_0^{-1}x_1x_0,\ \ x_3 = x_0x_2x_0^{-1},\ \ x_4 = x_0x_2x_0^{-1}\ \rangle\ ,$$

and we calculate G as follows (see Fig. 12):

$$\begin{aligned} G &\cong \langle\ x_0, x_1, x_2, x_3, x_4 \mid x_0 = x_1,\ \ x_2 = x_0^{-1}x_1x_0,\ \ x_3 = x_0x_2x_0^{-1},\ \ x_4 = x_0x_2x_0^{-1}\ \rangle \\ &\cong \langle\ x_0, x_1, x_2, x_3, x_4 \mid x_0 = x_1 = x_2,\ \ x_3 = x_0x_2x_0^{-1},\ \ x_4 = x_0x_2x_0^{-1}\ \rangle \\ &\cong \langle\ x_0, x_1, x_2, x_3, x_4 \mid x_0 = x_1 = x_2 = x_3,\ \ x_4 = x_0x_2x_0^{-1}\ \rangle \\ &\cong \langle\ x_0, x_1, x_2, x_3, x_4 \mid x_0 = x_1 = x_2 = x_3 = x_4\ \rangle \\ &\cong \mathbb{Z}\,. \end{aligned}$$

Proposition 5.5. *Let L be a pseudo-ribbon graph of degree n ($n \in \mathbb{N}$). There exists a subset $A_L \subset \{1, 2, \dots, n\}$ such that $G\left(\widetilde{\phi^n_{A_L}}(L)\right)$ is isomorphic to $\mathbb{Z}$.*

Proof. We can deform L to a pseudo-ribbon graph which is a tree by applying finitely many operations such as Fig. 13 and denote it by L' (see Fig. 14). We note that L' is not uniquely determined by L.

By Lemma 5.3, there exist a subset $A_{L'} \subset \{1, 2, \dots, n\}$ such that $G\left(\widetilde{\phi^n_{A_{L'}}}(L')\right)$ is

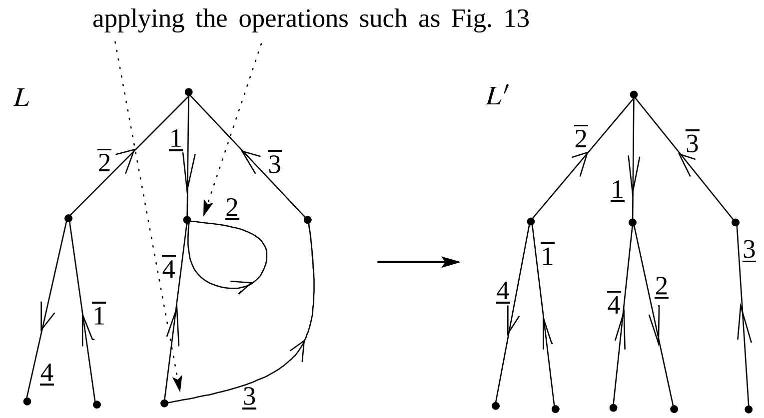

Fig. 14. An example of a graph which is not tree

isomorphic to $\mathbb{Z}$ and the group presentation of $G\big(\widetilde{\phi^n_{A_{L'}}}(L')\big)$ is

$$\begin{aligned} &\langle\, x_0, x_1, \ldots, x_n \mid x_1^{-1} x_{i_1}^{\epsilon_1} x_{j_1} x_{i_1}^{-\epsilon_1},\ x_2^{-1} x_{i_2}^{\epsilon_2} x_{j_2} x_{i_2}^{-\epsilon_2}, \ldots,\ x_n^{-1} x_{i_n}^{\epsilon_n} x_{j_n} x_{i_n}^{-\epsilon_n} \,\rangle \\ &\qquad\qquad (1 \leq m \leq n,\ i_m, j_m < m,\ \epsilon_m \in \{1, -1\}) \\ \cong\ &\langle\, x_0, x_1, \ldots, x_n \mid x_0 = x_1 = x_2 = \cdots = x_n \,\rangle\,. \end{aligned}$$

Since the group presentation of $G\big(\widetilde{\phi^n_{A_{L'}}}(L)\big)$ is obtained from that of $G\big(\widetilde{\phi^n_{A_{L'}}}(L')\big)$ by adding some relators such as $x_i = x_j$ $(i, j \in \{0, 1, \ldots, n\})$, $G\big(\widetilde{\phi^n_{A_{L'}}}(L)\big)$ is also isomorphic to $\mathbb{Z}$. □

6. The proof of Main theorem

In this section, we give the proof of Theorem 3.3 by using the construction of a pseudo-ribbon graph L_D from a pseudo-ribbon surface diagram D in Section 4.

By the construction, for any subset $A \subset \{1, 2, \ldots, n\}$, there exists a pseudo-ribbon surface diagram $D(A)$ such that $D(A)$ is deformed from D by some crossing changes and $L_{D(A)} = \widetilde{\phi^n_A}(L_D)$. By Remark 5.2, it holds that

$$\pi_1(\mathbb{R}^4 - F_{D(A)}) \cong G(L_{D(A)}) \cong G\left(\widetilde{\phi^n_A}(L_D)\right).$$

By Proposition 5.5, there exists a set of positive integers A_{L_D} such that

$$G\left(\widetilde{\phi^n_{A_{L_D}}}(L_D)\right) \cong \mathbb{Z}\,.$$

When we substitute A_{L_D} to the above A, it holds that

$$\pi_1(\mathbb{R}^4 - F_{D(A_{L_D})}) \cong G(L_{D(A_{L_D})}) \cong G\left(\widetilde{\phi^n_{A_{L_D}}}(L_D)\right) \cong \mathbb{Z}\,.$$

Thus the proof of Theorem 3.3 is completed. □

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