Ando, N.
Osaka J. Math.
**41** (2004), 865–876

# AN ISOLATED UMBILICAL POINT OF A WILLMORE SURFACE

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(Received April 9, 2003)

## 1. Introduction

Let $S$ be a surface in $\mathbf{R}^3$. Then it is known that if $S$ is a surface with constant mean curvature, then the index of an isolated umbilical point on $S$ is negative ([16]). If $S$ is special Weingarten, then the same result is obtained ([15]). In the present paper, we shall prove that the index of an isolated umbilical point on a Willmore surface does not exceed $1/2$.

We say that $S$ is a *Willmore surface* if $S$ is a stationary surface of the Willmore functional $\mathcal{W}$, where the *Willmore functional* is defined by the integral of the square of the mean curvature. It is known that $S$ is a Willmore surface if and only if $S$ satisfies the following partial differential equation ([12]):

$$\{\Delta + 2(H^2 - K)\}H = 0, \tag{1}$$

where $\Delta$ is the Laplace operator on $S$ and $K$, $H$ are the Gaussian and the mean curvatures of $S$, respectively. Equation (1) is the Euler-Lagrange equation for Willmore surfaces.

Willmore proved that $\mathcal{W} \geqq 4\pi$ for any compact surface in $\mathbf{R}^3$ and that the equality holds if and only if the surface is a round sphere ([36], [37]). In addition, he and Shiohama-Takagi proved that $\mathcal{W} \geqq 2\pi^2$ $(> 4\pi)$ for a torus represented as the boundary of a tubular neighborhood of a closed curve in $\mathbf{R}^3$ and that the equality holds if and only if the torus is a $\sqrt{2}$-anchor ring, i.e., the boundary of the tubular neighborhood with radius $a > 0$ of a circle with radius $\sqrt{2}a$ ([38], [27]). Willmore conjectured $\mathcal{W} \geqq 2\pi^2$ for any torus in $\mathbf{R}^3$ ([36]). Since White showed that if the surface is compact and orientable, then $\mathcal{W}$ is invariant under any conformal transformation of $\overline{\mathbf{R}}^3 := \mathbf{R}^3 \cup \{\infty\}$ ([35]), it has been expected that the equality in Willmore's conjecture holds if and only if the torus is conformally equivalent in $\overline{\mathbf{R}}^3$ to a $\sqrt{2}$-anchor ring. Li-Yau showed that Willmore's conjecture is true for tori with certain conformal structures close to the conformal structure of a $\sqrt{2}$-anchor ring ([21]); Montiel-Ros showed that Willmore's conjecture is also true for tori with more conformal structures ([22]). Simon proved that there exists an embedded torus in $\mathbf{R}^3$ at which $\mathcal{W}$ attains the infimum on all the immersed tori ([28], [29]). Recently, the author has had paper [26] by Schmidt the main theorem of which states that Willmore's conjecture is true for any torus immersed in $\mathbf{R}^3$.

Weiner proved that the image of any minimal surface in $S^3$ by a stereographic projection is a Willmore surface in $\mathbf{R}^3$ ([34]). Any compact two-dimensional manifold other than the projective plane may be realized in $S^3$ as a minimal surface ([20]), while the projective plane may not be realized in $S^3$ as any minimal surface ([1], [20]). Therefore we see that any compact two-dimensional manifold distinct from the projective plane may be realized in $\mathbf{R}^3$ as a Willmore surface. Pinkall showed that there exists a Hopf torus in $S^3$ which is not conformally equivalent in $S^3$ to any minimal surface and the image of which by a stereographic projection is a Willmore surface in $\mathbf{R}^3$ ([24]). In addition, Kusner found an example of a Willmore surface in $\mathbf{R}^3$ which is homeomorphic to the projective plane ([18], [19]). At this example, $\mathcal{W}$ attains $12\pi$, the infimum on all the projective planes immersed in $\mathbf{R}^3$. Bryant described the moduli space of the Willmore projective planes in $\mathbf{R}^3$ for each of which $\mathcal{W}$ is equal to $12\pi$ ([11]).

By Hopf-Poincaré's theorem together with Kusner's example of a Willmore projective plane, we see that our estimate of the index of an isolated umbilical point on a Willmore surface is sharp.

It is expected that the index of an isolated umbilical point on a surface does not exceed one. We call this conjecture the *index conjecture*. In relation to the index conjecture, the following two conjectures are known: Carathéodory's conjecture and Loewner's conjecture. *Carathéodory's conjecture* asserts that there exist at least two umbilical points on a compact, strictly convex surface in $\mathbf{R}^3$. If the index conjecture is true, then we see from Hopf-Poincaré's theorem that there exist at least two umbilical points on a compact, orientable surface of genus zero, and this immediately gives the affirmative answer to Carathéodory's conjecture. Let $F$ be a real-valued, smooth function of two real variables $x, y$, and set $\partial_{\bar{z}} := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2$. Then *Loewner's conjecture* for a positive integer $n \in \mathbf{N}$ asserts that if a vector field $\mathrm{Re}\left(\partial_{\bar{z}}^n F\right)(\partial/\partial x) + \mathrm{Im}\left(\partial_{\bar{z}}^n F\right)(\partial/\partial y)$ has an isolated zero point, then its index with respect to this vector field does not exceed $n$ ([17], [33]). Loewner's conjecture for $n = 1$ is affirmatively solved; Loewner's conjecture for $n = 2$ is equivalent to the index conjecture. We may find [9], [13], [30], [31] and [32] as recent papers in relation to Carathéodory's and Loewner's conjectures. We discussed the index of an isolated umbilical point on a surface in [2]–[7], and in [8], we introduced and studied a conjecture in relation to Loewner's conjecture.

We see from our estimate of the index in the present paper that the index conjecture is true for any isolated umbilical point on a Willmore surface. In the proof of the main theorem, we shall encounter a situation on a surface with an isolated umbilical point which has not appeared in our previous studies.

## 2. Willmore surfaces

Let $M$ be a connected, orientable two-dimensional manifold and $\iota\colon M \to \mathbf{R}^3$ an immersion of $M$ into $\mathbf{R}^3$. Let $H$ be the mean curvature of $M$ with respect to $\iota$ and $dA$ the area element of $M$ with respect to the metric $g$ induced by $\iota$. Then the *Willmore functional* $\mathcal{W}$ is given by

$$\mathcal{W}(\iota) := \int_M H^2\, dA.$$

Let $K$ be the Gaussian curvature of $M$ with respect to the metric $g$ and set

$$\widehat{\mathcal{W}}(\iota) := \int_M (H^2 - K)\, dA.$$

Then we obtain

$$\widehat{\mathcal{W}}(\iota) = \mathcal{W}(\iota) - \int_M K\, dA. \tag{2}$$

It is known that for any conformal transformation $X$ of $\overline{\mathbf{R}}^3$ such that $X \circ \iota$ is an immersion, the following holds ([35]):

$$\widehat{\mathcal{W}}(X \circ \iota) = \widehat{\mathcal{W}}(\iota). \tag{3}$$

If $M$ is compact, then by (2), (3) and Gauss-Bonnet's theorem, we obtain

$$\mathcal{W}(X \circ \iota) = \mathcal{W}(\iota).$$

Let $M$ and $\iota$ be as above. Let $\xi$ be a unit normal vector field on $M$ with respect to $\iota$ and $f$ a smooth function on $M$ with compact support. Let $\iota_f$ be a smooth map from $M \times \mathbf{R}$ into $\mathbf{R}^3$ satisfying $\iota_f(p, 0) = \iota(p)$, $(\partial\iota_f/\partial t)(p, 0) = f(p)\xi(p)$ for $p \in M$ and the condition that $\iota_f(p, t) = \iota_f(p, 0)$ for any $t \in \mathbf{R}$ and any point $p$ of $M$ outside the support of $f$. We set $\iota_{f,t}(p) := \iota_f(p, t)$ for $(p, t) \in M \times \mathbf{R}$. Then there exists an open interval $I$ containing 0 such that for each $t \in I$, $\iota_{f,t}$ is an immersion of $M$ into $\mathbf{R}^3$. We set

$$w_f(t) := \mathcal{W}(\iota_{f,t}),\ \ \widehat{w}_f(t) := \widehat{\mathcal{W}}(\iota_{f,t}).$$

An immersion $\iota$ is called *Willmore* if $(dw_f/dt)(0) = 0$ holds for any smooth function $f$ on $M$ with compact support; if $\iota$ is a Willmore immersion, then the pair $(M, \iota)$ or the image $\iota(M)$ of $M$ by $\iota$ is called a *Willmore surface*. An immersion $\iota$ is Willmore if and only if (1) holds, where $\Delta$ is the Laplace operator on $M$ with respect to the metric $g$ ([12]). Let $D$ be a domain in $M$ which contains the support of $f$

and the boundary of which consists of a finite number of closed curves. Then for $t \in I$, $w_f(t) - \widehat{w}_f(t)$ is represented as follows:

$$w_f(t) - \widehat{w}_f(t) = \int_{M \setminus D} K_t \, dA_t + \int_D K_t \, dA_t, \tag{4}$$

where $K_t$ and $dA_t$ are the Gaussian curvature and the area element of $M$ with respect to the metric induced by $\iota_{f,t}$, respectively. From Gauss-Bonnet's theorem, we see that the second term of the right hand side of (4) depends only on the boundary of $D$, which implies that this term does not depend on $t \in I$. In addition, since $D$ contains the support of $f$, the first term of the right hand side of (4) does not depend on $t \in I$ either. Therefore we see that $w_f - \widehat{w}_f$ is constant on $I$. In particular, we obtain

$$\frac{d\widehat{w}_f}{dt}(0) = \frac{dw_f}{dt}(0). \tag{5}$$

By (3) together with (5), we obtain

**Proposition 2.1.** *Let $\iota$ be an immersion of $M$ into $\mathbf{R}^3$ and $X$ a conformal transformation of $\overline{\mathbf{R}}^3$ such that $X \circ \iota$ is an immersion. Then $\iota$ is Willmore if and only if $X \circ \iota$ is Willmore.*

## 3. The index of an isolated umbilical point

Let $f$ be a smooth function of two variables $x$, $y$ and $\mathsf{G}_f$ the graph of $f$. We set

$$p_f := \frac{\partial f}{\partial x}, \ q_f := \frac{\partial f}{\partial y}, \ r_f := \frac{\partial^2 f}{\partial x^2}, \ s_f := \frac{\partial^2 f}{\partial x \partial y}, \ t_f := \frac{\partial^2 f}{\partial y^2}.$$

Then the Gaussian curvature $K_f$ and the mean curvature $H_f$ of $\mathsf{G}_f$ are represented as follows:

$$K_f := \frac{r_f t_f - s_f^2}{(1 + p_f^2 + q_f^2)^2}, \ H_f := \frac{r_f + t_f + p_f^2 t_f - 2 p_f q_f s_f + q_f^2 r_f}{2(1 + p_f^2 + q_f^2)^{3/2}}. \tag{6}$$

Let $\mathrm{D}_f$, $\mathrm{N}_f$, $\mathrm{PD}_f$ be symmetric tensor fields on $\mathsf{G}_f$ of type $(0, 2)$ represented in terms of the coordinates $(x, y)$ as follows:

$$\begin{aligned}
\mathrm{D}_f &:= s_f \, dx^2 + (t_f - r_f) \, dx \, dy - s_f \, dy^2, \\
\mathrm{N}_f &:= (s_f p_f^2 - p_f q_f r_f) \, dx^2 + (t_f p_f^2 - r_f q_f^2) \, dx \, dy + (p_f q_f t_f - s_f q_f^2) \, dy^2, \\
\mathrm{PD}_f &:= \frac{1}{1 + p_f^2 + q_f^2} (\mathrm{D}_f + \mathrm{N}_f).
\end{aligned}$$

A tangent vector $\boldsymbol{v}_0$ to $\mathsf{G}_f$ at a point is in a principal direction if and only if $\mathrm{PD}_f(\boldsymbol{v}_0, \boldsymbol{v}_0) = 0$ holds ([5]). For a tangent vector $\boldsymbol{v}$, we set

$$\tilde{\mathrm{D}}_f(\boldsymbol{v}) := \mathrm{D}_f(\boldsymbol{v}, \boldsymbol{v}), \quad \tilde{\mathrm{N}}_f(\boldsymbol{v}) := \mathrm{N}_f(\boldsymbol{v}, \boldsymbol{v}), \quad \widetilde{\mathrm{PD}}_f(\boldsymbol{v}) := \mathrm{PD}_f(\boldsymbol{v}, \boldsymbol{v}).$$

For $\phi \in \mathbf{R}$, we set

$$u_\phi := \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \ \mathbf{U}_\phi := \cos\phi \frac{\partial}{\partial x} + \sin\phi \frac{\partial}{\partial y}.$$

We set

$$\mathrm{grad}_f := \begin{pmatrix} p_f \\ q_f \end{pmatrix}, \ \mathrm{grad}_f^\perp := \begin{pmatrix} -q_f \\ p_f \end{pmatrix}, \ \mathrm{Hess}_f := \begin{pmatrix} r_f & s_f \\ s_f & t_f \end{pmatrix}.$$

Let $\langle\ ,\ \rangle$ be the scalar product in $\mathbf{R}^2$. Then for any $\phi \in \mathbf{R}$, the following hold ([5]):

$$\tilde{\mathrm{D}}_f(\mathbf{U}_\phi) = \langle \mathrm{Hess}_f\, u_\phi, u_{\phi+\pi/2} \rangle,$$
$$\tilde{\mathrm{N}}_f(\mathbf{U}_\phi) = \langle \mathrm{grad}_f, u_\phi \rangle \langle \mathrm{grad}_f^\perp, \mathrm{Hess}_f\, u_\phi \rangle.$$

For $l \in \mathbf{N} \cup \{\infty\}$, let $\mathcal{C}_o^{(\infty,l)}$ be the set of smooth functions defined on a connected neighborhood of $(0,0)$ in $\mathbf{R}^2$ such that $(\partial^{m+n} F/\partial x^m \partial y^n)(0,0) = 0$ for each $F \in \mathcal{C}_o^{(\infty,l)}$ and non-negative integers $m, n$ satisfying $0 \leqq m + n < l$. The following hold:

$$\mathcal{C}_o^{(\infty,l)} \supset \mathcal{C}_o^{(\infty,l+1)} \supset \mathcal{C}_o^{(\infty,\infty)} \neq \{0\},$$

where $l \in \mathbf{N}$. Let $F$ be an element of $\mathcal{C}_o^{(\infty,2)}$ such that $o := (0,0,0)$ is an umbilical point of the graph of $F$, that is, there exists a real number $a_F$ satisfying

$$F(x, y) = \frac{a_F(x^2 + y^2)}{2} + o(x^2 + y^2). \tag{7}$$

Let $\sigma_F$ be an element of $\mathcal{C}_o^{(\infty,2)}$ defined by

$$\sigma_F := \begin{cases} 0 & \text{if } a_F = 0, \\ \dfrac{1}{a_F} - \dfrac{|a_F|}{a_F}\sqrt{\dfrac{1}{a_F^2} - (x^2 + y^2)} & \text{if } a_F \neq 0. \end{cases}$$

Then we obtain $F - \sigma_F \in \mathcal{C}_o^{(\infty,3)}$. For an integer $l \geqq 2$, let $\mathcal{C}_o^{\langle\infty,l\rangle}$ be the subset of $\mathcal{C}_o^{(\infty,l)}$ such that each $F \in \mathcal{C}_o^{\langle\infty,l\rangle}$ satisfies (7) for some $a_F \in \mathbf{R}$ and $F - \sigma_F \notin \mathcal{C}_o^{(\infty,\infty)}$. For an integer $k \geqq 3$, let $\mathcal{P}^k$ be the set of the homogeneous polynomials of degree $k$. Then for each $F \in \mathcal{C}_o^{\langle\infty,2\rangle}$, there exist an integer $k_F \geqq 3$ and a nonzero element $g_F$ of $\mathcal{P}^{k_F}$ satisfying $F - \sigma_F - g_F \in \mathcal{C}_o^{(\infty,k_F+1)}$. Let $g$ be an element of $\mathcal{P}^k$.

Then set $\mathrm{Hess}_g(\theta) := \mathrm{Hess}_g(\cos\theta, \sin\theta)$ for $\theta \in \mathbf{R}$ and let $\eta_g$ be a continuous function on $\mathbf{R}$ such that for any $\theta \in \mathbf{R}$, $u_{\eta_g(\theta)}$ is an eigenvector of $\mathrm{Hess}_g(\theta)$, and let $S_g$ denote the set of the numbers at each of which $\mathrm{Hess}_g$ is represented by the unit matrix up to a constant.

Let $\mathcal{C}_o^{\infty,2}$ be the subset of $\mathcal{C}_o^{\langle\infty,2\rangle}$ such that on the graph $\mathsf{G}_F$ of each $F \in \mathcal{C}_o^{\infty,2}$, $o$ is an isolated umbilical point. For an element $F$ of $\mathcal{C}_o^{\infty,2}$, let $\rho_0$ be a positive number such that there exists no umbilical point of $\mathsf{G}_F$ on $\{0 < x^2 + y^2 < \rho_0^2\}$ and $\phi_F$ a continuous function on $(0, \rho_0) \times \mathbf{R}$ such that for each $(\rho, \theta) \in (0, \rho_0) \times \mathbf{R}$, a tangent vector $\cos\phi_F(\rho, \theta)\partial/\partial x + \sin\phi_F(\rho, \theta)\partial/\partial y$ of $\mathsf{G}_F$ at $(\rho\cos\theta, \rho\sin\theta)$ is in a principal direction. Then the following (a)–(c) hold ([5], [6]):

(a) For any $\theta_0 \in \mathbf{R} \setminus S_{g_F}$, there exists a number $\phi_{F,o}(\theta_0)$ satisfying the following:
   (i) $\lim_{\rho\to 0} \phi_F(\rho, \theta_0) = \phi_{F,o}(\theta_0)$,
   (ii) $u_{\phi_{F,o}(\theta_0)}$ is an eigenvector of $\mathrm{Hess}_{g_F}(\theta_0)$;

(b) For any $\theta_0 \in \mathbf{R}$, there exist numbers $\phi_{F,o}(\theta_0 + 0)$, $\phi_{F,o}(\theta_0 - 0)$ satisfying the following:
   (i) $\lim_{\theta\to\theta_0\pm 0} \phi_{F,o}(\theta) = \phi_{F,o}(\theta_0 \pm 0)$,
   (ii) $\Gamma_{F,o}(\theta_0) := \phi_{F,o}(\theta_0 + 0) - \phi_{F,o}(\theta_0 - 0)$ is an element of $\{n\pi/2\}_{n\in\mathbf{Z}}$;

(c) The *index* $\mathrm{ind}_o(\mathsf{G}_F)$ of $o$ on $\mathsf{G}_F$ is represented as follows:

$$\mathrm{ind}_o(\mathsf{G}_F) = \frac{\eta_{g_F}(\theta + 2\pi) - \eta_{g_F}(\theta)}{2\pi} + \frac{1}{2\pi} \sum_{\theta_0 \in S_{g_F} \cap [\theta, \theta+2\pi)} \Gamma_{F,o}(\theta_0). \tag{8}$$

For an integer $k \geqq 3$, set $\mathcal{P}_o^k := \mathcal{P}^k \cap \mathcal{C}_o^{\infty,2}$. Then for any $g \in \mathcal{P}_o^k$, the following hold: $\Gamma_{g,o}(\theta_0) = -\pi/2$ for any $\theta_0 \in S_g$ ([4]); $\mathrm{ind}_o(\mathsf{G}_g) \in \{1 - k/2 + i\}_{i=0}^{[k/2]}$ ([2]). Let $\mathcal{C}_{oo}^{\infty,2}$ be the subset of $\mathcal{C}_o^{\infty,2}$ such that for each $F \in \mathcal{C}_{oo}^{\infty,2}$, $o$ is an isolated umbilical point on each of $\mathsf{G}_F$ and $\mathsf{G}_{g_F}$. If $F$ is an element of $\mathcal{C}_o^{\langle\infty,2\rangle}$ satisfying $S_{g_F} = \emptyset$, then $F \in \mathcal{C}_{oo}^{\infty,2}$ holds ([5], [6]). We see that if $F \in \mathcal{C}_{oo}^{\infty,2}$ satisfies $S_{g_F} = \emptyset$, then the following hold:

$$\mathrm{ind}_o(\mathsf{G}_F) = \mathrm{ind}_o(\mathsf{G}_{g_F}) = \frac{\eta_{g_F}(\theta + 2\pi) - \eta_{g_F}(\theta)}{2\pi}. \tag{9}$$

For any $F \in \mathcal{C}_{oo}^{\infty,2}$, the following hold ([5], [6]):

(a) $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$ for any $\theta_0 \in S_{g_F}$;

(b) $\mathrm{ind}_o(\mathsf{G}_{g_F}) \leqq \mathrm{ind}_o(\mathsf{G}_F) \leqq 1$.

If $F$ is an element of $\mathcal{C}_o^{\infty,2}$ satisfying $\Gamma_{F,o}(\theta_0) \leqq \pi$ for any $\theta_0 \in S_{g_F}$, then $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1$ holds ([5], [6]). In general, it is expected that the index of an isolated umbilical point on a surface does not exceed one (which is called the *index conjecture* or the *local Carathéodory's conjecture*).

We presented one way of computing $\eta_g(\theta + 2\pi) - \eta_g(\theta)$ for any $g \in \mathcal{P}^k$ ([5]). For $\theta \in \mathbf{R}$, set $\tilde{g}(\theta) := g(\cos\theta, \sin\theta)$. A number $\theta_0 \in \mathbf{R}$ is called a *root* of $g$ if $(d\tilde{g}/d\theta)(\theta_0) = 0$. The set of the roots of $g$ is denoted by $R_g$. Let $R(\mathrm{Hess}_g)$ be the set of numbers such that each $\theta_0 \in R(\mathrm{Hess}_g)$ satisfies $\theta_0 \in \{\eta_g(\theta_0) + n\pi/2\}_{n\in\mathbf{Z}}$. For $\theta \in \mathbf{R}$, we set $\mathrm{grad}_g(\theta) := \mathrm{grad}_g(\cos\theta, \sin\theta)$. Then the following holds:

$$(k-1)\,\mathrm{grad}_g(\theta) = \mathrm{Hess}_g(\theta)u_\theta. \tag{10}$$

From (10), we obtain

$$\langle \mathrm{Hess}_g(\theta)u_\theta, u_{\theta+\pi/2}\rangle = (k-1)\frac{d\tilde{g}}{d\theta}(\theta). \tag{11}$$

Therefore we obtain $S_g \subset R_g$ and $R(\mathrm{Hess}_g) \subset R_g$. Suppose $R_g = \mathbf{R}$. Then $k$ is even and $g$ is represented by $(x^2+y^2)^{k/2}$ up to a constant. By direct computations, we obtain $S_g = \emptyset$. Therefore $o$ is an isolated umbilical point of $\mathsf{G}_g$. By (11), we see that $R(\mathrm{Hess}_g) = \mathbf{R}$, i.e., there exists a number $z_0 \in \{n\pi/2\}_{n\in\mathbf{Z}}$ satisfying $\eta_g(\theta) = \theta + z_0$ for any $\theta \in \mathbf{R}$. Therefore by (9), we obtain

$$\mathrm{ind}_o(\mathsf{G}_g) = \frac{\eta_g(\theta+2\pi) - \eta_g(\theta)}{2\pi} = 1.$$

In the following, suppose $R_g \neq \mathbf{R}$. Then for each $\theta_0 \in R_g$, there exists a positive integer $\mu$ satisfying $(d^{\mu+1}\tilde{g}/d\theta^{\mu+1})(\theta_0) \neq 0$. The minimum of such integers is denoted by $\mu_g(\theta_0)$. A root $\theta_0 \in R_g$ is said to be

(a) *related* if $\theta_0$ satisfies $\tilde{g}(\theta_0) = 0$ or if $\mu_g(\theta_0)$ is odd;

(b) *non-related* if $\theta_0$ satisfies $\tilde{g}(\theta_0) \neq 0$ and if $\mu_g(\theta_0)$ is even.

Suppose that $\theta_0 \in R_g$ is related. Then it is said that the *critical sign* of $\theta_0$ is positive (respectively, negative) if the following holds:

$$\tilde{g}(\theta_0)\frac{d^{\mu_g(\theta_0)+1}\tilde{g}}{d\theta^{\mu_g(\theta_0)+1}}(\theta_0) \leqq 0 \quad (\text{respectively, } > 0).$$

The critical sign of $\theta_0$ is denoted by $\text{c-sign}_g(\theta_0)$. The set $R_g \setminus R(\mathrm{Hess}_g)$ consists of the numbers at each of which $\mathrm{Hess}_g$ is represented by the unit matrix up to a nonzero constant; in addition, an element $\theta_0 \in R_g \setminus R(\mathrm{Hess}_g)$ is related and satisfies $\text{c-sign}_g(\theta_0) = -$ ([5]). It is said that the *sign* of $\theta_0 \in R(\mathrm{Hess}_g)$ is positive (respectively, negative) if there exists a neighborhood $U_{\theta_0}$ of $\theta_0$ in $\mathbf{R}$ satisfying

$$\{\theta - \eta_g(\theta) - (\theta_0 - \eta_g(\theta_0))\}(\theta - \theta_0) > 0 \quad (\text{respectively, } < 0)$$

for any $\theta \in U_{\theta_0} \setminus \{\theta_0\}$. For $\theta_0 \in R(\mathrm{Hess}_g)$, $\theta_0$ is related if and only if the sign of $\theta_0$ is positive or negative ([5]). If $\theta_0 \in R(\mathrm{Hess}_g)$ is related, then the sign of $\theta_0$ is denoted by $\mathrm{sign}_g(\theta_0)$. For a related root $\theta_0$ of $g$ satisfying $\text{c-sign}_g(\theta_0) = +$, $\theta_0 \in R(\mathrm{Hess}_g)$ and $\mathrm{sign}_g(\theta_0) = +$ hold ([5]). Referring to [3], we see that if $\theta_0$ is a related element of $R(\mathrm{Hess}_g)$ satisfying $\text{c-sign}_g(\theta_0) = -$, then the condition $\mathrm{sign}_g(\theta_0) = +$ (respectively, $-$) is equivalent to the following:

$$\frac{1}{\tilde{g}(\theta_0)}\frac{d^2\tilde{g}}{d\theta^2}(\theta_0) \in (k(k-2), \infty) \quad (\text{respectively, } [0, k(k-2))).$$

Let $n_{g,+}$ (respectively, $n_{g,-}$) denote the number of the related elements of $R(\mathrm{Hess}_g)$ in $[\theta, \theta+\pi)$ with positive (respectively, negative) sign. Then for any $\theta \in \mathbf{R}$, the following holds ([5]):

$$\frac{\eta_g(\theta+2\pi)-\eta_g(\theta)}{2\pi} = 1 - \frac{n_{g,+}-n_{g,-}}{2}. \tag{12}$$

## 4. The main theorem

We shall prove

**Theorem 4.1.** *Let $F$ be an element of $\mathcal{C}_o^{(\infty,2)}$ satisfying* (7) *for some $a_F \in \mathbf{R}$ and suppose that the graph $\mathsf{G}_F$ of $F$ is a Willmore surface such that there exists no totally umbilical neighborhood of $o$ in $\mathsf{G}_F$. Then the following hold*:
(a) $F \in \mathcal{C}_o^{\langle\infty,2\rangle}$;
(b) *If $o$ is an isolated umbilical point of $\mathsf{G}_F$, then* $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1/2$.

REMARK. Noticing Proposition 2.1 and that whether a one-dimensional subspace of the tangent plane at a point of a surface is a principal direction is invariant under any conformal transformation of $\overline{\mathbf{R}}^3$, we may suppose $F \in \mathcal{C}_o^{(\infty,3)}$ in Theorem 4.1.

REMARK. Although $F$ is an element of $\mathcal{C}_o^{(\infty,2)}$ such that $o$ is an isolated umbilical point of $\mathsf{G}_F$, $F \in \mathcal{C}_o^{\langle\infty,2\rangle}$ does not always hold. Let $f$ be a smooth function on a neighborhood of $(0,0)$ in $\mathbf{R}^2$ satisfying $f(0,0) = 0$ and $f > 0$ on a punctured neighborhood of $(0,0)$. Then $\exp(-1/f)$ is a smooth function defined on a punctured neighborhood of $(0,0)$ and smoothly extended to $(0,0)$ so that all the partial derivatives of $\exp(-1/f)$ at $(0,0)$ are equal to zero. Then we obtain $\exp(-1/f) \in \mathcal{C}_o^{(\infty,\infty)}$. Suppose that for each positive number $c > 0$, there exists a punctured neighborhood of $(0,0)$ on which the norm of the gradient vector field of $\log f$ is bounded from below by the number $c$. Then $o$ is an isolated umbilical point on the graph of $\exp(-1/f)$ ([7]). However, since $\exp(-1/f) \in \mathcal{C}_o^{(\infty,\infty)}$, we obtain $\exp(-1/f) \notin \mathcal{C}_o^{\langle\infty,2\rangle}$. (a) of Theorem 4.1 is crucial to the proof of (b) of Theorem 4.1.

Proof of (a) of Theorem 4.1. Let $\Delta_F$ be the Laplace operator on $\mathsf{G}_F$, and $K_F$, $H_F$ the Gaussian and the mean curvatures of $\mathsf{G}_F$, respectively. Then $H_F$ satisfies the following elliptic partial differential equation:

$$\{\Delta_F + 2(H_F^2 - K_F)\}H_F = 0. \tag{13}$$

If $H_F \equiv 0$, then $\mathsf{G}_F$ is a minimal surface and $F$ is real-analytic. Since $\mathsf{G}_F$ is not totally umbilical, we obtain $F \not\equiv 0$ and this implies $F \in \mathcal{C}_o^{\langle\infty,3\rangle}$. If $H_F \not\equiv 0$, then $H_F$ is a non-trivial solution of (13) and referring to [14] as in [15], we see that not all the partial derivatives of $H_F$ at $(0,0)$ are equal to zero. This implies $F \in \mathcal{C}_o^{\langle\infty,3\rangle}$.

Hence we obtain (a) of Theorem 4.1. □

Proof of (b) of Theorem 4.1. Let $F$ be an element of $\mathcal{C}_o^{\langle\infty,3\rangle}$ such that the graph $\mathsf{G}_F$ of $F$ is a Willmore surface. Then there exist an integer $k_F \geqq 3$ and a nonzero homogeneous polynomial $g_F \in \mathcal{P}^{k_F}$ satisfying $F - g_F \in \mathcal{C}_o^{\langle\infty,k_F+1\rangle}$, and noticing (6) and (13), we see that $g_F$ satisfies $\Delta_0^2 g_F \equiv 0$, where $\Delta_0 := (\partial/\partial x)^2 + (\partial/\partial y)^2$. Therefore there exist spherical harmonic functions $h_{k_F}$, $h_{k_F-2}$ of degree $k_F$, $k_F - 2$, respectively such that $g_F$ is represented as

$$g_F = h_{k_F} + (x^2 + y^2)h_{k_F-2}.$$

Suppose $S_{g_F} = \emptyset$. Then $F \in \mathcal{C}_{oo}^{\infty,2}$ holds. Noticing that the number of the zero points of $\tilde{g}_F$ in $[\theta, \theta + \pi)$ is more than or equal to $k_F - 2$, we obtain

$$k_F - 2 \leqq \sharp\{R_{g_F} \cap [\theta, \theta + \pi)\} \leqq k_F$$

and

$$(n_{g_F,+}, n_{g_F,-}) \in \{(k_F - 2, 0), (k_F - 1, 1), (k_F, 0)\}.$$

Therefore by (9), (12) and $k_F \geqq 3$, we obtain

$$\mathrm{ind}_o(\mathsf{G}_F) \leqq 1 - \frac{k_F - 2}{2} = 2 - \frac{k_F}{2} \leqq \frac{1}{2}.$$

Suppose $S_{g_F} \neq \emptyset$ and $F \in \mathcal{C}_{oo}^{\infty,2}$. Then we obtain $\sharp\{S_{g_F} \cap [\theta, \theta+\pi)\} = 1$, $(n_{g_F,+}, n_{g_F,-}) = (k_F - 1, 0)$ and $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$ for any $\theta_0 \in S_{g_F}$. Therefore by (8), (12) and $k_F \geqq 3$, we obtain

$$\mathrm{ind}_o(\mathsf{G}_F) \leqq 1 - \frac{k_F - 1}{2} + \frac{1}{2} = 2 - \frac{k_F}{2} \leqq \frac{1}{2}.$$

Suppose $S_{g_F} \neq \emptyset$, $F \in \mathcal{C}_o^{\infty,2}$ and $F \notin \mathcal{C}_{oo}^{\infty,2}$. Then there exists an element $\theta_0 \in S_{g_F}$ satisfying $\tilde{g}_F(\theta_0) = 0$ and $\mu_{g_F}(\theta_0) = 2$. We obtain $\sharp\{S_{g_F} \cap [\theta, \theta+\pi)\} = 1$ and $(n_{g_F,+}, n_{g_F,-}) = (k_F - 1, 0)$. We shall prove $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$, which implies $\mathrm{ind}_o(\mathsf{G}_F) \leqq 1/2$. We may suppose $\theta_0 = 0$ and represent $g_F$ as

$$g_F(x, y) = g_0(x, y)y^3, \tag{14}$$

where $g_0$ is a homogeneous polynomial of degree $k_F - 3$ satisfying $g_0(x, 0) \neq 0$ for any $x \in \mathbf{R} \setminus \{0\}$. We set

$$a_F := s_F + s_F p_F^2 - p_F q_F r_F,$$
$$2b_F := t_F - r_F + t_F p_F^2 - r_F q_F^2,$$

$$c_F := -s_F - s_F q_F^2 + p_F q_F t_F.$$

Then the following holds:

$$(1 + p_F^2 + q_F^2)\,\mathrm{PD}_F = a_F\,dx^2 + 2b_F\,dx\,dy + c_F\,dy^2.$$

We set

$$\tilde{b}_F(\rho, \theta) := b_F(\rho\cos\theta, \rho\sin\theta)$$

for $(\rho, \theta) \in (-\rho_0, \rho_0) \times \mathbf{R}$, where $\rho_0 > 0$ is a positive number such that there exists no umbilical point of $\mathsf{G}_F$ on $\{0 < x^2 + y^2 < \rho_0^2\}$. There exists a smooth function $\tilde{b}_F^{(k_F-2)}$ on $\mathbf{R}$ satisfying

$$\tilde{b}_F(\rho, \theta) - \rho^{k_F-2}\tilde{b}_F^{(k_F-2)}(\theta) = o(\rho^{k_F-2}).$$

From (14), we obtain $(d\tilde{b}_F^{(k_F-2)}/d\theta)(0) \neq 0$. Therefore by the implicit function theorem, we see that there exist a neighborhood $V_0$ of $(0, 0)$ in $\mathbf{R}^2$ and a curve $C_0$ in $V_0$ through $(0, 0)$ satisfying
(a) $C_0 = \{(\rho, \theta) \in V_0\,;\, \tilde{b}_F(\rho, \theta)/\rho^{k_F-2} = 0\}$;
(b) $C_0$ is not tangent to the $\theta$-axis at $(0, 0)$.
Then noticing the behavior of the two continuous distributions around $o$ defined by

$$b_F\,dx^2 + (c_F - a_F)\,dx\,dy - b_F\,dy^2 = 0,$$

we obtain $-\pi/2 \leqq \Gamma_{F,o}(\theta_0) \leqq \pi/2$. □

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