

## NOTE ON A 2-DIMENSIONAL VERSAL $D_8$ -COVER

Dedicated to Professor Makoto Namba on his sixtieth birthday

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(Received April 9, 2003)

### Introduction

Let  $G$  be a finite group. Let  $X$  and  $Y$  be normal projective varieties.  $X$  is called a  $G$ -cover of  $Y$  if there exists a finite surjective morphism  $\pi: X \rightarrow Y$  such that the induced morphism  $\pi^*: \mathbb{C}(Y) \rightarrow \mathbb{C}(X)$  gives a Galois extension with  $\text{Gal}(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$ , where  $\mathbb{C}(X)$  and  $\mathbb{C}(Y)$  denote the rational function fields of  $X$  and  $Y$ , respectively.

DEFINITION 0.1. A  $G$ -cover  $\varpi: X \rightarrow M$  is said to be versal if it satisfies the following property:

For any  $G$ -cover  $\pi: Y \rightarrow Z$ , there exist a rational map  $\nu: Z \cdots \rightarrow M$  and a Zariski open set  $U$  in  $Z$  such that

- (i)  $\nu|_U: U \rightarrow M$  is a morphism, and
- (ii)  $\pi^{-1}(U)$  is birational to  $U \times_M X$  over  $U$ .

One could say the investigation of versal  $G$ -covers is a geometric study of generic or versal  $G$ -polynomials (see [2] and [4]). The notion of versal  $G$ -covers implicitly appeared in [7], [8], and is defined explicitly in [12], [13]. It is known that there exists a versal  $G$ -cover for any finite group  $G$  ([8, Theorem 2.4]). The dimension of the versal  $G$ -cover given by Namba, however, is equal to  $\sharp(G)$ . Hence it does not seem to be tractable in practical use. We need to find a tractable model for  $G$ . So far it has been done for some cases by ad-hoc methods in ([12], [13]).

In this note, we consider versal  $D_8$ -covers, where  $D_8$  is the dihedral group of order 8, i.e.,  $D_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1 \rangle$ . It is known that any versal  $D_8$ -cover has dimension at least 2 (see [2]), and one of such models was given in [12]. The purpose of this article is to give another new model, which is described as follows:

Let  $\varphi_{141}: X_{141} \rightarrow \mathbb{P}^1$  be the rational elliptic surface obtained by blowing-up base

points of the (2, 2)-pencil on  $\mathbb{P}^1 \times \mathbb{P}^1$  given by

$$\{ \lambda_0(s_0 - s_1)^2(t_0 - t_1)^2 + \lambda_1(s_0s_1t_0t_1) = 0 \}_{[\lambda_0:\lambda_1] \in \mathbb{P}^1}.$$

As for  $X_{141}$ , the following facts are well-known:

- (i)  $X_{141}$  is so-called the elliptic modular surface attached to

$$\Gamma_1(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{4} \right\},$$

and  $\varphi_{141}$  has three singular fibers and their types are of  $I_1^*$ ,  $I_4$  and  $I_1$  (see [3, p.350]).

- (ii) The group of sections,  $MW(X_{141})$ , is isomorphic to  $\mathbb{Z}/4\mathbb{Z}$  (see [6] or [9]).

Let  $\sigma_{\varphi_{141}}$  be the involution on  $X_{141}$  induced by the inversion with respect to the group law and let  $\tau_s$  be the translation by a 4-torsion section  $s$ .  $\sigma_{\varphi_{141}}$  and  $\tau_s$  generate a finite fiber preserving automorphism group isomorphic to  $D_8$ . Let  $\Sigma_{141} := X_{141}/\langle \sigma_{\varphi_{141}}, \tau_s \rangle$  be the quotient surface and we denote its quotient morphism by  $\pi_{141}: X_{141} \rightarrow \Sigma_{141}$ . Now we are in position to state our result:

**Theorem 0.2.**  $\pi_{141}: X_{141} \rightarrow \Sigma_{141}$  is a versal  $D_8$ -cover.

In comparison with the model in [12], this model has a nice description with respect to the action of the Galois group.

### 1. Preliminaries

Let  $S$  be a smooth minimal projective surface, and let  $\Lambda$  be a pencil of curves on  $S$  such that a general member is irreducible. Let  $\bar{\varphi}_\Lambda: S \cdots \rightarrow \mathbb{P}^1$  be the rational map determined by  $\Lambda$ , and let  $q: S_\Lambda \rightarrow S$  be the resolution of the indeterminacy of  $\bar{\varphi}_\Lambda$ . We denote the induced morphism from  $S_\Lambda$  to  $\mathbb{P}^1$  by  $\varphi_\Lambda$ . We may assume that  $\varphi_\Lambda$  is relatively minimal. Let us begin with the following lemma.

**Lemma 1.1.** *Let  $\sigma$  be an automorphism of  $S$ . Suppose that  $\bar{\varphi}_\Lambda^\sigma = \bar{\varphi}_\Lambda$  (we regard  $\bar{\varphi}_\Lambda$  as an element of  $\mathbb{C}(S_\Lambda)$ ). Then  $\sigma$  gives rise to a fiber preserving automorphism of  $S_\Lambda$  (By abuse of notation, we also denote it by  $\sigma$ ).*

*Proof.* Since  $\mathbb{C}(S) \cong \mathbb{C}(S_\Lambda)$ ,  $\bar{\varphi}_\Lambda^\sigma = \bar{\varphi}_\Lambda$  implies  $\varphi_\Lambda^\sigma = \varphi_\Lambda$ . Hence a general fiber of  $\varphi_\Lambda$  goes to that of  $\varphi_\Lambda$  under  $\sigma$ . Let  $\bar{\sigma}$  be the induced birational map from  $S_\Lambda$  to itself induced by  $\sigma$ . Let  $\mu: \hat{S}_\Lambda \rightarrow S_\Lambda$  be a succession of blowing-ups so that (i)  $\hat{\sigma} := \bar{\sigma} \circ \mu$  becomes a birational morphism and (ii) the number of the blowing-ups is minimal. It is well-known that  $\hat{\sigma}$  is a composition of blowing-downs. Let  $E_1, \dots, E_r$  be the exceptional divisors for  $\mu$ , and let  $F_1, \dots, F_s$  be those for  $\hat{\sigma}$ . We may assume that  $F_1$  is the  $(-1)$  curve for the first blow-down. Since  $F_1$  can not be any of  $E_i$

$(i = 1, \dots, r)$ ,  $F_1 E_i \geq 0$  ( $i = 1, \dots, r$ ). Since

$$-1 = F_1 K_{S_\Lambda} = F_1 \left( \mu^* K_{S_\Lambda} + \sum_{j=1}^r m_j E_j \right) = F_1 \mu^* K_{S_\Lambda} + F_1 \left( \sum_{j=1}^r m_j E_j \right),$$

we have  $F_1 \mu^* K_{S_\Lambda} \leq -1$ . Hence  $\mu(F_1)$  is not any irreducible component in a fiber of  $\varphi_\Lambda$ . Since the image of  $\mu(F_1)$  by  $\sigma$  is a point, this implies that a general fiber of  $\varphi_\Lambda$  does not go to that of  $\varphi_\Lambda$  under  $\hat{\sigma}$ , which leads us to a contradiction.  $\square$

EXAMPLE 1.2. Let  $(x, y)$  be an inhomogeneous coordinate of  $\mathbb{P}^1 \times \mathbb{P}^1$ . The pencil on  $\mathbb{P}^1 \times \mathbb{P}^1$  in Introduction is given by

$$\Lambda: \{ \lambda_0(x-1)^2(y-1)^2 + \lambda_1 xy = 0 \}_{[\lambda_0, \lambda_1] \in \mathbb{P}^1}.$$

Let  $\sigma$  and  $\tau$  be the automorphisms of  $\mathbb{P}^1 \times \mathbb{P}^1$  given by

$$(x^\sigma, y^\sigma) = (y, x), \quad (x^\tau, y^\tau) = \left( y, \frac{1}{x} \right).$$

$\sigma$  and  $\tau$  generate a finite automorphism group isomorphic to  $D_8$ . Since the rational function on  $\mathbb{P}^1 \times \mathbb{P}^1$ ,

$$\frac{(x-1)^2(y-1)^2}{xy},$$

is invariant under  $\sigma$  and  $\tau$ , one can apply Lemma 1.1 to this case. In fact, it follows that  $(\mathbb{P}^1 \times \mathbb{P}^1)_\Lambda = X_{141}$ ,  $\varphi_\Lambda = \varphi_{141}$ , and  $\sigma, \tau$  fiber preserving automorphisms of  $X_{141}$ . By Example 4.7, Chapter III in [11], both  $\sigma$  and  $\tau$  are the compositions of isogenies and translations. In particular, the corresponding isogenies should be isomorphisms as an elliptic curve. By Theorem 10.1, Chapter III in [11], the automorphisms of  $X_{141}$  as an elliptic curve are only the identity and  $\sigma_{\varphi_{141}}$ . Since  $\sigma \circ \tau$  has a fixed point  $(1, 0)$ , the induced fiber preserving automorphism on  $X_{141}$  is non-trivial and fixes a section,  $O$ . Hence we may assume that  $\sigma \circ \tau = \sigma_{\varphi_{141}}$  on  $X_{141}$ , by regarding  $O$  as the zero. Also if  $\tau$  is the composition of  $\sigma_{\varphi_{141}}$  and a translation, then  $\tau^2 = \text{id}$  on  $X_{141}$ . This implies that we may assume that  $\tau$  is a translation by a 4-torsion section  $s$ .

EXAMPLE 1.3. Let  $N = \mathbb{Z}^{\oplus 2}$  and let  $\Delta_i$  ( $i = 1, \dots, 6$ ) be 2-dimensional cones in  $N \otimes \mathbb{R}$  given by

$$\begin{aligned} \Delta_1 &= \mathbb{R}_{\geq 0} \mathbf{e}_1 + \mathbb{R}_{\geq 0} (\mathbf{e}_1 + \mathbf{e}_2) & \Delta_2 &= \mathbb{R}_{\geq 0} (\mathbf{e}_1 + \mathbf{e}_2) + \mathbb{R}_{\geq 0} \mathbf{e}_2 \\ \Delta_3 &= \mathbb{R}_{\geq 0} (-\mathbf{e}_1) + \mathbb{R}_{\geq 0} \mathbf{e}_2 & \Delta_4 &= \mathbb{R}_{\geq 0} (-\mathbf{e}_1) + \mathbb{R}_{\geq 0} (-\mathbf{e}_1 - \mathbf{e}_2) \\ \Delta_5 &= \mathbb{R}_{\geq 0} (-\mathbf{e}_1 - \mathbf{e}_2) + \mathbb{R}_{\geq 0} (-\mathbf{e}_2) & \Delta_6 &= \mathbb{R}_{\geq 0} (-\mathbf{e}_2) + \mathbb{R}_{\geq 0} \mathbf{e}_1, \end{aligned}$$

where  $e_1 = {}^t(1, 0)$ ,  $e_2 = {}^t(0, 1)$ . Let  $\Sigma = \bigcup_{i=1}^6 \Delta_i$  and  $X_{D_{12}} = T_N \text{emb}(\Sigma)$  (see [10, §1.2]). The subgroup,  $G$ , of  $\text{GL}(2, \mathbb{Z})$  generated by

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

is isomorphic to  $D_{12}$ . Since  $G$  preserves  $\Sigma$  as above,  $D_{12}$  acts on  $X_{D_{12}}$ . An explicit description is as follows:

Let  $\{f_1, f_2\}$  be the basis of  $\text{Hom}(N, \mathbb{Z})$  dual to  $\{e_1, e_1 + e_2\}$  and let  $x = e(f_1)$  and  $y = e(f_2)$ .  $(x, y)$  gives an affine coordinate of the affine open set,  $U_{\Delta_1}$ , determined by  $\Delta_1$ . Then

$$(x^\sigma, y^\sigma) = (y, x), \quad (x^\tau, y^\tau) = \left(\frac{1}{y}, xy\right).$$

Consider the pencil on  $X$  which is given on  $U_{\Delta_1}$  by

$$\{\lambda_0(xy) + \lambda_1(xy + 1)(x + 1)(y + 1) = 0\}_{[\lambda_0, \lambda_1] \in \mathbb{P}^1}$$

on  $U_{\sigma_1}$ . Since

$$\frac{xy}{(xy + 1)(x + 1)(y + 1)}$$

is  $D_{12}$ -invariant, one can apply Lemma 1.1 to this case. In this case,  $X_\Lambda$  coincides with the rational elliptic surface  $\varphi_{6321}: X_{6321} \rightarrow \mathbb{P}^1$  (the notation is due to [6]). The following facts on  $X_{6321}$  are well-known:

(i)  $X_{6321}$  is so-called the elliptic modular surface attached to

$$\Gamma_1(6) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{6} \right\},$$

and has four singular fibers and their types are of  $I_6, I_3, I_2$  and  $I_1$  (see [1] and [6]).

(ii) The group of sections,  $MW(X_{6321})$ , is isomorphic to  $\mathbb{Z}/6\mathbb{Z}$  (see [6]).

In our case,  $\varphi_\Lambda = \varphi_{6321}$ , and  $\sigma$  and  $\tau$  generate a fiber preserving automorphism group of  $X_{6321}$  isomorphic to  $D_{12}$ . By the same argument to that in Example 1.2, we may assume that it coincides with one given by the inversion with respect to the group law and the translation by a 6-torsion section.

We here raise a question concerning Example 1.3.

**Question 1.4.** Let  $M_{D_{12}}$  be the quotient of  $X_{D_{12}}$  with respect to the  $D_{12}$ -action in Example 1.3. Is the  $D_{12}$ -cover  $X_{D_{12}} \rightarrow M_{D_{12}}$  versal? In other words, is  $X_{6321} \rightarrow \Sigma_{6321}$  a versal  $D_{12}$ -cover, where  $\Sigma_{6321}$  is the quotient by the inversion with respect to the group law and the translation by 6-torsion?

**2. Proof of Theorem 0.2**

Let  $\sigma$  and  $\tau$  be the automorphisms of  $\mathbb{P}^1 \times \mathbb{P}^1$  as in Example 1.2. They give rise to a finite automorphism group isomorphic to  $D_8$ . Put  $\Sigma_{D_8} := \mathbb{P}^1 \times \mathbb{P}^1 / \langle \sigma, \tau \rangle$ , and we denote the quotient morphism by  $\varpi_{D_8} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \Sigma_{D_8}$ . Theorem 0.2 follows from Example 1.2 and the proposition below.

**Proposition 2.1.**  $\varpi_{D_8} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \Sigma_{D_8}$  is versal.

We need two lemmas to prove Proposition 2.1.

**Lemma 2.2.** Let  $\pi : Y \rightarrow Z$  be a  $D_8$ -cover. Then there exist non-constant rational functions  $\psi_1$  and  $\psi_2$  such that

- (i)  $(\psi_1^\sigma, \psi_2^\sigma) = (\psi_2, \psi_1)$ ,
- (ii)  $(\psi_1^\tau, \psi_2^\tau) = (\psi_2, 1/\psi_1)$ ,
- (iii)  $\psi_1/\psi_2 \notin \mathbb{C}$  and
- (iv)  $\psi_1\psi_2 \neq 1$ .

*Proof.* By the normal basis theorem (see [5, p.229]), there exists  $\theta \in \mathbb{C}(Y)$  such that  $\{\theta^i, \theta^{\sigma^i}\}$  ( $i = 0, 1, 2, 3$ ) form a basis of  $\mathbb{C}(Y)$  as a vector  $\mathbb{C}(Z)$ -space. Put

$$\psi_1 = \frac{\theta + \theta^\sigma + \theta^{\tau^3} + \theta^{\sigma\tau^3}}{\theta^{\tau^2} + \theta^{\sigma\tau^2} + \theta^\tau + \theta^{\sigma\tau}}, \quad \psi_2 = \frac{\theta + \theta^\sigma + \theta^\tau + \theta^{\sigma\tau}}{\theta^{\tau^2} + \theta^{\sigma\tau^2} + \theta^{\tau^3} + \theta^{\sigma\tau^3}}.$$

Both  $\psi_1$  and  $\psi_2$  are non-constant rational functions since  $\{\theta^g \mid g \in D_8\}$  is a basis over  $\mathbb{C}(Z)$ , and the statements (i) and (ii) are straightforward. If  $\psi_1 = c\psi_2$  for some  $c \in \mathbb{C}$ , we have  $\psi_1 = \pm\psi_2$  by (i). If this happens, then we infer that  $\psi_2^2 = \pm 1$  by (ii), but this is impossible as  $\psi_2$  is non-constant. Suppose that  $\psi_1\psi_2 = 1$ . Then  $\psi_2/\psi_1 = 1$  by (i). This contradicts to (iii). □

**Lemma 2.3.** Let  $\psi_1$  and  $\psi_2$  be the rational functions as in Lemma 2.2. Then  $\mathbb{C}(Y) = \mathbb{C}(Z)(\psi_1, \psi_2)$ .

*Proof.* Choose a rational number  $c$  not equal to  $\pm 1$  and we put  $\psi = \psi_1 + c\psi_2$ . It is enough to see that  $\psi \neq \psi^g$  for all  $g(\neq 1) \in D_8$ .

- (i)  $\psi \neq \psi^\tau$ . If  $\psi = \psi^\tau$ , we have

$$\psi_1 - \psi_2 = c \left( \frac{1 - \psi_1\psi_2}{\psi_1} \right).$$

By Lemma 2.2,  $1 - \psi_1\psi_2 \neq 0$ ,  $\psi_1 - \psi_2 \neq 0$ . So

$$c = \frac{\psi_1 - \psi_2}{1 - \psi_1\psi_2} \psi_1 \neq 0.$$

On the other hand, we have

$$\left(\frac{\psi_1 - \psi_2}{1 - \psi_1\psi_2}\psi_1\right)^\sigma = \frac{\psi_2 - \psi_1}{1 - \psi_1\psi_2}\psi_2 = -\frac{\psi_1 - \psi_2}{1 - \psi_1\psi_2}\psi_2.$$

As  $c^\sigma = c$ , we have  $\psi_1 = -\psi_2$ , but this contradicts to Lemma 2.2, (iii).

(ii)  $\psi \neq \psi^{\tau^2}$ . If  $\psi = \psi^{\tau^2}$ , we have

$$c = \frac{(\psi_1^2 - 1)\psi_2}{(1 - \psi_2^2)\psi_1}.$$

By Lemma 2.2,  $c \neq 0$ . On the other hand, we have

$$\left(\frac{(\psi_1^2 - 1)\psi_2}{(1 - \psi_2^2)\psi_1}\right)^\sigma = \frac{(1 - \psi_2^2)\psi_1}{(\psi_1^2 - 1)\psi_2} = \frac{1}{c}.$$

As  $c^\sigma = c$ , we have  $c^2 = 1$ . This contradicts to our choice of  $c$ .

(iii)  $\psi \neq \psi^{\tau^3}$ . If  $\psi = \psi^{\tau^3}$ , we have

$$c = \frac{\psi_1\psi_2 - 1}{\psi_1 - \psi_2} \frac{1}{\psi_2}.$$

By the similar argument to the first case, we infer  $\psi_1 = -\psi_2$ , but this is impossible.

(iv)  $\psi \neq \psi^\sigma$ . If  $\psi = \psi^\sigma$ , we have  $c = 1$ , but this contradicts to our choice of  $c$ .

(v)  $\psi \neq \psi^{\sigma\tau}$ . If  $\psi = \psi^{\sigma\tau}$ ,  $\psi_1^2 = 1$ . This contradicts to Lemma 2.2.

(vi)  $\psi \neq \psi^{\sigma\tau^2}$ . If  $\psi = \psi^{\sigma\tau^2}$ , we have  $c = -\psi_1/\psi_2$ . As  $c = c^\sigma$ , we have  $\psi_1/\psi_2 = \psi_2/\psi_1$ .

This implies that  $\psi_1 = \pm\psi_2$ , but this contradicts to Lemma 2.2 (iii).

(vii)  $\psi \neq \psi^{\sigma\tau^3}$ . If  $\psi = \psi^{\sigma\tau^3}$ , we have  $\psi_2^2 = 1$ , but this is impossible. □

**Proof of Proposition 2.1.** Let  $\pi: Y \rightarrow Z$  be an arbitrary  $D_8$ -cover. By Lemmas 2.2 and 2.3, there exist non-constant rational functions  $\psi_1$  and  $\psi_2$  such that

(i)  $\mathbb{C}(Y) = \mathbb{C}(Z)(\psi_1, \psi_2)$  and (ii)  $(\psi_1^\sigma, \psi_2^\sigma) = (\psi_2, \psi_1)$  and  $(\psi_1^\tau, \psi_2^\tau) = (\psi_2, 1/\psi_1)$ .

Define the  $D_8$ -equivalent rational map  $\Psi: Y \cdots \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  by

$$p \in Y \mapsto (\psi_1(p), \psi_2(p)) \in \mathbb{P}^1 \times \mathbb{P}^1.$$

This shows Proposition 2.1. □

Now Theorem 0.2 follows from Proposition 2.1 and Example 1.2. We end this section with the following example.

EXAMPLE 2.4. Let

$$\rho: D_8 \rightarrow \mathrm{GL}(2, \mathbb{C})$$

be the representation given by

$$\rho(\sigma) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let  $\tilde{\rho} = 1_{D_8} \oplus \rho$ , and define the  $D_8$ -action on  $\mathbb{P}^2$  by

$$g([z_0, z_1, z_2]) = [z_0, z_1, z_2](\tilde{\rho}(g))^{-1}, \quad [z_0, z_1, z_2] \in \mathbb{P}^2.$$

Put  $X_1 = \mathbb{P}^2$  and  $M_1 = \mathbb{P}^2/D_8$  (note that  $X_1 \rightarrow M_1$  is the versal  $D_8$ -cover by [12, Proposition 4.1]). Let  $u$  and  $v$  be the rational functions of  $\mathbb{P}^2$  given by  $z_1/z_0$  and  $z_2/z_0$ , respectively. Then one can check

$$\{\theta^g\}_{g \in D_8}, \quad \theta = \frac{1}{1 - u - 2v}$$

form a basis over  $\mathbb{C}(M_1) = \mathbb{C}(u, v)^{D_8}$ . To see this, let

$$A := \begin{pmatrix} \theta & \theta^\tau & \theta^{\tau^2} & \theta^{\tau^3} & \theta^\sigma & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} \\ \theta^\tau & \theta^{\tau^2} & \theta^{\tau^3} & \theta & \theta^{\sigma\tau^3} & \theta^\sigma & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} \\ \theta^{\tau^2} & \theta^{\tau^3} & \theta & \theta^\tau & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^\sigma & \theta^{\sigma\tau} \\ \theta^{\tau^3} & \theta & \theta^\tau & \theta^{\tau^2} & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^\sigma \\ \theta^\sigma & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta & \theta^\tau & \theta^{\tau^2} & \theta^{\tau^3} \\ \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^\sigma & \theta & \theta^\tau & \theta^{\tau^2} & \theta^{\tau^3} \\ \theta^{\sigma\tau^2} & \theta^{\sigma\tau^3} & \theta^\sigma & \theta^{\sigma\tau} & \theta^{\tau^2} & \theta^{\tau^3} & \theta & \theta^\tau \\ \theta^{\sigma\tau^3} & \theta^\sigma & \theta^{\sigma\tau} & \theta^{\sigma\tau^2} & \theta^\tau & \theta^{\tau^2} & \theta^{\tau^3} & \theta \end{pmatrix},$$

and check that  $\det A \neq 0$ . The explicit forms of  $\psi_1$  and  $\psi_2$  with respect to the normal basis  $\{\theta^g\}_{g \in D_8}$  are as follows:

$$\begin{aligned} \psi_1 &= -\frac{(-2 + 6u^3 + 9u - 9uv^2 - 13u^2 + 5v^2)}{(2 + 6u^3 + 9u - 9uv^2 + 13u^2 - 5v^2)} \\ &\quad \times \frac{(1 + u + 2v)(1 + 2u + v)(1 + 2u - v)(1 + u - 2v)}{(-1 + u + 2v)(-1 + 2u + v)(-1 - v + 2u)(-1 + u - 2v)} \\ \psi_2 &= -\frac{(2 + 9u^2v - 5u^2 - 6v^3 + 13v^2 - 9v)}{(-2 + 9u^2v + 5u^2 - 6v^3 - 13v^2 - 9v)} \\ &\quad \times \frac{(1 + u + 2v)(1 + 2u + v)(-1 - v + 2u)(-1 + u - 2v)}{(-1 + u + 2v)(-1 + 2u + v)(1 + 2u - v)(1 + u - 2v)} \end{aligned}$$

Hence we have a  $D_8$ -equivalent rational map from  $\mathbb{P}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . Note that the existence of this rational map gives another proof for Proposition 2.1.

ACKNOWLEDGEMENT. The author thanks for the referee for his/her comments on the first version of this article.

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