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THE CHARACTER TABLE OF THE HECKE ALGEBRA $\mathcal{H}(GL_2(F_q), A)$, WHERE A IS THE SPLIT TORUS

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Introduction

The Hecke algebra $\mathcal{H}(G, A)$ of a group G relative to its subgroup A is a generalization of the group algebra $\mathbb{C}G$ of G, whose structure and representations are interesting mathematical objects as well as those of $\mathbb{C}G$. As is well known, the Hecke algebras of $GL_2(F)$ where F is a p-adic field relative to its open subgroups take a significant part in Number Theory or more precisely in the theory of modular forms.

On the other hand, Hecke algebras of finite groups have been studied in connection with the irreducible decomposition of various induced representations (cf. [2], [6], [7], [10]). Recently, it has emerged that they play an important role in Graph Theory. In fact, certain families of double cosets of a finite group G with respect to its subgroup A yield vertex transitive graphs with vertex set G/A and the spectra of those graphs are determined with the help of the irreducible characters of the Hecke algebra $\mathcal{H}(G, A)$ (see for example [5]). In this setting, A. Terras et al. ([1]) and R. Evans ([3]) have studied the structure and characters of $\mathcal{H}(GL_2(F_q), K)$ where F_q is a finite field and K is the anisotropic torus of $GL_2(F_q)$. In our previous paper ([9]), we have considered the structure of $\mathcal{H}(GL_2(F_q), A)$ where A is the split torus of $GL_2(F_q)$ and described the multiplication table with respect to the standard basis of it. The aim of the present article is to determine all the irreducible characters of $\mathcal{H}(GL_2(F_q), A)$ and describe the character table with respect to the standard basis of it. Throughout the paper, we assume that q is a power of an odd prime.

The paper is organized as follows. §1 contains several results concerning a finite field F_q, which are useful for computing the character values of $\mathcal{H}(G, A)$. Here we put $G = GL_2(F_q)$ for simplicity. In §2 we give a complete set $\mathcal{R}$ of representatives of the double coset space $A\backslash G/A$ and the standard basis $\{\varepsilon[g];\, g \in \mathcal{R}\}$ of $\mathcal{H}(G, A)$. In §3 we give the irreducible decomposition of the induced character 1_A^G (see Theorem 3.3). As a by-product, we get the set $\hat{G}^A$ of all irreducible characters of $\mathcal{H}(G, A)$. In §4 we describe the character table $(\chi(\varepsilon[g]))_{g\in\mathcal{R},\chi\in\hat{G}^A}$ of $\mathcal{H}(G, A)$ in Main Theorem. In order to calculate the value of $\chi(\varepsilon[g])$, it is essential to decide the conjugacy class of ag for $a \in A$ and $g \in \mathcal{R}$, which is performed in Lemma 4.3.

The results of the paper and ([9]) will be applied to the construction of vertex transitive graphs over G/A and the determination of the spectra of those graphs, which will be discussed in a subsequent paper. We also mention that our results about the Hecke algebra $\mathcal{H}(G, A)$ will be useful for the study of the Hecke algebra of $GL_2(F)$ relative to its certain open subgroup where F is a p-adic field.

1. A finite field with q elements

Let $F = F_q$ be a finite field with q elements where q is a power of an odd prime p. Let $F^\times = F - \{0\}$ be the multiplicative group of F. Then $F^\times$ is a cyclic group of order $q-1$. Fix a generator ρ of $F^\times$, so that $F^\times = \{\rho^k\,;\, k = 0, 1, \dots, q-2\}$. Let $F_0^\times$ be the subgroup of $F^\times$ consisting of squares of $F^\times$, and put $F_1^\times = F^\times - F_0^\times$. Then $F_0^\times = \{\rho^{2j}\,;\, j = 0, 1, \dots, (q-3)/2\}$, $F_1^\times = \{\rho^{2j+1}\,;\, j = 0, 1, \dots, (q-3)/2\}$, and hence $F_1^\times = \rho F_0^\times$. Since $-1 = \rho^{(q-1)/2}$, it follows that $-1 \in F_0^\times$ if and only if $q \equiv 1 \pmod 4$. In the following if $t = \rho^{2j} \in F_0^\times$, then we write $\sqrt{t}$ for ρ^j. Let $\hat{F}^\times$ be the set of all characters of $F^\times$. Define the character λ_k of $F^\times$ by $\lambda_k(\rho^j) = e^{2\pi ikj/(q-1)}$ where $k = 0, 1, \dots, q-2$ and $i = \sqrt{-1}$. Then $\hat{F}^\times = \{\lambda_k\,;\, k = 0, 1, \dots, q-2\}$. In particular we write $1_F = \lambda_0$ (the trivial character of $F^\times$) and $\sigma_F = \lambda_{(q-1)/2}$. The character σ_F has the property that $\sigma_F(t) = 1$ for $t \in F_0^\times$ and $\sigma_F(t) = -1$ for $t \in F_1^\times$. We extend σ_F to a multiplicative function on F by putting $\sigma_F(0) = 0$.

Let $E = F(\sqrt{\rho}) = \{\zeta = x + y\sqrt{\rho}\,;\, x, y \in F\}$ be the quadratic extension of F. Then E is a finite field with q^2 elements. It is well known that $\zeta^q = x - y\sqrt{\rho}$ for $\zeta = x + y\sqrt{\rho}$. Let $N : E \to F$ be the norm map. Then $N(\zeta) = \zeta\zeta^q = x^2 - y^2\rho$ for $\zeta = x + y\sqrt{\rho}$. Let $E^\times$ be the multiplicative group of E. Then $E^\times$ is a cyclic group of order $q^2 - 1$. Choose a generator τ of $E^\times$ satisfying $\tau^{q+1} = \rho$ and $\tau^l \in F^\times$ $(l = 1, \dots, q)$. Note that $N\colon E^\times \to F^\times$ is a surjective homomorphism. For $t \in F^\times$ we put $E_t^\times = \{\zeta \in E^\times\,;\, N(\zeta) = t\}$. Then it is easy to check that $E_1^\times = \{\tau^{j(q-1)}\,;\, j = 0, 1, \dots, q\}$, $E_\rho^\times = \{\tau\zeta\,;\, \zeta \in E_1^\times\}$, $E_{t^2}^\times = \{t\zeta\,;\, \zeta \in E_1^\times\}$ and $E_{t^2\rho}^\times = \{t\zeta\,;\, \zeta \in E_\rho^\times\}$ for $t \in F^\times$. Let $\hat{E}^\times$ be the set of all characters of $E^\times$. Define the character $\theta_k (k = 0, 1, \dots, q^2 - 2)$ for $E^\times$ by $\theta_k(\tau^j) = e^{2\pi ikj/(q^2-1)}$. Then $\hat{E}^\times = \{\theta_k\,;\, k = 0, 1, \dots, q^2 - 2\}$. Note that $\theta_k^q = \theta_k$ if and only if $\theta_k = \lambda_k \circ N$ where $\lambda_k \in \hat{F}^\times$, and $\theta_k|_{F^\times} = 1_F$ if and only if $k = l(q-1)$ where $l = 0, 1, \dots, q$. The following lemmas will be used later in the proof of the main theorem.

Lemma 1.1. *Put* $1 + F_0^\times = \{1 + t\,;\, t \in F_0^\times\}$ *and* $1 + F_1^\times = \{1 + t\,;\, t \in F_1^\times\}$.
(i) *If* $q \equiv 1 \pmod 4$, *then*

$$|(1 + F_1^\times) \cap F_0^\times| = |(1 + F_1^\times) \cap F_1^\times| = |(1 + F_0^\times) \cap F_1^\times| = \frac{q-1}{4},$$
$$|(1 + F_0^\times) \cap F_0^\times| = \frac{q-5}{4}.$$

(ii) *If* $q \equiv 3 \pmod 4$, *then*

$$\left|(1+F_1^\times)\cap F_0^\times\right| = \left|(1+F_1^\times)\cap F_1^\times\right| = \left|(1+F_0^\times)\cap F_0^\times\right| = \frac{q-3}{4},$$
$$\left|(1+F_0^\times)\cap F_1^\times\right| = \frac{q+1}{4}.$$

Proof. First we show

$$\left|(1+F_1^\times)\cap F_0^\times\right| = \left|(1+F_1^\times)\cap F_1^\times\right|. \tag{1.1}$$

Let $u \in (1+F_1^\times)\cap F_0^\times$. Then $u-1 \in F_1^\times$ and hence $(u-1)^{-1} \in F_1^\times$. Since $1+(u-1)^{-1} = u(u-1)^{-1}$ and $u \in F_0^\times$, it follows that $1+(u-1)^{-1} \in (1+F_1^\times)\cap F_1^\times$. Conversely let $v \in (1+F_1^\times)\cap F_1^\times$. Then $v-1 \in F_1^\times$ and hence $(v-1)^{-1} \in F_1^\times$. Since $1+(v-1)^{-1} = v(v-1)^{-1}$ and $v \in F_1^\times$, it follows that $1+(v-1)^{-1} \in (1+F_1^\times)\cap F_0^\times$. Consequently the map $f\colon (1+F_1^\times)\cap F_0^\times \to (1+F_1^\times)\cap F_1^\times$ defined by $f(u) = 1+(u-1)^{-1}$ is a bijection. Thus (1.1) holds. Since $-1 \in F_1^\times$ if and only if $q \equiv 3 \pmod 4$, namely, $0 \in 1+F_1^\times$ if and only if $q \equiv 3 \pmod 4$, it follows that

$$\left|1+F_1^\times\right| = \left|(1+F_1^\times)\cap F_0^\times\right| + \left|(1+F_1^\times)\cap F_1^\times\right| + \begin{cases} 0 & (q \equiv 1 \pmod 4), \\ 1 & (q \equiv 3 \pmod 4). \end{cases}$$

Since $|1+F_1^\times| = |F_1^\times| = (q-1)/2$, it follows from (1.1) that

$$\left|(1+F_1^\times)\cap F_0^\times\right| = \left|(1+F_1^\times)\cap F_1^\times\right| = \begin{cases} \dfrac{q-1}{4} & (q \equiv 1 \pmod 4), \\ \dfrac{q-3}{4} & (q \equiv 3 \pmod 4). \end{cases} \tag{1.2}$$

Note that

$$(1+F_0^\times)\cup(1+F_1^\times) = 1+F^\times = F-\{1\} = (F_0^\times-\{1\})\cup F_1^\times\cup\{0\}.$$

This yields that

$$((1+F_0^\times)\cap F_0^\times)\cup((1+F_1^\times)\cap F_0^\times) = F_0^\times-\{1\} \tag{1.3}$$

and

$$((1+F_0^\times)\cap F_1^\times)\cup((1+F_1^\times)\cap F_1^\times) = F_1^\times. \tag{1.4}$$

From (1.2) and (1.3), we have

$$\left|(1+F_0^\times)\cap F_0^\times\right| = \begin{cases} \dfrac{q-5}{4} & (q \equiv 1 \pmod 4), \\ \dfrac{q-3}{4} & (q \equiv 3 \pmod 4), \end{cases}$$

and from (1.2) and (1.4), we have

$$\left|(1+F_0^\times)\cap F_1^\times\right| = \begin{cases} \dfrac{q-1}{4} & (q\equiv 1 \pmod 4),\\ \dfrac{q+1}{4} & (q\equiv 3 \pmod 4).\end{cases} \qquad \square$$

Lemma 1.2. *Let $r \in F^\times$. Define the subsets $F_0(r)$ and $F_1(r)$ of F by*

$$F_0(r) = \left\{u \in F\,;\, u^2 - r \in F_0^\times\right\}, \quad F_1(r) = \left\{u \in F\,;\, u^2 - r \in F_1^\times\right\}.$$

Then we have

$$|F_0(r)| = \begin{cases} \dfrac{q-3}{2} & (r\in F_0^\times),\\ \dfrac{q-1}{2} & (r \in F_1^\times),\end{cases} \quad |F_1(r)| = \begin{cases} \dfrac{q-1}{2} & (r\in F_0^\times),\\ \dfrac{q+1}{2} & (r \in F_1^\times).\end{cases}$$

Proof. Since $F_0(s^2r) = sF_0(r)$ and $F_1(s^2r) = sF_1(r)$ for $s \in F^\times$, it follows that $|F_0(r)| = |F_0(1)|$, $|F_1(r)| = |F_1(1)|$ if $r \in F_0^\times$ and $|F_0(r)| = |F_0(\rho)|$, $|F_1(r)| = |F_1(\rho)|$ if $r \in F_1^\times$. Therefore it is enough to consider the cases $r = 1$ and $r = \rho$. Note that if $q \equiv 1 \pmod 4$, then $-1 \in F_0^\times$ and hence $0 \in F_0(1)$, while if $q \equiv 3 \pmod 4$, then $-1 \in F_1^\times$ and hence $0 \in F_1(1)$. Assume $q \equiv 1 \pmod 4$. If $u \in F_0(1) - \{0\}$, then $u^2 \in (1+F_0^\times)\cap F_0^\times$. Conversely if $u^2 \in (1+F_0^\times)\cap F_0^\times$, then $\pm u \in F_0(1) - \{0\}$. Therefore by Lemma 1.1, we have

$$|F_0(1) - \{0\}| = 2\left|(1+F_0^\times)\cap F_0^\times\right| = \frac{q-5}{2}$$

and hence $|F_0(1)| = (q-3)/2$. Assume $q \equiv 3 \pmod 4$. If $u \in F_0(1)$, then $u \in F^\times$ and $u^2 \in (1+F_0^\times)\cap F_0^\times$. Conversely if $u^2 \in (1+F_0^\times)\cap F_0^\times$, then $\pm u \in F_0(1)$. Consequently by Lemma 1.1, we have

$$|F_0(1)| = 2\left|(1+F_0^\times)\cap F_0^\times\right| = \frac{q-3}{2}.$$

Similar argument yields that $|F_1(1)| = (q-1)/2$. Next we consider $F_0(\rho)$ and $F_1(\rho)$. Note that if $q \equiv 1 \pmod 4$, then $-\rho \in F_1^\times$ and hence $0 \in F_1(\rho)$, while if $q \equiv 3 \pmod 4$, then $-\rho \in F_0^\times$ and hence $0 \in F_0(\rho)$. Assume $q \equiv 1 \pmod 4$. If $u \in F_0(\rho)$, then $u^2 \in (\rho+F_0^\times)\cap F_0^\times$. Note that $(\rho+F_0^\times)\cap F_0^\times = \rho((1+F_1^\times)\cap F_1^\times)$. Conversely if $u^2 \in (\rho+F_0^\times)\cap F_0^\times$, then $\pm u \in F_0(\rho)$. Therefore by Lemma 1.1, we have

$$|F_0(\rho)| = 2\left|(1+F_1^\times)\cap F_1^\times\right| = \frac{q-1}{2}.$$

Similarly we get $|F_1(\rho) - \{0\}| = 2|(\rho+F_1^\times)\cap F_0^\times| = 2|(1+F_0^\times)\cap F_1^\times|$, and hence

$|F_1(\rho)| = (q+1)/2$. The case $q \equiv 3 \pmod 4$ is treated in the same way. So we omit it. $\square$

Lemma 1.3. *Let $r \in F^\times$.*
(i) *Put $f_r(t) = 2^{-1}(t + rt^{-1})$ for $t \in F^\times$. Then if $r \in F_0^\times$, the map $f_r \colon F^\times - \{\pm\sqrt{r}\} \to F_0(r)$ is a two to one surjection, while if $r \in F_1^\times$, the map $f_r \colon F^\times \to F_0(r)$ is a two to one surjection.*
(ii) *Put $g_r(z) = 2^{-1}(z + z^q)$ for $z \in E_r^\times$. Then if $r \in F_0^\times$, the map $g_r \colon E_r^\times - \{\pm\sqrt{r}\} \to F_1(r)$ is a two to one surjection, while if $r \in F_1^\times$, the map $g_r \colon E_r^\times \to F_1(r)$ is a two to one surjection.*

Proof. (i) If $f_r(t_1) = f_r(t_2)$ $(t_1, t_2 \in F^\times)$, then $t_2 = t_1$ or $t_2 = rt_1^{-1}$ and hence f_r is a two to one mapping. Moreover $f_r(t)^2 - r = (2^{-1}(t - rt^{-1}))^2$, so that $f_r(t) \in F_0(r)$ unless $t^2 = r$. Thus $f_r(F^\times - \{\pm\sqrt{r}\}) \subset F_0(r)$ if $r \in F_0^\times$, and $f_r(F^\times) \subset F_0(r)$ if $r \in F_1^\times$. Since f_r is two to one, $|f_r(F^\times - \{\pm\sqrt{r}\})| = (q-3)/2$ and $|f_r(F^\times)| = (q-1)/2$. Whereas by Lemma 1.2, $|F_0(r)| = (q-3)/2$ if $r \in F_0^\times$ and $|F_0(r)| = (q-1)/2$ if $r \in F_1^\times$. Therefore f_r is a surjection in each case.
(ii) If $g_r(z_1) = g_r(z_2)$ for $z_1 = x_1 + y_1\sqrt{\rho}$, $z_2 = x_2 + y_2\sqrt{\rho} \in E_r^\times$, then $x_1 = x_2$. Moreover since $x_1^2 - y_1^2\rho = x_2^2 - y_2^2\rho = r$, we have $y_2 = \pm y_1$. Hence $g_r(z_1) = g_r(z_2)$ implies $z_2 = z_1$ or $z_2 = z_1^q$. Thus g_r is a two to one mapping. Since $g_r(z)^2 - r = (2^{-1}(z - z^q))^2 = y^2\rho$ for $z = x + y\sqrt{\rho} \in E_r^\times$, it follows that if $z \in E_r^\times - F^\times$ then $g_r(z) \in F_1(r)$. Note that $E_r^\times - F^\times = E_r^\times - \{\pm\sqrt{r}\}$ if $r \in F_0^\times$, while $E_r^\times - F^\times = E_r^\times$ if $r \in F_1^\times$. Therefore we have $g_r(E_r^\times - \{\pm\sqrt{r}\}) \subset F_1(r)$ if $r \in F_0^\times$, while $g_r(E_r^\times) \subset F_1(r)$ if $r \in F_1^\times$. Since g_r is two to one and $|E_r^\times| = q+1$, it follows that $|g_r(E_r^\times - \{\pm\sqrt{r}\})| = (q-1)/2$ and $|g_r(E_r^\times)| = (q+1)/2$. Whereas by Lemma 1.2, $|F_1(r)| = (q-1)/2$ if $r \in F_0^\times$ and $|F_1(r)| = (q+1)/2$ if $r \in F_1^\times$. Thus g_r is a surjection in each case. $\square$

Lemma 1.4. *Let $\theta_{l(q-1)}$ $(l = 0, 1, \ldots, q)$ be the characters of $E^\times$, which have the property $\theta_{l(q-1)}|_{F^\times} = 1_F$. Then*

$$\sum_{\zeta \in E_1^\times - \{-1\}} \theta_{l(q-1)}(1+\zeta) = \begin{cases} q & (l = 0), \\ (-1)^{l+1} & (l = 1, \ldots, q). \end{cases}$$

Proof. Recall that $E_1^\times = \{\tau^{j(q-1)};\ j = 0, 1, \ldots, q\}$. Since $\zeta \in E_1^\times - \{-1\}$, we can write $\zeta = \tau^{j(q-1)}$ where $0 \le j \le q$ with $j \ne (q+1)/2$. Therefore we have $1 + \zeta = \tau^{-j}(\tau^j + \tau^{jq})$. Since $\zeta \ne -1$, it follows that $\tau^j + \tau^{jq} \in F^\times$ and hence $\theta_{l(q-1)}(\tau^j + \tau^{jq}) = 1$. Consequently

$$\sum_{\zeta \in E_1^\times - \{-1\}} \theta_{l(q-1)}(1+\zeta) = \sum_{0 \le j \le q,\, j \ne (q+1)/2} \theta_{l(q-1)}(\tau^{-j}),$$

which equals

$$\sum_{0\le j\le q} e^{-2\pi ijl/(q+1)} - (-1)^l.$$

Since

$$\sum_{0\le j\le q} e^{-2\pi ijl/(q+1)} = \begin{cases} q+1 & (l=0), \\ 0 & (l=1,\dots,q), \end{cases}$$

we obtain the lemma. □

2. The Hecke algebra $\mathcal{H}(G, A)$

Let $G = GL_2(F)$ be the general linear group of 2×2 non-singular matrices over F. The order $|G|$ of G is known to be equal to $q(q+1)(q-1)^2$. There are several important subgroups of G appearing in this paper:

$$\begin{aligned} A &= \left\{ a(x,y) = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} ; x, y \in F^\times \right\}, \\ U &= \left\{ u(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} ; x \in F \right\}, \\ K &= \left\{ \kappa(z) = \begin{pmatrix} x & y\rho \\ y & x \end{pmatrix} ; z = x + y\sqrt{\rho} \in E^\times \right\}, \\ Z(G) &= \left\{ a(x,x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} ; x \in F^\times \right\} \quad \text{(the center of } G\text{)}. \end{aligned}$$

Note that A is isomorphic to $F^\times \times F^\times$ so that $|A| = (q-1)^2$, U is isomorphic to the additive group F so that $|U| = q$, K is isomorphic to $E^\times$ so that $|K| = q^2 - 1$. It is known that

$$G = UA \cup UwUA. \tag{2.1}$$

where

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

In fact if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c = 0$, then $g \in UA$, while $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \in F^\times$, then we can verify

$$g = u(ac^{-1})wu(cd(\det g)^{-1})a(c, c^{-1}\det g) \in UwUA. \tag{2.3}$$

From (2.1) it follows that the coset space G/A can be written as

$$G/A = \{u(x)A\,;\, x \in F\} \cup \{u(y)wu(z)A\,;\, y, z \in F\}.$$

Now we consider the double coset space $A\backslash G/A$.

Theorem 2.1. *Let $\mathcal{R}$ be the subset of G which is defined by*

$$\mathcal{R} = \{e, w, u(1), wu(1), u(1)wu(r)\} \quad (r \in F) \tag{2.4}$$

where e is the identity matrix. Then $\mathcal{R}$ is a complete set of representatives of $A\backslash G/A$, namely,

$$A\backslash G/A = \{AgA;\, g \in \mathcal{R}\} \tag{2.5}$$

and consequently $|A\backslash G/A| = q + 4$.

Proof. It is enough to see $A\backslash G/A \subset \{AgA;\, g \in \mathcal{R}\}$. Assume $g = u(x)a(s,t) \in UA$. Then $AgA = Au(x)A$. Since

$$u(x) = a(x,1)u(1)a(x^{-1},1) \quad \text{for } x \in F^\times$$

we have $Au(x)A = A$ for $x = 0$ and $Au(x)A = Au(1)A$ for $x \in F^\times$. Assume $g = u(y)wu(z)a(s,t) \in UwUA$. Then $AgA = Au(y)wu(z)A$. In particular if $y = z = 0$, then $Au(y)wu(z)A = AwA$. If $y = 0$ and $z \neq 0$, then $Awu(z)A = Awa(z,1)u(1)a(z^{-1},1)A = Awa(z,1)w^{-1}wu(1)A$. But since $wa(z,1)w^{-1} \in A$, so we obtain $Awu(z)A = Awu(1)A$. Similarly if $y \neq 0$ and $z = 0$, we have $Au(y)wA = Au(1)wA$. Finally if both y and $z \in F^\times$, then we will show

$$Au(y)wu(z)A = Au(1)wu(yz)A. \tag{2.6}$$

Since $y \in F^\times$,

$$Au(y)wu(z)A = Aa(y,1)u(1)a(y^{-1},1)wu(z)A.$$

Moreover since $w^{-1}a(y^{-1},1)w = a(1,y^{-1})$, it follows that

$$Au(1)ww^{-1}a(y^{-1},1)wu(z)A = Au(1)wa(1,y^{-1})u(z)A.$$

Using $a(1,y^{-1})u(z)a(1,y) = u(yz)$, we have

$$Au(1)wa(1,y^{-1})u(z)A = Au(1)wu(yz)A.$$

Thus we obtain (2.6). Since $G = UA \cup UwUA$, the assertion $A\backslash G/A \subset \{AgA;\, g \in \mathcal{R}\}$ is completed. □

For $g \in G$, we put

$$\operatorname{ind}(g) = |A/A_g| \quad \text{where } A_g = A \cap gAg^{-1}. \tag{2.7}$$

We notice that $\mathrm{ind}(g)$ is equal to the number of left A-cosets in the double coset AgA and hence it depends only on the double coset AgA. A simple computation yields that $A_e = A_w = A$ while $A_g = Z(G)$ for $g \in \mathcal{R} - \{e, w\}$. Therefore we have

$$\mathrm{ind}(g) = \begin{cases} 1 & (g = e, w), \\ q - 1 & (g \in \mathcal{R} - \{e, w\}). \end{cases} \tag{2.8}$$

Let $\mathbb{C}G$ be the group algebra of G over $\mathbb{C}$. Define $\varepsilon \in \mathbb{C}G$ by

$$\varepsilon = |A|^{-1} \sum_{a \in A} a. \tag{2.9}$$

Then ε is an idempotent of $\mathbb{C}G$, which satisfies $\varepsilon^2 = \varepsilon$, $a\varepsilon = \varepsilon a' = \varepsilon$ for $a, a' \in A$. This means that $\varepsilon\mathbb{C}G\varepsilon$ is a subalgebra of $\mathbb{C}G$, which we call the Hecke algebra of G relative to A. From now on, we write $\mathcal{H}(G, A)$ instead of $\varepsilon\mathbb{C}G\varepsilon$. Clearly $\mathcal{H}(G, A)$ is spanned by $\varepsilon g \varepsilon$ $(g \in G)$. Put

$$\varepsilon[g] = \mathrm{ind}(g)\varepsilon g \varepsilon \quad \text{for } g \in \mathcal{R}. \tag{2.10}$$

Note that $\varepsilon[e] = \varepsilon$ is the identity element of $\mathcal{H}(G, A)$ and $\varepsilon[g]$ depends only on the double coset AgA. It can be easily seen that $\{\varepsilon[g]; g \in \mathcal{R}\}$ forms a linear basis of $\mathcal{H}(G, A)$, which we call the standard basis. We remark ([8]) that

$$\varepsilon[g] = |A|^{-1} \sum_{h \in AgA} h. \tag{2.11}$$

The multiplication table of $\mathcal{H}(G, A)$ is given in ([9]).

3. Irreducible decomposition of the induced character $\mathbf{1}_A^G$

In this section, we provide the irreducible decomposition of the induced character 1_A^G, which is induced from the principal character 1_A of A to G. Let $\hat{G}$ be the set of all irreducible characters of G, and let $\hat{G}^A$ be the subset of $\hat{G}$ consisting of those $\chi \in \hat{G}$ which appear in the irreducible decomposition of 1_A^G. Throughout the paper, we denote by $[g]$ the conjugacy class of $g \in G$. Let $[G]$ be the set of all conjugacy classes of G. Then it is known ([4]) that

$$[G] = [G]_{\mathrm{I}} \cup [G]_{\mathrm{II}} \cup [G]_{\mathrm{III}} \cup [G]_{\mathrm{IV}}$$

where

$$[G]_{\mathrm{I}} = \{[a(x, x)]; x \in F^\times\}, \tag{3.1}$$

$$[G]_{\mathrm{II}} = \left\{ \left[\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} \right] = [a(x, x)u(x^{-1})]; x \in F^\times \right\}, \tag{3.2}$$

Table 1.

$[\hat{G}]$ / G	U_k $(0 \le k < q-1)$	V_k $(0 \le k < q-1)$	$W_{k,l}$ $(0 \le l < q-1)$	$X_n = X_{nq}$ $\left(1 \le n < q^2-1, q+1 \nmid n\right)$
$[a(x,x)]$ $(x \in F^\times)$	$\lambda_k^2(x)$	$q\lambda_k^2(x)$	$(q+1)\lambda_k(x)\lambda_l(x)$	$(q-1)\theta_n(x)$
$[a(x,x)u(x^{-1})]$ $(x \in F^\times)$	$\lambda_k^2(x)$	0	$\lambda_k(x)\lambda_l(x)$	$-\theta_n(x)$
$[a(x,y)]$ $(x, y \in F^\times,\ x \ne y)$	$\lambda_k(xy)$	$\lambda_k(xy)$	$\lambda_k(x)\lambda_l(y) + \lambda_k(y)\lambda_l(x)$	0
$[\kappa_k(z)]$ $(z \in E^\times - F^\times)$	$\lambda_k(zz^q)$	$-\lambda_k(zz^q)$	0	$-(\theta_n(z) + \theta_n(z^q))$

$$[G]_{\mathrm{III}} = \{[a(x,y)] = [a(y,x)];\, x, y \in F^\times, x \ne y\}, \tag{3.3}$$

$$[G]_{\mathrm{IV}} = \{[\kappa(z)] = [\kappa(z^q)];\, z \in E^\times - F^\times\}. \tag{3.4}$$

Furthermore the numbers of elements in the conjugacy classes are given by

$$\begin{aligned} &|[a(x,x)]| = 1, \quad |[a(x,x)u(x^{-1})]| = q^2 - 1, \\ &|[a(x,y)]| = q(q+1), \quad |[\kappa(z)]| = q(q-1). \end{aligned} \tag{3.5}$$

Here we bring out the character table of G for convenience sake (Table 1). Now we decide the character values of 1_A^G.

Lemma 3.1. *The induced character 1_A^G takes the following values in $[G]$.*

$$\begin{aligned} &1_A^G([a(x,x)]) = q(q+1) && \text{for } [a(x,x)] \in [G]_{\mathrm{I}} \\ &1_A^G([a(x,x)u(x^{-1})]) = 0 && \text{for } [a(x,x)u(x^{-1})] \in [G]_{\mathrm{II}}, \\ &1_A^G([a(x,y)]) = 2 && \text{for } [a(x,y)] \in [G]_{\mathrm{III}}, \\ &1_A^G([\kappa(z)]) = 0 && \text{for } [\kappa(z)] \in [G]_{\mathrm{IV}}. \end{aligned}$$

Proof. The value of 1_A^G on the conjugacy class $[g]$ is given by

$$1_A^G([g]) = \frac{|G|}{|A|}\frac{|[g] \cap A|}{|[g]|} = q(q+1)\frac{|[g] \cap A|}{|[g]|}.$$

It is an easy task to check $[a(x,x)] \cap A = \{a(x,x)\}$, $[a(x,x)u(x^{-1})] \cap A = \phi$, $[a(x,y)] \cap A = \{a(x,y), a(y,x)\}$ and $[\kappa(z)] \cap A = \phi$. From this and (3.6), the lemma follows immediately. $\square$

REMARK 3.2. Lemma 3.1 yields that

$$1_A^G([a(x,x)g]) = 1_A^G([g]) \text{ for any } a(x,x) \in Z(G).$$

Theorem 3.3. *The irreducible decomposition of the induced character 1_A^G is given by*

$$1_A^G = U_0 + 2V_0 + V_{(q-1)/2} + \sum_{1 \le k \le (q-3)/2} W_{k,q-1-k} + \sum_{1 \le l \le (q-1)/2} X_{l(q-1)}$$

and hence

$$\hat{G}^A = \left\{ U_0, V_0, V_{(q-1)/2}, W_{k,q-1-k} \quad \left(1 \le k \le \frac{q-3}{2}\right), X_{l(q-1)} \quad \left(1 \le l \le \frac{q-1}{2}\right) \right\}.$$

Proof. To show the theorem, it is enough to compute the inner product

$$\begin{aligned}\left(\chi, 1_A^G\right)_G &= |G|^{-1} \sum_{g \in G} \chi(g) 1_A^G(g) \\ &= |G|^{-1} \sum_{[g] \in [G]} |[g]| \chi([g]) 1_A^G([g])\end{aligned}$$

for each $\chi \in \hat{G}$. Applying the above lemma, we obtain

$$\left(\chi, 1_A^G\right)_G = |G|^{-1} \left\{ q(q+1) \sum_{[g] \in [G]_{\mathrm{I}}} |[g]| \chi([g]) + 2 \sum_{[g] \in [G]_{\mathrm{III}}} |[g]| \chi([g]) \right\}.$$

Since $|G| = q(q+1)(q-1)^2$, $|[g]| = 1$ for $[g] \in [G]_{\mathrm{I}}$ and $|[g]| = q(q+1)$ for $[g] \in [G]_{\mathrm{III}}$, we have

$$\left(\chi, 1_A^G\right)_G = (q-1)^{-2} \left\{ \sum_{[g] \in [G]_{\mathrm{I}}} \chi([g]) + 2 \sum_{[g] \in [G]_{\mathrm{III}}} \chi([g]) \right\}.$$

Using (3.1) and (3.3), we get

$$\left(\chi, 1_A^G\right)_G = (q-1)^{-2} \left\{ \sum_{x \in F^\times} \chi([a(x,x)]) + \sum_{x,y \in F^\times, x \ne y} \chi([a(x,y)]) \right\}. \tag{3.6}$$

Before starting the case by case consideration, we remark that for $\lambda_k, \lambda_l \in \hat{F}^\times$ the following identity holds.

$$\sum_{x \in F^\times} \lambda_k(x) \lambda_l(x) = \begin{cases} q-1 & (k+l \equiv 0 \pmod{q-1}), \\ 0 & (\text{otherwise}). \end{cases} \tag{3.7}$$

CASE 1. $\chi = U_k$ $(0 \le k < q-1)$.

Applying Table 1 to (3.6), we have

$$\left(U_k, 1_A^G\right)_G = (q-1)^{-2}\left\{\sum_{x\in F^\times}\lambda_k^2(x) + \sum_{x,y\in F^\times, x\neq y}\lambda_k(x)\lambda_k(y)\right\}.$$

Since

$$\sum_{x,y\in F^\times, x\neq y}\lambda_k(x)\lambda_k(y) = \left(\sum_{x\in F^\times}\lambda_k(x)\right)^2 - \sum_{x\in F^\times}\lambda_k^2(x),$$

it follows that

$$\left(U_k, 1_A^G\right)_G = (q-1)^{-2}\left(\sum_{x\in F^\times}\lambda_k(x)\right)^2.$$

Applying (3.7) with $l = 0$, we get

$$\left(U_k, 1_A^G\right)_G = \begin{cases} 1 & (k=0), \\ 0 & (\text{otherwise}). \end{cases}$$

CASE 2. $\chi = V_k$ $(0 \le k < q-1)$.

Applying Table 1 to (3.6), we have

$$\left(V_k, 1_A^G\right)_G = (q-1)^{-2}\left\{q\sum_{x\in F^\times}\lambda_k^2(x) + \sum_{x,y\in F^\times, x\neq y}\lambda_k(x)\lambda_k(y)\right\}.$$

As in Case 1, we obtain

$$\left(V_k, 1_A^G\right)_G = (q-1)^{-2}\left\{(q-1)\sum_{x\in F^\times}\lambda_k^2(x) + \left(\sum_{x\in F^\times}\lambda_k(x)\right)^2\right\}.$$

Using (3.7) with $k = l$, we have

$$\sum_{x\in F^\times}\lambda_k^2(x) = \begin{cases} q-1 & \left(k = 0, \dfrac{q-1}{2}\right), \\ 0 & (\text{otherwise}). \end{cases}$$

Therefore we get

$$(V_k, 1_A^G)_G = \begin{cases} 2 & (k=0), \\ 1 & \left(k = \dfrac{q-1}{2}\right), \\ 0 & (\text{otherwise}). \end{cases}$$

CASE 3. $\chi = W_{k,l}$ $(0 \le k < l < q-1)$.
Applying Table 1 to (3.6), we have

$$\left(W_{k,l}, 1_A^G\right)_G = (q-1)^{-2}\left\{(q+1)\sum_{x\in F^\times}\lambda_k(x)\lambda_l(x) + \sum_{x,y\in F^\times, x\neq y}(\lambda_k(x)\lambda_l(y)+\lambda_k(y)\lambda_l(x))\right\}.$$

Since

$$\sum_{x,y\in F^\times, x\neq y}(\lambda_k(x)\lambda_l(y)+\lambda_k(y)\lambda_l(x)) = 2\left\{\left(\sum_{x\in F^\times}\lambda_k(x)\right)\left(\sum_{y\in F^\times}\lambda_l(y)\right) - \sum_{x\in F^\times}\lambda_k(x)\lambda_l(x)\right\},$$

it follows that

$$\left(W_{k,l}, 1_A^G\right)_G = (q-1)^{-2}\left\{(q-1)\sum_{x\in F^\times}\lambda_k(x)\lambda_l(x) + 2\left(\sum_{x\in F^\times}\lambda_k(x)\right)\left(\sum_{y\in F^\times}\lambda_l(y)\right)\right\}.$$

But since $l \neq 0$, we have $\sum_{y\in F^\times}\lambda_l(y) = 0$, and hence

$$\left(W_{k,l}, 1_A^G\right)_G = (q-1)^{-1}\sum_{x\in F^\times}\lambda_k(x)\lambda_l(x).$$

Applying (3.7) where $0 \le k < l < q-1$, we obtain

$$\left(W_{k,l}, 1_A^G\right)_G = \begin{cases} 1 & \left(l = q-1-k,\ 1 \le k \le \dfrac{q-3}{2}\right), \\ 0 & \text{(otherwise).} \end{cases}$$

CASE 4. $\chi = X_n$ $(1 \le n < q^2-1,\ q+1 \nmid n,\ X_n = X_{nq})$.
Applying Table 1 to (3.6), we have

$$\left(X_n, 1_A^G\right)_G = (q-1)^{-1}\sum_{x\in F^\times}\theta_n(x).$$

Note that

$$\sum_{x\in F^\times}\theta_n(x) = \begin{cases} q-1 & (\theta_n|_{F^\times} = 1_F), \\ 0 & \text{(otherwise).} \end{cases}$$

But we know $\theta_n|_{F^\times} = 1_F$ if and only if n is of the form $n = l(q-1)$ where $l = 1, 2, \ldots, q$. Since $l(q-1)q \equiv (q-(l-1))(q-1) \pmod{q^2-1}$ and hence $X_{l(q-1)} = X_{(q-(l-1))(q-1)}$, we get

$$(X_n, 1_A^G)_G = \begin{cases} 1 & \left(n = l(q-1),\ 1 \le l \le \dfrac{q-1}{2}\right), \\ 0 & \text{(otherwise).} \end{cases} \qquad \square$$

Before proceeding the next section, we recall some properties of the characters of $\mathcal{H}(G, A)$ (see [2]). Every irreducible character $\chi \in \hat{G}$ can be regarded as an irreducible character of $\mathbb{C}G$, by extending it linearly. The restriction of χ to the subalgebra $\mathcal{H}(G, A)$ is either 0 or an irreducible character of $\mathcal{H}(G, A)$. Moreover every irreducible character of $\mathcal{H}(G, A)$ is obtained by the nonzero restriction of some irreducible character of G. Since

(3.8)
$$\chi(\varepsilon) = |A|^{-1} \sum_{a \in A} \chi(a) = (\chi, 1_A)_A = \left(\chi, 1_A^G\right)_G$$

where the last equality comes from the Frobenius reciprocity law, the restriction of $\chi \in \hat{G}$ to $\mathcal{H}(G, A)$ is nonzero if and only if $\chi \in \hat{G}^A$.

4. The Character Table of $\mathcal{H}(G, A)$

In this section, we write down the character table of $\mathcal{H}(G, A)$. Here we mean that the character table of $\mathcal{H}(G, A)$ is the matrix

$$(\chi(\varepsilon[g]))_{g \in \mathcal{R}, \chi \in \hat{G}^A}$$

where $\{\varepsilon[g]; g \in \mathcal{R}\}$ is the standard basis of $\mathcal{H}(G, A)$ introduced in (2.10) and $\hat{G}^A = \left\{\chi \in \hat{G};\ \left(\chi, 1_A^G\right)_G \neq 0\right\}$, which is given explicitly in Theorem 3.3.

Main Theorem. *Let $G = GL_2(F_q)$ where F_q is a finite field with q elements. We assume that q is a power of an odd prime, and we put $F = F_q$ for simplicity. Let A be the subgroup of G consisting of diagonal matrices of G. The character table of the Hecke algebra $\mathcal{H}(G, A)$ is given by* Table 2 *described below.*

Before proving Main Theorem, we require some preliminary results. First we transform $\chi(\varepsilon[g])$ into more convenient form. Since $\varepsilon[g] = \mathrm{ind}(g)\varepsilon g \varepsilon$ $(g \in \mathcal{R})$, it follows that

$$\chi(\varepsilon[g]) = \mathrm{ind}(g)\chi(\varepsilon g \varepsilon) = \mathrm{ind}(g)\chi(\varepsilon^2 g) = \mathrm{ind}(g)\chi(\varepsilon g)$$

and hence

(4.1)
$$\chi(\varepsilon[g]) = \mathrm{ind}(g)|A|^{-1} \sum_{a \in A} \chi(ag).$$

Since every element $a \in A$ can be written uniquely as

(4.2)
$$a = a(x, x)a(y, 1) \quad (x, y \in F^\times)$$

and since every $\chi \in \hat{G}^A$ has the property

(4.3)
$$\chi(a(x, x)g) = \chi(g) \quad (x \in F^\times)$$

Table 2.

	U_0	V_0	$V_{(q-1)/2}$	$W_{k,q-1-k}$ $(1 \leq k \leq (q-3)/2)$	$X_{l(q-1)}$ $(1 \leq l \leq (q-1)/2)$
ε	1	2	1	1	1
$\varepsilon[w]$	1	0	$\begin{cases} 1 & (q \equiv 1 \mod 4) \\ -1 & (q \equiv 3 \mod 4) \end{cases}$	$(-1)^k$	$(-1)^{l+1}$
$\varepsilon[u(1)]$	$q-1$	$q-2$	-1	-1	-1
$\varepsilon[wu(1)]$	$q-1$	-1	$\begin{cases} -1 & (q \equiv 1 \mod 4) \\ 1 & (q \equiv 3 \mod 4) \end{cases}$	$(-1)^{k+1}$	$(-1)^l$
$\varepsilon[u(1)w]$	$q-1$	-1	$\begin{cases} -1 & (q \equiv 1 \mod 4) \\ 1 & (q \equiv 3 \mod 4) \end{cases}$	$(-1)^{k+1}$	$(-1)^l$
$\varepsilon[u(1)wu(1)]$	$q-1$	$q-2$	-1	-1	-1
$\varepsilon[u(1)wu(r)]$ $(r \in F^\times - \{1\})$	$q-1$	-2	$\displaystyle\sum_{s \in F^\times - \{1,r\}} \sigma_F(\phi_r(s))$	$\displaystyle\sum_{t \in F^\times - \{1,r\}} \lambda_k(\phi_r(t))$	$\displaystyle -\sum_{\zeta \in E_{1-r}^\times} \theta_{l(q-1)}(1+\zeta)$

Where $\phi_r(s) = s(s-1)(s-r)^{-1}$ and $E_{1-r}^\times = \{\zeta \in E^\times\,;\, \zeta\zeta^q = 1-r\}$.

(see Remark 3.2), it follows that

$$\chi(\varepsilon[g]) = \mathrm{ind}(g)(q-1)^{-1} \sum_{y \in F^\times} \chi(a(y,1)g). \tag{4.4}$$

In order to compute $\chi(\varepsilon[g])$ explicitly, it is necessary to investigate the conjugacy class of $a(y,1)g$. The following lemma is useful for that purpose. Let $\mathrm{tr}(g)$ and $\det(g)$ be the trace and the determinant of g respectively. Put

$$\Delta(g) = (\mathrm{tr}(g))^2 - 4\det(g). \tag{4.5}$$

Lemma 4.1. *The conjugacy class* $[g]$ *of* $g \in G$ *is characterize as follows.*
(i) $[g] \in [G]_{\mathrm{I}}$ *if and only if* $g \in Z(G)$.
(ii) $[g] \in [G]_{\mathrm{II}}$ *if and only if* $g \in G - Z(G)$ *and* $\Delta(g) = 0$. *In this case*

$$[g] = \left[\begin{pmatrix} 2^{-1}\mathrm{tr}(g) & 1 \\ 0 & 2^{-1}\mathrm{tr}(g) \end{pmatrix}\right]. \tag{4.6}$$

(iii) $[g] \in [G]_{\mathrm{III}}$ *if and only if* $\Delta(g) \in F_0^\times$. *In this case*

$$[g] = \left[a\left(2^{-1}(\mathrm{tr}(g) + \delta(g)), 2^{-1}(\mathrm{tr}(g) - \delta(g))\right)\right] \tag{4.7}$$

where $\delta(g) \in F^\times$ *such that* $\delta(g)^2 = \Delta(g)$.

(iv) $[g] \in [G]_{\mathrm{IV}}$ *if and only if* $\Delta(g) \in F_1^{\times}$. *In this case*

$$[g] = \left[\kappa\left(2^{-1}(\mathrm{tr}(g) + \delta(g)\sqrt{\rho})\right)\right] \tag{4.8}$$

where $\delta(g) \in F^{\times}$ *such that* $\delta(g)^2\rho = \Delta(g)$.

Proof. The proof of the lemma is a simple exercise of linear algebra. So we omit it. □

The next lemma slightly simplifies the proof of Main Theorem.

Lemma 4.2. *The following two identities hold.*

$$\chi(\varepsilon[u(1)wu(1)]) = \chi(\varepsilon[u(1)]) \tag{4.9}$$

and

$$\chi(\varepsilon[wu(1)]) = \chi(\varepsilon[u(1)w]). \tag{4.10}$$

Proof. Since $\det(a(y,1)u(1)wu(1)) = \det(a(y,1)u(1))$ and $\mathrm{tr}(a(y,1)u(1)wu(1)) = \mathrm{tr}(a(y,1)u(1))$, it follows from Lemma 4.1 that $a(y,1)u(1)wu(1)$ and $a(y,1)u(1)$ belong to the same conjugacy class. Noting $\mathrm{ind}(u(1)wu(1)) = \mathrm{ind}(u(1))$, we conclude from (4.4) that (4.9) holds. Since the characters are conjugation invariant, we obtain from (4.1)

$$\chi(\varepsilon[wu(1)]) = \mathrm{ind}(wu(1))|A|^{-1}\sum_{a\in A}\chi\left(w^{-1}(awu(1))w\right).$$

Since $w^{-1}aw \in A$ for $a \in A$, and $\mathrm{ind}(wu(1)) = \mathrm{ind}(u(1)w)$, we have

$$\chi(\varepsilon[wu(1)]) = \mathrm{ind}(u(1)w)|A|^{-1}\sum_{a\in A}\chi(au(1)w),$$

which equals $\chi(\varepsilon[u(1)w])$. □

From (3.8) and Theorem 3.3, we have already seen that

$$\chi(\varepsilon[e]) = \chi(\varepsilon) = \left(\chi, 1_A^G\right)_G = \begin{cases} 2 & (\chi = V_0) \\ 1 & (\chi \in \hat{G}^A - \{V_0\}). \end{cases} \tag{4.11}$$

Set

$$\mathcal{S} = \mathcal{R} - \{e, wu(1), u(1)wu(1)\} = \{w, u(1), u(1)wu(r)\} \quad (r \in F - \{1\}). \tag{4.12}$$

Then from (4.9), (4.10) and (4.11), we have only to compute $\chi(\varepsilon[g])$ for $g \in \mathcal{S}$. Note that if $g \in \mathcal{S}$, then $a(y,1)g$ does note belong to $Z(G)$. Define for $g \in \mathcal{S}$ the subsets of $F^{\times}$ by

$$F_{\mathrm{II}}^{\times}(g) = \{y \in F^{\times}\,;\, a(y,1)g \in [G]_{\mathrm{II}}\},$$
$$F_{\mathrm{III}}^{\times}(g) = \{y \in F^{\times}\,;\, a(y,1)g \in [G]_{\mathrm{III}}\},$$
$$F_{\mathrm{IV}}^{\times}(g) = \{y \in F^{\times}\,;\, a(y,1)g \in [G]_{\mathrm{IV}}\}.$$

Note that $F^{\times} = F_{\mathrm{II}}^{\times}(g) \cup F_{\mathrm{III}}^{\times}(g) \cup F_{\mathrm{IV}}^{\times}(g)$ for $g \in \mathcal{S}$. Furthermore if we put

(4.13) $$\Delta_g(y) = \Delta(a(y,1)g),$$

then we can deduce from Lemma 4.1 that

(4.14) $$F_{\mathrm{II}}^{\times}(g) = \{y \in F^{\times}\,;\, \Delta_g(y) = 0\},$$

(4.15) $$F_{\mathrm{III}}^{\times}(g) = \{y \in F^{\times}\,;\, \Delta_g(y) \in F_0^{\times}\},$$

(4.16) $$F_{\mathrm{IV}}^{\times}(g) = \{y \in F^{\times}\,;\, \Delta_g(y) \in F_1^{\times}\}.$$

Moreover we can rewrite (4.4) as

(4.17)
$$\chi(\varepsilon[g]) = \frac{\mathrm{ind}(g)}{q-1}\left\{\sum_{y \in F_{\mathrm{II}}^{\times}(g)} \chi(a(y,1)g) + \sum_{y \in F_{\mathrm{III}}^{\times}(g)} \chi(a(y,1)g) + \sum_{y \in F_{\mathrm{IV}}^{\times}(g)} \chi(a(y,1)g)\right\}.$$

Lemma 4.1 enables us to clarify the structure of $F_{\mathrm{II}}^{\times}(g)$, $F_{\mathrm{III}}^{\times}(g)$ and $F_{\mathrm{IV}}^{\times}(g)$.

Lemma 4.3. *Let $g \in \mathcal{S}$.*
(i) *If $g = w$, then $F_{\mathrm{II}}^{\times}(w) = \phi$, $F_{\mathrm{III}}^{\times}(w) = \{y \in F^{\times}\,;\, -y \in F_0^{\times}\}$, and $F_{\mathrm{IV}}^{\times}(w) = \{y \in F^{\times}\,;\, -y \in F_1^{\times}\}$. Moreover if $y \in F_{\mathrm{III}}^{\times}(w)$, then $a(y,1)w \in [a(\sqrt{-y}, -\sqrt{-y})]$, while if $y \in F_{\mathrm{IV}}^{\times}(w)$, then $a(y,1)w \in t[\kappa(\eta\sqrt{\rho})]$, where $\eta \in F^{\times}$ such that $\eta^2\rho = -y$.*
(ii) *If $g = u(1)$, then $F_{\mathrm{II}}^{\times}(u(1)) = \{1\}$, $F_{\mathrm{III}}^{\times}(u(1)) = F^{\times} - \{1\}$, and $F_{\mathrm{IV}}^{\times}(u(1)) = \phi$. Moreover if $y \in F_{\mathrm{III}}^{\times}(u(1))$, then $a(y,1)u(1) \in [a(y,1)]$.*
(iii) *If $g = u(1)w$, then $F_{\mathrm{II}}^{\times}(u(1)w) = \{4\}$, $F_{\mathrm{III}}^{\times}(u(1)w) = \{2(1+\xi)\,;\, \xi \in F_0(1)\}$, and $F_{\mathrm{IV}}^{\times}(u(1)w) = \{2(1+\xi)\,;\, \xi \in F_1(1)\}$. Moreover if $\xi \in F_0(1)$, then*

$$a(2(1+\xi),1)u(1)w \in [a(1+\xi+\eta, 1+\xi-\eta)]\,,$$

where $\eta \in F^{\times}$ such that $\eta^2 = \xi^2 - 1$. While if $\xi \in F_1(1)$,

$$a(2(1+\xi),1)u(1)w \in [\kappa(1+\xi+\eta\sqrt{\rho})],$$

where $\eta \in F^{\times}$ such that $\eta^2\rho = \xi^2 - 1$.

(iv) *If* $g = u(1)wu(r)$ *where* $r \neq 0, 1$, *then*

$$F_{\mathrm{II}}^{\times}(u(1)wu(r)) = \begin{cases} \{\left(1+\sqrt{1-r}\,\right)^2, \left(1-\sqrt{1-r}\,\right)^2\} & (1-r \in F_0^{\times}), \\ \phi & (1-r \in F_1^{\times}). \end{cases}$$

$$F_{\mathrm{III}}^{\times}(u(1)wu(r)) = \{2(1+\xi) - r\,;\, \xi \in F_0(1-r), \xi \neq 2^{-1}(r-2)\},$$

$$F_{\mathrm{IV}}^{\times}(u(1)wu(r)) = \{2(1+\xi) - r\,;\, \xi \in F_1(1-r)\}.$$

Moreover if $\xi \in F_0(1-r) - \{2^{-1}(r-2)\}$, *then*

$$a(2(1+\xi) - r, 1)u(1)wu(r) \in [a(1+\xi+\eta, 1+\xi-\eta)]$$

where $\eta \in F^{\times}$ *such that* $\eta^2 = \xi^2 - (1-r)$. *While if* $\xi \in F_1(1-r)$, *then*

$$a(2(1+\xi) - r, 1)u(1)wu(r) \in \left[\kappa(1+\xi+\eta\sqrt{\rho}\,)\right]$$

where $\eta \in F^{\times}$ *such that* $\eta^2\rho = \xi^2 - (1-r)$.

Here we recall that $F_0(c) = \{\xi \in F\,;\, \xi^2 - c \in F_0^{\times}\}$ *and* $F_1(c) = \{\xi \in F\,;\, \xi^2 - c \in F_1^{\times}\}$.

Proof. (i) If $g = w$, then $\mathrm{tr}(a(y,1)w) = 0$, $\det(a(y,1)w) = y$ and hence $\Delta_w(y) = -4y$. Thus $F_{\mathrm{II}}^{\times}(w) = \phi$, $F_{\mathrm{III}}^{\times}(w) = \{y \in F^{\times};\ -y \in F_0^{\times}\}$ and $F_{\mathrm{IV}}^{\times}(w) = \{y \in F^{\times};\ -y \in F_1^{\times}\}$. If $-y \in F_0^{\times}$ (resp. $F_1^{\times}$), we may take $\delta(a(y,1)w) = 2\sqrt{-y}$ in (4.7) (resp. $\delta(a(y,1)w) = 2\eta$ in (4.8) where $\eta \in F^{\times}$ such that $\eta^2\rho = -y$), from which $a(y,1)w \in [a(\sqrt{-y}, -\sqrt{-y})]$ (resp. $a(y,1)w \in [\kappa(\eta\sqrt{\rho})]$).
(ii) If $g = u(1)$, then $\mathrm{tr}(a(y,1)u(1)) = y+1$, $\det(a(y,1)u(1)) = y$ and hence $\Delta_{u(1)}(y) = (y-1)^2$. Thus $F_{\mathrm{II}}^{\times}(u(1)) = \{1\}$, $F_{\mathrm{III}}^{\times}(u(1)) = F^{\times} - \{1\}$ and $F_{\mathrm{IV}}^{\times}(u(1)) = \phi$. Moreover if $y \in F_{\mathrm{III}}^{\times}(u(1))$, we may choose $\delta(a(y,1)u(1)) = y-1$ in (4.7), so that $a(y,1)u(1) \in [a(y,1)]$.
(iii) If $g = u(1)w$, then $\mathrm{tr}(a(y,1)u(1)w) = \det(a(y,1)u(1)w) = y$ and hence $\Delta_{u(1)w}(y) = y^2 - 4y = (y-2)^2 - 4$. Thus $F_{\mathrm{II}}^{\times}(u(1)w) = \{4\}$. If we put $y = 2(1+\xi)$ with $\xi \neq -1$, then $\Delta_{u(1)w}(y) = 4(\xi^2-1)$. Therefore $\Delta_{u(1)w}(y) \in F_0^{\times}$ (resp. $F_1^{\times}$) if and only if $\xi \in F_0(1)$ (resp. $F_1(1)$). Note that -1 does note belong to $F_0(1) \cup F_1(1)$. Consequently $F_{\mathrm{III}}^{\times}(u(1)w) = \{2(1+\xi); \xi \in F_0(1)\}$ and $F_{\mathrm{IV}}^{\times}(u(1)w) = \{2(1+\xi); \xi \in F_1(1)\}$. Moreover if $\xi \in F_0(1)$ (resp. $F_1(1)$), we may take $\delta(a(2(1+\xi),1)u(1)w) = 2\eta$ where $\eta \in F^{\times}$ such that $\eta^2 = \xi^2 - 1$ (resp. $\eta^2\rho = \xi^2 - 1$) in (4.7) (resp. (4.8)) and hence $a(2(1+\xi),1)u(1)w \in [a(1+\xi+\eta, 1+\xi-\eta)]$ (resp. $[\kappa(1+\xi+\eta\sqrt{\rho})]$).
(iv) If $g=u(1)wu(r)$ where $r \neq 0,1$, then $\det(a(y,1)u(1)wu(r)) = y$, $\mathrm{tr}(a(y,1)u(1)wu(r)) = y+r$ and hence $\Delta_{u(1)wu(r)}(y) = (y+r)^2 - 4y = (y+r-2)^2 - 4(1-r)$. Thus $\Delta_{u(1)wu(r)}(y) = 0$ has solutions if and only if $1-r \in F_0^{\times}$. If this is the case, the solutions are $y = (1 \pm \sqrt{1-r}\,)^2$, and hence $F_{\mathrm{II}}^{\times}(u(1)wu(r)) = \{(1+\sqrt{1-r}\,)^2, (1-\sqrt{1-r}\,)^2\}$ in case $1-r \in F_0^{\times}$, otherwise $F_{\mathrm{II}}^{\times}(u(1)wu(r)) = \phi$. Putting $y = 2(1+\xi) - r$ where $\xi \neq 2^{-1}(r-2)$, we get $\Delta_{u(1)wu(r)}(y) = 4(\xi^2 - (1-r))$ and hence $\Delta_{u(1)wu(r)}(y) \in F_0^{\times}$ (resp. $F_1^{\times}$) if and

only if $\xi \in F_0(1-r)$ (resp. $F_1(1-r)$). Note that $2^{-1}(r-2) \in F_0(1-r)$. Thus we have $F_{\mathrm{III}}^{\times}(u(1)wu(r)) = \{2(1+\xi) - r\,;\, \xi \in F_0(1-r) - \{2^{-1}(r-2)\}\}$ and $F_{\mathrm{IV}}^{\times}(u(1)wu(r)) = \{2(1+\xi) - r\,;\, \xi \in F_1(1-r)\}$. Furthermore if $\xi \in F_0(1-r) - \{2^{-1}(r-2)\}$ (resp. $\xi \in F_1(1-r)$), then we may take $\delta(a(2(1+\xi)-r, 1)u(1)wu(r)) = 2\eta$ where $\eta \in F^{\times}$ such that $\eta^2 = \xi^2 - (1-r)$ in (4.7) (resp. $\eta^2\rho = \xi^2 - (1-r)$ in (4.8)), and consequently $a(2(1+\xi)-r, 1)u(1)wu(r) \in [a(1+\xi+\eta, 1+\xi-\eta)]$ (resp. $[\kappa(1+\xi+\eta\sqrt{\rho})]$).

$\square$

Proof of Main Theorem. The proof is proceeding by case by case computation for $\chi \in \hat{G}^A$.

CASE 1. $\chi = U_0$. Since $U_0 = 1_G$ is the principal character of G, we conclude from (4.1) and (2.8) that

$$U_0(\varepsilon[g]) = \operatorname{ind}(g) = \begin{cases} 1 & (g = e, w), \\ q-1 & (g \in \mathcal{R} - \{e, w\}). \end{cases} \tag{4.18}$$

CASE 2. $\chi = V_0$. Let $g \in \mathcal{S}$. From Table 1, we have $V_0 = 0$ on $[G]_{\mathrm{II}}$, $V_0 = 1$ on $[G]_{\mathrm{III}}$ and $V_0 = -1$ on $[G]_{\mathrm{IV}}$. Hence by (4.17)

$$V_0(\varepsilon[g]) = \operatorname{ind}(g)(q-1)^{-1}\left\{\left|F_{\mathrm{III}}^{\times}(g)\right| - \left|F_{\mathrm{IV}}^{\times}(g)\right|\right\}.$$

If $g = w$, then $\operatorname{ind}(w) = 1$, $|F_{\mathrm{III}}^{\times}(w)| = |F_0^{\times}| = (q-1)/2$ and $|F_{\mathrm{IV}}^{\times}(w)| = |F_1^{\times}| = (q-1)/2$ from Lemma 4.3. Thus $V_0(\varepsilon[w]) = 0$.

If $g = u(1)$, then $\operatorname{ind}(u(1)) = q-1$, $|F_{\mathrm{III}}^{\times}(u(1))| = |F^{\times} - \{1\}| = q-2$ and $|F_{\mathrm{IV}}^{\times}(u(1))| = 0$ from Lemma 4.3. Hence $V_0(\varepsilon[u(1)]) = q-2$.

If $g = u(1)w$, then $\operatorname{ind}(u(1)w) = q-1$, $|F_{\mathrm{III}}^{\times}(u(1)w)| = |F_0(1)| = (q-3)/2$ and $|F_{\mathrm{IV}}^{\times}(u(1)w)| = |F_1(1)| = (q-1)/2$ from Lemma 4.3 and Lemma 1.2. Thus we have $V_0(\varepsilon[u(1)w]) = -1$.

If $g = u(1)wu(r)$ with $r \neq 0, 1$, then $\operatorname{ind}(u(1)wu(r)) = q-1$, $|F_{\mathrm{III}}^{\times}(u(1)wu(r))| = |F_0(1-r)| - 1$ and $|F_{\mathrm{IV}}^{\times}(u(1)wu(r))| = |F_1(1-r)|$ from Lemma 4.3. Again by Lemma 1.2, we know $|F_0(1-r)| = (q-3)/2$ and $|F_1(1-r)| = (q-1)/2$ for $1-r \in F_0^{\times}$, whereas $|F_0(1-r)| = (q-1)/2$ and $|F_1(1-r)| = (q+1)/2$ for $1-r \in F_1^{\times}$. Therefore in any case we have $V_0(\varepsilon[u(1)wu(r)]) = -2$.

CASE 3. $\chi = V_{(q-1)/2}$. Let $g \in \mathcal{S}$. Since $\lambda_{(q-1)/2} = \sigma_F$, it follows from Table 1, that $V_{(q-1)/2} = 0$ on $[G]_{\mathrm{II}}$, $V_{(q-1)/2}([a(x,y)]) = \sigma_F(xy) = \sigma_F(\det(a(x,y)))$ and $V_{(q-1)/2}([\kappa(z)]) = -\sigma_F(zz^q) = -\sigma_F(\det(\kappa(z)))$. Therefore by (4.17) we have

$$V_{(q-1)/2}(\varepsilon[g]) = \operatorname{ind}(g)(q-1)^{-1}\left\{\sum_{y \in F_{\mathrm{III}}^{\times}(g)} \sigma_F(\det(a(y,1)g)) - \sum_{y \in F_{\mathrm{IV}}^{\times}(g)} \sigma_F(\det(a(y,1)g))\right\}.$$

But since $\det(a(y,1)g) = y$ for $g \in \mathcal{S}$, it follows that

$$V_{(q-1)/2}(\varepsilon[g]) = \mathrm{ind}(g)(q-1)^{-1}\left\{\sum_{y\in F_{\mathrm{III}}^{\times}(g)} \sigma_F(y) - \sum_{y\in F_{\mathrm{IV}}^{\times}(g)} \sigma_F(y)\right\}.$$

If $g = w$, then by Lemma 4.3 we have

$$\begin{aligned} V_{(q-1)/2}(\varepsilon[w]) &= (q-1)^{-1}\left\{\sum_{x\in F_0^{\times}} \sigma_F(-x) - \sum_{x\in F_1^{\times}} \sigma_F(-x)\right\} \\ &= (q-1)^{-1}\sigma_F(-1)\left\{\sum_{x\in F_0^{\times}} \sigma_F(x) - \sum_{x\in F_1^{\times}} \sigma_F(x)\right\}, \end{aligned}$$

which equals

$$\sigma_F(-1) = \begin{cases} 1 & (q \equiv 1 \pmod 4), \\ -1 & (q \equiv 3 \pmod 4). \end{cases}$$

If $g = u(1)$, then by Lemma 4.3 we have

$$V_{(q-1)/2}(\varepsilon[u(1)]) = \sum_{y\in F^{\times}-\{1\}} \sigma_F(y) = -\sigma_F(1) = -1.$$

If $g = u(1)w$, then by Lemma 4.3 we have

$$V_{(q-1)/2}(\varepsilon[u(1)w]) = \sum_{\xi\in F_0(1)} \sigma_F(2(1+\xi)) - \sum_{\xi\in F_1(1)} \sigma_F(2(1+\xi)).$$

Since $F_0(1) \cup F_1(1) = F - \{\pm 1\}$, it follows that

$$\sum_{\xi\in F_0(1)} \sigma_F(2(1+\xi)) + \sum_{\xi\in F_1(1)} \sigma_F(2(1+\xi)) = \sum_{\xi\in F-\{\pm 1\}} \sigma_F(2(1+\xi)).$$

The right-side is equal to

$$\sum_{x\in F^{\times}-\{4\}} \sigma_F(x) = -\sigma_F(4) = -1.$$

Consequently we obtain

$$V_{(q-1)/2}(\varepsilon[u(1)w]) = 2\sum_{\xi\in F_0(1)} \sigma_F(2(1+\xi)) + 1.$$

On the other hand, by Lemma 1.3 we have

$$2\sum_{\xi\in F_0(1)}\sigma_F(2(1+\xi)) = \sum_{t\in F^\times-\{\pm 1\}}\sigma_F\left(2\left(1+2^{-1}\left(t+t^{-1}\right)\right)\right).$$

Since $2(1+2^{-1}(t+t^{-1})) = (t+1)^2t^{-1}$, it follows that

$$2\sum_{\xi\in F_0(1)}\sigma_F(2(1+\xi)) = \sum_{t\in F^\times-\{\pm 1\}}\sigma_F\left(t^{-1}\right),$$

which equals $-(\sigma_F(1)+\sigma_F(-1)) = -(1+\sigma_F(-1))$. Thus we have $V_{(q-1)/2}(\varepsilon[u(1)w]) = -\sigma_F(-1)$.

Assume $g = u(1)wu(r)$ with $r \neq 0, 1$. Using Lemma 4.3, we have

$$V_{(q-1)/2}(\varepsilon[u(1)wu(r)]) = M - N$$

where we put for simplicity

$$M = \sum_{\xi\in F_0(1-r)-\{2^{-1}(r-2)\}}\sigma_F(2(1+\xi)-r),\quad N = \sum_{\xi\in F_1(1-r)}\sigma_F(2(1+\xi)-r).$$

Since

$$F_0(1-r)\cup F_1(1-r) = \begin{cases} F-\left\{\pm\sqrt{1-r}\right\} & (1-r\in F_0^\times),\\ F & (1-r\in F_1^\times),\end{cases}$$

we have, by putting $x = 2(1+\xi)-r$

$$M+N = \begin{cases} \displaystyle\sum_{\xi\in F-\{\pm\sqrt{1-r},2^{-1}(r-2)\}}\sigma_F(2(1+\xi)-r) = \sum_{x\in F^\times-\{(1\pm\sqrt{1-r})^2\}}\sigma_F(x) & (1-r\in F_0^\times),\\ \displaystyle\sum_{\xi\in F-\{2^{-1}(r-2)\}}\sigma_F(2(1+\xi)-r) = \sum_{x\in F^\times}\sigma_F(x) & (1-r\in F_1^\times).\end{cases}$$

Thus we have

$$M+N = \begin{cases} -2 & (1-r\in F_0^\times),\\ 0 & (1-r\in F_1^\times)\end{cases}$$

and consequently

$$V_{(q-1)/2}(\varepsilon[u(1)wu(r)]) = \begin{cases} 2M+2 & (1-r\in F_0^\times),\\ 2M & (1-r\in F_1^\times).\end{cases}$$

Applying Lemma 1.3, we obtain

$$2M = \begin{cases} \displaystyle\sum_{t\in F^\times - \{\pm\sqrt{1-r}, -1, r-1\}} \sigma_F\left(2+t+(1-r)t^{-1}-r\right) & (1-r \in F_0^\times), \\ \displaystyle\sum_{t\in F^\times - \{-1, r-1\}} \sigma_F\left(2+t+(1-r)t^{-1}-r\right) & (1-r \in F_1^\times), \end{cases}$$

because $f_{1-r}^{-1}(2^{-1}(r-2)) = \{-1, r-1\}$. Note that

$$2+t+(1-r)t^{-1}-r = (t+1)(t+1-r)t^{-1}$$

and it takes the values $(1\pm\sqrt{1-r}\,)^2 \in F_0^\times$ at $t = \pm\sqrt{1-r}$. Then we have

$$2M = \begin{cases} \displaystyle\sum_{t\in F^\times - \{-1, r-1\}} \sigma_F\left((t+1)(t+1-r)t^{-1}\right) - 2 & (1-r \in F_0^\times), \\ \displaystyle\sum_{t\in F^\times - \{-1, r-1\}} \sigma_F\left((t+1)(t+1-r)t^{-1}\right) & (1-r \in F_1^\times). \end{cases}$$

Consequently we conclude that

$$V_{(q-1)/2}(\varepsilon[u(1)wu(r)]) = \sum_{t\in F^\times - \{-1, r-1\}} \sigma_F\left((t+1)(t+1-r)t^{-1}\right).$$

Replacing $t+1$ by s, we get

$$V_{(q-1)/2}(\varepsilon[u(1)wu(r)]) = \sum_{s\in F^\times - \{1, r\}} \sigma_F\left(s(s-r)(s-1)^{-1}\right).$$

Furthermore since $\sigma_F(x) = \sigma_F(x^{-1})$ for $x \in F^\times$, it follows that

$$V_{(q-1)/2}(\varepsilon[u(1)wu(r)]) = \sum_{s\in F^\times - \{1, r\}} \sigma_F\left(s(s-1)(s-r)^{-1}\right).$$

CASE 4. $\chi = W_{k,q-1-k}$. Let $g \in \mathcal{S}$. Since $\lambda_{q-1-k}(x) = \lambda_{-k}(x) = \lambda_k(x^{-1})$ ($x \in F^\times$), we conclude from Table 1 that $W_{k,q-1-k} = 1$ on $[G]_{\mathrm{II}}$, $W_{k,q-1-k}([a(x,y)]) = \lambda_k(xy^{-1}) + \lambda_k(x^{-1}y)$ on $[G]_{\mathrm{III}}$ and $W_{k,q-1-k} = 0$ on $[G]_{\mathrm{IV}}$. Hence by (4.17)

$$W_{k,q-1-k}(\varepsilon[g]) = \mathrm{ind}(g)(q-1)^{-1}\left\{\left|F_{\mathrm{II}}^\times(g)\right| + W_k(g)\right\}$$

where we put for simplicity

$$W_k(g) = \sum_{y\in F_{\mathrm{III}}^\times(g)} W_{k,q-1-k}(a(y,1)g).$$

If $g = w$, then we have $|F_{\mathrm{II}}^{\times}(w)| = 0$ and $a(y,1)w \in [a(\sqrt{-y}, -\sqrt{-y})]$ for $y \in F_{\mathrm{III}}^{\times}(w)$, namely, for $y \in -F_0^{\times}$. Therefore

$$W_{k,q-1-k}(a(y,1)w) = \lambda_k(\sqrt{-y})\lambda_{-k}(-\sqrt{-y}) + \lambda_k(-\sqrt{-y})\lambda_{-k}(\sqrt{-y}),$$

which is equal to $2\lambda_k(-1) = 2(-1)^k$. Thus $W_k(w) = 2(-1)^k|F_0^{\times}| = (-1)^k(q-1)$, and hence $W_{k,q-1-k}(\varepsilon[w]) = (-1)^k$.

If $g = u(1)$, then we have $|F_{\mathrm{II}}^{\times}(u(1))| = 1$ and $a(y,1)u(1) \in [a(y,1)]$ for $y \in F_{\mathrm{III}}^{\times}(u(1)) = F^{\times} - \{1\}$. Therefore $W_{k,q-1-k}(a(y,1)u(1)) = \lambda_k(y) + \lambda_k(y^{-1})$ and hence

$$W_k(u(1)) = \sum_{y \in F^{\times} - \{1\}} \left(\lambda_k(y) + \lambda_k\left(y^{-1}\right)\right) = -2\lambda_k(1) = -2.$$

Consequently we have $W_{k,q-1-k}(\varepsilon[u(1)]) = -1$.

If $g = u(1)w$, then $|F_{\mathrm{II}}^{\times}(u(1)w)| = 1$ and $F_{\mathrm{III}}^{\times}(u(1)w) = \{2(1+\xi);\, \xi \in F_0(1)\}$ from Lemma 4.3. Moreover we have

$$\begin{aligned} &W_{k,q-1-k}(a(2(1+\xi),1)u(1)w) \\ &= \lambda_k\left((1+\xi+\eta)(1+\xi-\eta)^{-1}\right) + \lambda_k\left((1+\xi+\eta)^{-1}(1+\xi-\eta)\right). \end{aligned}$$

But by Lemma 1.3, we can write $\xi = f_1(t) = 2^{-1}(t + t^{-1})$ and $\eta = 2^{-1}(t - t^{-1})$ where $t \in F^{\times} - \{\pm 1\}$, and hence $(1+\xi+\eta)(1+\xi-\eta)^{-1} = t$. Since the map $f_1\colon F^{\times} - \{\pm 1\} \to F_0(1)$ is a 2 to 1 surjection, it follows that

$$W_k(u(1)w) = \frac{1}{2} \sum_{t \in F^{\times} - \{\pm 1\}} \left(\lambda_k(t) + \lambda_k\left(t^{-1}\right)\right) = \sum_{t \in F^{\times} - \{\pm 1\}} \lambda_k(t),$$

which is equal to $-(\lambda_k(1)+\lambda_k(-1)) = -1+(-1)^{k+1}$. Thus we have $W_{k,q-1-k}(\varepsilon[u(1)w]) = (-1)^{k+1}$.

Assume $g = u(1)wu(r)$ with $r \neq 0, 1$. First we consider the case $1-r \in F_0^{\times}$. From Lemma 4.3, we know that $|F_{\mathrm{II}}^{\times}(u(1)wu(r))| = 2$ and $F_{\mathrm{III}}^{\times}(u(1)wu(r)) = \{2(1+\xi)-r;\, \xi \in F_0(1-r) - \{2^{-1}(r-2)\}\}$. Moreover

$$\begin{aligned} &W_{k,q-1-k}(a(2(1+\xi)-r,1)u(1)wu(r)) \\ &= \lambda_k\left((1+\xi+\eta)(1+\xi-\eta)^{-1}\right) + \lambda_k\left((1+\xi+\eta)^{-1}(1+\xi-\eta)\right). \end{aligned}$$

By Lemma 1.3, we can write $\xi \in F_0(1-r)$ as $\xi = f_{1-r}(t) = 2^{-1}(t + (1-r)t^{-1})$ and $\eta = 2^{-1}(t-(1-r)t^{-1})$ where $t \in F^{\times} - \{\pm\sqrt{1-r}\}$ and hence $(1+\xi+\eta)(1+\xi-\eta)^{-1} = t(t+1)(t+1-r)^{-1}$. Since $f_{1-r}\colon F^{\times} - \{\pm\sqrt{1-r}\} \to F_0(1-r)$ is a 2 to 1 surjection and moreover $f_{1-r}^{-1}(2^{-1}(r-2)) = \{-1, r-1\}$, it follows that

$$\begin{aligned} &W_k(u(1)wu(r)) \\ &= \frac{1}{2} \sum_{t \in F^{\times} - \{\pm\sqrt{1-r}, -1, r-1\}} \left\{\lambda_k\left(t(t+1)(t+1-r)^{-1}\right) + \lambda_k\left(t^{-1}(t+1)^{-1}(t+1-r)\right)\right\}. \end{aligned}$$

If we replace t by $(1-r)s^{-1}$, we have $t^{-1}(t+1)^{-1}(t+1-r) = s(s+1)(s+1-r)^{-1}$. This implies that

$$W_k(u(1)wu(r)) = \sum_{t\in F^\times - \{\pm\sqrt{1-r}, -1, r-1\}} \lambda_k\left(t(t+1)(t+1-r)^{-1}\right).$$

If we notice that $t(t+1)(t+1-r)^{-1} = 1$ for $t = \pm\sqrt{1-r}$, we can deduce that

$$\begin{aligned} W_{k,q-1-k}(\varepsilon[u(1)wu(r)]) &= 2 + W_k(u(1)wu(r)) \\ &= \sum_{t\in F^\times - \{-1, r-1\}} \lambda_k\left(t(t+1)(t+1-r)^{-1}\right), \end{aligned}$$

which implies

$$W_{k,q-1-k}(\varepsilon[u(1)wu(r)]) = \sum_{t\in F^\times - \{1, r\}} \lambda_k\left(t(t-1)(t-r)^{-1}\right).$$

The case $1-r \in F_1^\times$ is quite similar and the result is the same as in the case $1-r \in F_0^\times$.

CASE 5. $\chi = X_{l(q-1)}$. Let $g \in \mathcal{S}$. Since $\theta_{l(q-1)}(x) = 1$ ($x \in F^\times$), it follows from Table 1 that $X_{l(q-1)} = -1$ on $[G]_{\mathrm{II}}$ and $X_{l(q-1)} = 0$ on $[G]_{\mathrm{III}}$. Consequently from (4.17), we have

$$X_{l(q-1)}(\varepsilon[g]) = \mathrm{ind}(g)(q-1)^{-1}\left\{-\left|F_{\mathrm{II}}^\times(g)\right| + X_l(g)\right\}$$

where we put for simplicity

$$X_l(g) = \sum_{y\in F_{\mathrm{IV}}^\times(g)} X_{l(q-1)}(a(y,1)g).$$

If $g = w$, then $|F_{\mathrm{II}}^\times(w)| = 0$ and $a(y,1)w \in [\kappa(\eta\sqrt{\rho})]$ for $y \in F_{\mathrm{IV}}^\times(w)$ from Lemma 4.3 and hence

$$X_{l(q-1)}(a(y,1)w) = -(\theta_{l(q-1)}(\eta\sqrt{\rho}) + \theta_{l(q-1)}(-\eta\sqrt{\rho})).$$

Since $\theta_{l(q-1)}|_{F^\times} = 1_F$ and $\theta_{l(q-1)}(\sqrt{\rho}) = (-1)^l$, we have

$$X_l(w) = -2\sum_{y\in F^\times, -y\in F_1^\times} (-1)^l = -2(-1)^l\left|F_1^\times\right| = (q-1)(-1)^{l+1}$$

and consequently we have $X_{l(q-1)}(\varepsilon[w]) = (-1)^{l+1}$.

If $g = u(1)$, then $|F_{\mathrm{II}}^\times(u(1))| = 1$ and $F_{\mathrm{IV}}^\times(u(1)) = \phi$ from Lemma 4.3. Therefore we have $X_{l(q-1)}(\varepsilon[u(1)]) = -1$.

Assume $g = u(1)wu(r)$ with $r \neq 1$. Then from Lemma 4.3 $F_{\mathrm{IV}}^{\times}(u(1)wu(r)) = \{2(1+\xi) - r\,;\, \xi \in F_1(1-r)\}$ and $a(2(1+\xi)-r)u(1)wu(r) \in [\kappa(1+\xi+\eta\sqrt{\rho})]$ where $\eta \in F^{\times}$ such that $\eta^2\rho = \xi^2 - (1-r)$. Here we use the results of Lemma 1.3. Put $\xi = g_{1-r}(\zeta) = 2^{-1}(\zeta + \zeta^q)$ where $\zeta \in E_{1-r}^{\times}$. Then $g_{1-r}\colon E_{1-r}^{\times} - \{\pm\sqrt{1-r}\} \to F_1(1-r)$ is a 2 to 1 surjection if $1-r \in F_0^{\times}$, while $g_{1-r}\colon E_{1-r}^{\times} \to F_1(1-r)$ is a 2 to 1 surjection if $1-r \in F_1^{\times}$. Moreover we can take $\eta\sqrt{\rho} = 2^{-1}(\zeta - \zeta^q)$. Therefore we have $1+\xi+\eta\sqrt{\rho} = 1+\zeta$ and consequently

$$X_{l(q-1)}([\kappa(1+\xi+\eta\sqrt{\rho})]) = -(\theta_{l(q-1)}(1+\zeta) + \theta_{l(q-1)}(1+\zeta^q)).$$

Thus we obtain

$$X_l(u(1)wu(r)) = \begin{cases} -\dfrac{1}{2} \displaystyle\sum_{\zeta \in E_{1-r}^{\times} - \{\pm\sqrt{1-r}\}} \{\theta_{l(q-1)}(1+\zeta) + \theta_{l(q-1)}(1+\zeta^q)\} & (1-r \in F_0^{\times}), \\ -\dfrac{1}{2} \displaystyle\sum_{\zeta \in E_{1-r}^{\times}} \{\theta_{l(q-1)}(1+\zeta) + \theta_{l(q-1)}(1+\zeta^q)\} & (1-r \in F_1^{\times}). \end{cases}$$

If we substitute ζ for ζ^q in the second term, which does not change $E_{1-r}^{\times} - \{\pm\sqrt{1-r}\}$ and $E_{1-r}^{\times}$ respectively, we have

$$X_l(u(1)wu(r)) = \begin{cases} -\displaystyle\sum_{\zeta \in E_{1-r}^{\times} - \{\pm\sqrt{1-r}\}} \theta_{l(q-1)}(1+\zeta) & (1-r \in F_0^{\times}), \\ -\displaystyle\sum_{\zeta \in E_{1-r}^{\times}} \theta_{l(q-1)}(1+\zeta) & (1-r \in F_1^{\times}). \end{cases}$$

If $r = 0$, then $|F_{\mathrm{II}}^{\times}(u(1)w)| = 1$ and hence

$$X_{l(q-1)}(\varepsilon[u(1)w]) = -1 - \sum_{\zeta \in E_1^{\times} - \{\pm 1\}} \theta_{l(q-1)}(1+\zeta) = -\sum_{\zeta \in E_1^{\times} - \{-1\}} \theta_{l(q-1)}(1+\zeta).$$

Using Lemma 1.4, we obtain $X_{l(q-1)}(\varepsilon[u(1)w]) = (-1)^l$.

If $r \neq 0$ and $1-r \in F_0^{\times}$, then $|F_{\mathrm{II}}^{\times}(u(1)wu(r))| = 2$ and hence

$$X_{l(q-1)}(\varepsilon[u(1)wu(r)]) = -2 - \sum_{\zeta \in E_{1-r}^{\times} - \{\pm\sqrt{1-r}\}} \theta_{l(q-1)}(1+\zeta).$$

Since $\theta_{l(q-1)}(\pm\sqrt{1-r}) = 1$, we have

$$X_{l(q-1)}(\varepsilon[u(1)wu(r)]) = -\sum_{\zeta \in E_{1-r}^{\times}} \theta_{l(q-1)}(1+\zeta).$$

If $1 - r \in F_1^{\times}$, then $|F_{\mathrm{II}}^{\times}(u(1)wu(r))| = 0$ and hence

$$X_{l(q-1)}(\varepsilon[u(1)wu(r)]) = -\sum_{\zeta \in E_{1-r}^{\times}} \theta_{l(q-1)}(1+\zeta).$$

Thus we have completed the proof of Main Theorem. □

EXAMPLE 4.4. The character table of $\mathcal{H}(GL_2(F_5), A)$

	U_0	V_0	V_2	$W_{1,3}$	X_4	X_8
ε	1	2	1	1	1	1
$\varepsilon[w]$	1	0	1	−1	1	−1
$\varepsilon[u(1)]$	4	3	−1	−1	−1	−1
$\varepsilon[wu(1)]$	4	−1	−1	1	−1	1
$\varepsilon[u(1)w]$	4	−1	−1	1	−1	1
$\varepsilon[u(1)wu(1)]$	4	3	−1	−1	−1	−1
$\varepsilon[u(1)wu(2)]$	4	−2	2	2	−1	−3
$\varepsilon[u(1)wu(3)]$	4	−2	−2	0	4	0
$\varepsilon[u(1)wu(4)]$	4	−2	2	−2	−1	3

EXAMPLE 4.5. The character table of $\mathcal{H}(GL_2(F_7), A)$

	U_0	V_0	V_3	$W_{1,5}$	$W_{2,4}$	X_6	X_{12}	X_{18}
ε	1	2	1	1	1	1	1	1
$\varepsilon[w]$	1	0	−1	−1	1	1	−1	1
$\varepsilon[u(1)]$	6	5	−1	−1	−1	−1	−1	−1
$\varepsilon[wu(1)]$	6	−1	1	1	−1	−1	1	−1
$\varepsilon[u(1)w]$	6	−1	1	1	−1	−1	1	−1
$\varepsilon[u(1)wu(1)]$	6	5	−1	−1	−1	−1	−1	−1
$\varepsilon[u(1)wu(2)]$	6	−2	0	−3	1	$-2\sqrt{2}$	4	$2\sqrt{2}$
$\varepsilon[u(1)wu(3)]$	6	−2	−4	2	−2	$2+\sqrt{2}$	2	$2-\sqrt{2}$
$\varepsilon[u(1)wu(4)]$	6	−2	0	0	4	$-2+2\sqrt{2}$	0	$-2-2\sqrt{2}$
$\varepsilon[u(1)wu(5)]$	6	−2	4	−2	−2	$2+\sqrt{2}$	−2	$2-\sqrt{2}$
$\varepsilon[u(1)wu(6)]$	6	−2	0	3	1	$-2\sqrt{2}$	−4	$2\sqrt{2}$

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