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A FINITE UNIVERSAL SAGBI BASIS FOR THE KERNEL OF A DERIVATION

SHIGERU KURODA

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1. Introduction

To find and to calculate generating sets for invariant rings is a fundamental problem in invariant theory with a long tradition. With the progress of computers, the significance of computational methods in this field has increased. The SAGBI bases are the sets of generators of a subalgebra of a polynomial ring which have certain computational property. These are the natural “Subalgebra Analogue to Gröbner Bases for Ideals” introduced at the end of 1980’s by Robbiano and Sweedler [20] and Kapur and Madlener [8], independently. There are indeed some applications of the SAGBI bases to invariant theory. The algorithm of Stillman and Tsai [23] gives a method for computing generating sets for certain invariant rings by using this notion. However, compared with the theory of Gröbner bases, that of SAGBI bases has made a slow progress, and many basic problems remaining unsolved. The purpose of this paper is to investigate the properties of a SAGBI basis for the kernel of a derivation on a polynomial ring.

The kernel of a derivation on a polynomial ring is closely related to an invariant ring. It is an important object in the study of invariant theory and the fourteenth problem of Hilbert. It is well-known that some kind of derivation corresponds to an action of one-dimensional additive group, and the kernel and the invariant subring are the same. Moreover, various counterexamples to the fourteenth problem of Hilbert can be described as the kernel of a derivation. Nagata’s counterexample [17] and Roberts’ counterexample [22] were described as this by Derksen [2] and by Deveney and Finston [4], respectively. Nowicki showed that the invariant subring for a linear action of a connected linear algebraic group on a polynomial ring is obtained as the kernel of a derivation [18]. Recently, new counterexamples to the fourteenth problem of Hilbert were constructed by using the kernel of a derivation by several people (cf. [1], [6], [10], [13]). We believe that a computational methods will give us further progress in this field.

In this paper, k is always a field of characteristic zero except Section 6. Let

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$k[\mathbf{x}] = k[x_1, \ldots, x_n]$ and $k[\mathbf{x}, \mathbf{x}^{-1}] = k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ be the polynomial and the Laurent polynomial rings in n variables over k, respectively, and $k(\mathbf{x})$ the filed of fractions of $k[\mathbf{x}]$. For each $a = (a_1, \ldots, a_n) \in \mathbf{Z}^n$, we denote by $\mathbf{x}^a$ the monomial $x_1^{a_1} \cdots x_n^{a_n}$. Let Ω denote the set of total orders $\preceq$ on $\mathbf{Z}^n$ such that $a \preceq b$ implies $a + c \preceq b + c$ for any a, b, $c \in \mathbf{Z}^n$, and Ω_0 the set of $\preceq \in \Omega$ such that the zero vector is the minimum among $(\mathbf{Z}_{\geq 0})^n$ for the order relation $\preceq$. Here, we denote by $\mathbf{Z}_{\geq 0}$ the set of nonnegative integers. An element of Ω_0 is called a *monomial order* on $k[\mathbf{x}]$. When an order $\preceq$ is given, we write $a \prec b$ if $a \preceq b$ and $a \neq b$ for a, $b \in \mathbf{Z}^n$. We sometimes denote by $\mathbf{x}^a \preceq \mathbf{x}^b$ instead of $a \preceq b$. The *lexicographic order* $\preceq$ on $k[\mathbf{x}]$ with $x_{i_1} \prec x_{i_2} \prec \cdots \prec x_{i_n}$ is the monomial order on $k[\mathbf{x}]$ which is defined by $a \preceq b$ if $0 < c_{i_l}$ for the maximal integer l with $c_{i_l} \neq 0$ for a, $b \in \mathbf{Z}^n$, where $b - a = (c_1, \ldots, c_n)$.

Let $\preceq$ be an element of Ω. For $f = \sum_{a \in \mathbf{Z}^n} \mu_a \mathbf{x}^a \in k[\mathbf{x}, \mathbf{x}^{-1}]$, we define the *support* $\mathrm{supp}(f)$ of f by

$$\mathrm{supp}(f) = \{a \mid \mu_a \neq 0\}. \tag{1.1}$$

The convex hull of $\mathrm{supp}(f)$ in $\mathbf{R}^n$ is denoted by $\mathrm{New}(f)$, and called the *Newton polytope* of f. If $f \neq 0$, then we set $v_\preceq(f)$ to be the maximal element of $\mathrm{supp}(f)$ for $\preceq$. The maximum exists, since $\mathrm{supp}(f)$ is a nonempty finite subset of $\mathbf{Z}^n$. For any f, $g \in k[\mathbf{x}, \mathbf{x}^{-1}] \setminus \{0\}$, it follows that $v_\preceq(fg) = v_\preceq(f) + v_\preceq(g)$. We define the *initial term* $\mathrm{in}_\preceq(f)$ of f by

$$\mathrm{in}_\preceq(f) = \mu_{v_\preceq(f)} \mathbf{x}^{v_\preceq(f)} \tag{1.2}$$

if $f \neq 0$, while we define $\mathrm{in}_\preceq(0) = 0$. Then, it follows that

$$\mathrm{in}_\preceq(fg) = \mathrm{in}_\preceq(f)\,\mathrm{in}_\preceq(g) \tag{1.3}$$

for any f, $g \in k[\mathbf{x}, \mathbf{x}^{-1}]$. For a k-vector subspace V of $k[\mathbf{x}]$, we define the *initial vector space* $\mathrm{in}_\preceq(V)$ to be the k-vector space generated by $\{\mathrm{in}_\preceq(f) \mid f \in V\}$. If A is a k-subalgebra of $k[\mathbf{x}]$, then $\mathrm{in}_\preceq(A)$ is a k-algebra. It is called the *initial algebra* of A. A subset $\mathcal{S}$ of A is said to be a *SAGBI basis* for A if it is a generating set for A over k such that

$$\mathrm{in}_\preceq(A) = k[\{\mathrm{in}_\preceq(s) \mid s \in \mathcal{S}\}]. \tag{1.4}$$

We say that $\mathcal{S}$ is a *universal SAGBI basis* for A if it is a SAGBI basis for A with respect to any $\preceq \in \Omega$. We remark that, if $\preceq$ is in Ω_0, then the condition (1.4) implies that $\mathcal{S}$ generates A over k by [20, Proposition 1.16]. Hence, if this is the case, then $\mathcal{S}$ is a SAGBI basis for A. In particular, a subset $\mathcal{S}$ is a universal SAGBI basis for A if and only if the subsemigroup $\{v_\preceq(f) \mid f \in A \setminus \{0\}\}$ of $\mathbf{Z}^n$ is generated by $\{v_\preceq(f) \mid f \in \mathcal{S} \setminus \{0\}\}$ for any $\preceq \in \Omega$, since it is equivalent to the condition that (1.4) holds for any $\preceq \in \Omega$.

By definition, there exist the following implications:

$$\begin{gathered}A \text{ has a finite universal SAGBI basis.}\\ \Downarrow \\ A \text{ has a finite SAGBI basis for some } \preceq\, \in \Omega. \\ \Downarrow \\ A \text{ is finitely generated over } k.\end{gathered}$$

However, the converse of each implication is not always true. Actually, Robbiano and Sweedler [20, Example 4.11] showed that $\{x_1, x_1x_2 - x_2^2, x_1x_2^2\}$ is a SAGBI basis for $k[x_1, x_1x_2 - x_2^2, x_1x_2^2]$ with respect to $\preceq\, \in \Omega$ with $x_1 \prec x_2$, but this k-algebra does not have a finite SAGBI basis for $\preceq\, \in \Omega$ with $x_2 \prec x_1$. We also give such examples as the kernel of a derivation in Section 5. We showed in [11, Theorem 2.2] that certain finitely generated invariant rings do not have finite SAGBI bases for any $\preceq\, \in \Omega$. This theorem also says that each of these invariant rings has uncountable cardinality of distinct initial algebras. Therefore, we may ask the following questions for a finitely generated k-subalgebra A of $k[\mathbf{x}]$.

Question 1. Does A have a finite SAGBI basis?

Question 2. How many distinct initial algebras does A have?

These questions are generally difficult to answer. In some case, Question 1 is closely related to the fourteenth problem of Hilbert as we will see in Section 5. In the present paper, we will give a sufficient condition on derivations for their kernels to have finite universal SAGBI bases, and an upper bound for the number of distinct initial algebras of them.

For a commutative k-algebra A, a k-linear map $D\colon A \to A$ is called a *k-derivation* on A if $D(fg) = D(f)g + fD(g)$ for any f, $g \in A$. For a k-vector subspace V of A, we denote by

$$V^D = \{f \in V \mid D(f) = 0\}. \tag{1.5}$$

If V is a k-subalgebra of A, then V^D is a k-subalgebra of V. We will study the kernel $k[\mathbf{x}]^D$ of a k-derivation D on $k[\mathbf{x}]$. We note that $k[\mathbf{x}]^D$ is not necessarily finitely generated (cf. [1], [2], [6], [10], [13]), and this is a kind of the fourteenth problem of Hilbert.

We define the *support* supp(D) of D by

$$\operatorname{supp}(D) = \bigcup_{i=1}^{n} \operatorname{supp}(x_i^{-1}D(x_i)). \tag{1.6}$$

The convex hull of $\operatorname{supp}(D)$ in $\mathbf{R}^n$ is denoted by $\operatorname{New}(D)$, and called the *Newton polytope* of D. For each $\delta \in \operatorname{supp}(D)$ and $1 \leq i \leq n$, there exists $\kappa_{\delta,i} \in k$ such that

$$x_i^{-1} D(x_i) = \sum_{\delta \in \operatorname{supp}(D)} \kappa_{\delta,i} \mathbf{x}^{\delta}. \tag{1.7}$$

Then, define a homomorphism $\lambda_\delta \colon \mathbf{Z}^n \to k$ of additive groups by

$$\lambda_\delta((a_1, \ldots, a_n)) = a_1 \kappa_{\delta,1} + \cdots + a_n \kappa_{\delta,n}. \tag{1.8}$$

We define a subset $\operatorname{supp}^\circ(D)$ of $\operatorname{supp}(D)$ as follows. Set $S_0 = \operatorname{supp}(D)$ and

$$S_{i+1} = \{\delta \in S_i \mid \delta' - \delta \notin \ker \lambda_\delta \text{ for some } \delta' \in S_i\} \tag{1.9}$$

for each $i \in \mathbf{Z}_{\geq 0}$, inductively. Then, define $\operatorname{supp}^\circ(D)$ to be the set of $\delta \in \operatorname{supp}(D)$ contained in the convex hull of $\bigcap_{i=0}^{\infty} S_i$ in $\mathbf{R}^n$. For a subset $S \subset \mathbf{R}^n$, the dimension $\dim S$ of S is defined as the dimension of the $\mathbf{R}$-vector subspace of $\mathbf{R}^n$ generated by $\{s - t \mid s, t \in S\}$ if $S \neq \emptyset$, and -1 if $S = \emptyset$. Since $\operatorname{supp}^\circ(D)$ cannot be a single point, the dimension of $\operatorname{supp}^\circ(D)$ is not zero for any D. As we see in Section 2, there exist various k-derivations D such that $\operatorname{supp}^\circ(D) \neq \operatorname{supp}(D)$.

In [12, Theorem 1.3], we showed that $k[\mathbf{x}]^D$ is finitely generated over k if the dimension of $\operatorname{supp}(D)$ is at most two. We will show a stronger theorem below in Section 2.

Theorem 1.1. *Assume that D is a k-derivation on $k[\mathbf{x}]$. If the dimension of* $\operatorname{supp}^\circ(D)$ *is at most two, then $k[\mathbf{x}]^D$ has a finite universal SAGBI basis.*

There exist various k-derivations D such that the dimension of $\operatorname{supp}(D)$ is greater than two but that of $\operatorname{supp}^\circ(D)$ is at most two. Hence, Theorem 1.1 can be applied for far more cases than [12, Theorem 1.3].

A k-derivation D on $k[\mathbf{x}]$ is said to be *triangular* if $D(x_i)$ is in $k[x_1, \ldots, x_{i-1}]$ for each i. In this case, we have further the following.

Theorem 1.2. *Assume that D is a triangular derivation on $k[\mathbf{x}]$. If the dimension of* $\operatorname{supp}^\circ(D)$ *is at most two, then there exists a universal SAGBI basis for $k[\mathbf{x}]^D$ with at most n elements.*

We will describe the universal SAGBI basis mentioned in Theorem 1.2 explicitly in Section 3. In Section 4, we discuss the number of distinct initial algebras of $k[\mathbf{x}]^D$, and show the following.

Theorem 1.3. *Assume that D is a k-derivation on $k[\mathbf{x}]$. If the dimension of* $\operatorname{supp}^\circ(D)$ *is two, then the cardinality of $\{\operatorname{in}_{\preceq}(k[\mathbf{x}]^D) \mid \preceq \in \Omega\}$ is at most double* *the number of the vertices of the convex hull of* $\mathrm{supp}^\circ(D)$ *in* $\mathbf{R}^n$. *If the dimension of* $\mathrm{supp}^\circ(D)$ *is one, then the cardinality of* $\{\mathrm{in}_{\preceq}(k[\mathbf{x}]^D) \mid \preceq\, \in \Omega\}$ *is at most two.*

In Section 5, we show that the kernel of certain locally nilpotent derivation is finitely generated but has infinitely generated initial algebras. In Section 6, we investigate a method for describing the kernel of a derivation in terms of Newton polytopes.

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2. A finite universal SAGBI basis

First, we review [12, Lemma 2.1] and its proof. Let A be a finitely generated normal domain over k, and K the field of fractions of A. We assume that K is a regular extension of k, i.e., $K \otimes_k \overline{k}$ is a field for the algebraic closure $\overline{k}$ of k. In that lemma, we showed the following. Let L be a subfield of K containing k, and $g_1, \dots, g_r$ be elements of $K \setminus \{0\}$. Then, the k-subalgebra

$$R = \sum_{i_1,\dots,i_r \in \mathbf{Z}} \left(L g_1^{i_1} \cdots g_r^{i_r} \cap A \right) \tag{2.1}$$

of A is finitely generated over k if L is a simple extension of k. Actually, we have a more precise statement as follows.

Lemma 2.1. *Assume that* $L = k(u_0/u_1)$ *for some* $u_0,\ u_1 \in A$. *Then, we may find a finite subset* $\Sigma_0 \subset \mathbf{P}^1_{\overline{k}}$ *of closed points such that, for any finite subset* $\Sigma \subset \mathbf{P}^1_{\overline{k}}$ *of closed points containing* Σ_0, *there exist* $f_1, \dots, f_s \in R \otimes_k \overline{k}$ *with the following property. Assume that* f *is in* $L g_1^{i_1} \cdots g_r^{i_r} \cap A$ *for some* $i_1, \dots, i_r \in \mathbf{Z}$. *Then, there exists* $h \in \overline{k}[u_0, u_1] \setminus \{0\}$ *of the form*

$$h = \prod_{j=1}^{q} (\alpha_j u_0 - \beta_j u_1)^{m_j}$$

with $(\alpha_j : \beta_j) \in \mathbf{P}^1_{\overline{k}} \setminus \Sigma$ *and* $m_j \in \mathbf{Z}_{\geq 0}$ *for* $j = 1, \dots, q$ *such that* $u_0^i u_1^{m-i} f/h$ *is equal to a product of powers of* $f_1, \dots, f_s$ *multiplied by an element of* $\overline{k} \setminus \{0\}$ *for* $0 \leq i \leq m$, *where* $m = \sum_{j=1}^{q} m_j$.

Proof. We set $\overline{L} = L \otimes_k \overline{k}$, $\overline{A} = A \otimes_k \overline{k}$, $\overline{K} = A \otimes_k \overline{K}$ and $\overline{R} = R \otimes_k \overline{k}$. First, assume that u_0/u_1 is transcendental over k. Let $\phi\colon \operatorname{Spec} \overline{A} \dashrightarrow \mathbf{P}^1_{\overline{k}}$ be the dominant rational map defined by the inclusion map $\overline{L} \to \overline{K}$. Then, we may consider the homomorphism

$$\phi^*\colon \operatorname{Div}(\mathbf{P}^1_{\overline{k}}) \to \operatorname{Div}(\operatorname{Spec} \overline{A})$$

of the divisor groups of $\mathbf{P}^1_{\overline{k}}$ and $\operatorname{Spec} \overline{A}$. Since the complement of the image of ϕ is a finite set, $\ker\phi^*$ is finitely generated. In the proof of [12, Lemma 2.1], we showed the following.

There exists a finite subset $\Sigma \subset \mathbf{P}^1_k$ of closed points as follows:
(i) $\ker\phi^*$ is contained in the subgroup of $\mathrm{Div}(\mathbf{P}^1_{\bar{k}})$ generated by Σ, where we regard Σ as a set of prime divisors.
(ii) Let p be the generic point of a prime divisor which appears in $(g_i) \in \mathrm{Div}(\mathrm{Spec}\,\bar{A})$ for some $1 \leq i \leq r$. Then, $\phi(p)$ is in Σ, unless it is the generic point of $\mathbf{P}^1_{\bar{k}}$.

If Σ is a finite subset of $\mathbf{P}^1_{\bar{k}}$ of closed points as above, then there exist a finite number of elements $f_1, \ldots, f_s \in \bar{R}$ with the following property. Assume that f is an element of $\bar{L}g_1^{i_1}\cdots g_r^{i_r} \cap \bar{A} \setminus \{0\}$ for some $(i_l)_l \in \mathbf{Z}^r$ such that the supports of zeros and poles of the rational function $f/(g_1^{i_1}\cdots g_r^{i_r})$ on $\mathbf{P}^1_{\bar{k}}$ are contained in Σ. Then, f is equal to a product of powers of $f_1, \ldots, f_s$ multiplied by an element of $\bar{k} \setminus \{0\}$.

Let Σ_0 be a finite subset of $\mathbf{P}^1_{\bar{k}}$ of closed points satisfying (i) and (ii) which contains the supports of zeros and poles of u_0/u_1. We show that Σ_0 satisfies the desired property. Assume that Σ is a finite subset of $\mathbf{P}^1_{\bar{k}}$ of closed points containing Σ_0. Then, Σ also satisfies (i) and (ii). Hence, there exist a finite number of elements $f_1, \ldots, f_s \in \bar{R}$ as above. Assume that f is in $Lg_1^{i_1}\cdots g_r^{i_r} \cap A \setminus \{0\}$. Put $h' = f/(g_1^{i_1}\cdots g_r^{i_r})$, and set $(h') = \sum_{p\in\mathbf{P}^1_{\bar{k}}} m_p p$ and $E = \sum_{p\in\Sigma} m_p p$. For each closed point $p \in \mathbf{P}^1_{\bar{k}}$, we assign $(\alpha_p, \beta_p) \in \bar{k}^2 \setminus \{0\}$ so that $h\prod_{p\in\mathbf{P}^1_{\bar{k}}}(\alpha_p u_0 - \beta_p u_1)^{-m'_p}$ is in $\bar{k}\setminus\{0\}$ for every $h \in \bar{L}\setminus\{0\}$ with $(h) = \sum_{p\in\mathbf{P}^1_{\bar{k}}} m'_p p$, and identify p with the ratio $(\alpha_p : \beta_p)$. Then,

$$h = \prod_{p\in\mathbf{P}^1_{\bar{k}}\setminus\Sigma} (\alpha_p u_0 - \beta_p u_1)^{m_p}$$

is in $\bar{k}[u_0, u_1] \setminus \{0\}$, since h' is in $H^0(\mathbf{P}^1_{\bar{k}}, \mathcal{O}_{\mathbf{P}^1_{\bar{k}}}(-E))$. Set $m = \sum_{p\in\mathbf{P}^1_{\bar{k}}\setminus\Sigma} m_p$, and take any $0 \leq i \leq m$. Then, the supports of zeros and poles of $u_0^i u_1^{m-i} h'/h$ are contained in Σ. Hence, those of $u_0^i u_1^{m-i} f/h$ are also in Σ. Moreover, $u_0^i u_1^{m-i} f/h$ is in $\bar{L}g_1^{i_1}\cdots g_r^{i_r} \cap \bar{A}$. Actually,

$$H^0(\mathbf{P}^1_{\bar{k}}, \mathcal{O}_{\mathbf{P}^1_{\bar{k}}}(-E))g_1^{i_1}\cdots g_r^{i_r} \subset \bar{L}g_1^{i_1}\cdots g_r^{i_r} \cap \bar{A},$$

and $u_0^i u_1^{m-i} h'/h$ is in $H^0(\mathbf{P}^1_{\bar{k}}, \mathcal{O}_{\mathbf{P}^1_{\bar{k}}}(-E))$. Thus, $u_0^i u_1^{m-i} f/h$ is equal to a product of powers of $f_1, \ldots, f_s$ multiplied by an element of $\bar{k} \setminus \{0\}$ by assumption. Therefore, the assertion is true if u_0/u_1 is transcendental over k.

Now, assume that u_0/u_1 is algebraic over k. Then, $L = k$, since K is a regular extension of k. In this case, the proof of [12, Lemma 2.1] says that there exist a finite number of elements $f_1, \ldots, f_s \in \bar{R}$ such that every element of $\bar{k}g_1^{i_1}\cdots g_r^{i_r} \cap \bar{A}$ is equal to a product of powers of $f_1, \ldots, f_s$ multiplied by an element of $\bar{k} \setminus \{0\}$. Hence, the assertion holds for $\Sigma_0 = \emptyset$ and $h = 1$. □

Now, let Γ be an additive group, $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ a Γ-graded finitely generated normal k-subalgebra of $k[\mathbf{x}]$, and D a k-derivation defined on an extension of A. Here, we say that a k-algebra R is Γ-*graded* if $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$ for some k-vector spaces $R_\gamma \subset R$ such that $R_\gamma R_\mu \subset R_{\gamma+\mu}$ for every γ, $\mu \in \Gamma$. An element $f \in R$ is said to be Γ-*homogeneous* if f is in R_γ for some $\gamma \in \Gamma$. Since A is a domain, the set $A_H = \bigcup_{\gamma\in\Gamma} A_\gamma \setminus \{0\}$ of nonzero Γ-homogeneous elements is multiplicatively closed. We set $B = A_H^{-1}A$ to be the localization of A by A_H. Then, the Γ-grading $B = \bigoplus_{\gamma\in\Gamma} B_\gamma$ is defined by setting

$$B_\gamma = \left\{ \frac{f}{g} \;\middle|\; (f, g) \in A_{\mu+\gamma} \times (A_\mu \setminus \{0\}) \text{ for some } \mu \in \Gamma \right\} \tag{2.2}$$

for each $\gamma \in \Gamma$. Note that B_0 is a field containing k. For a k-domain R, we denote by $\operatorname{trans.deg}_k R$ the transcendence degree of R over k.

Theorem 2.2. *Assume that* $A^D = \bigoplus_{\gamma\in\Gamma} A^D_\gamma$ *and* $\operatorname{in}_\preceq(A^D) = \bigoplus_{\gamma\in\Gamma} \operatorname{in}_\preceq(A^D_\gamma)$ *for any* $\preceq \in \Omega$. *If* $\operatorname{trans.deg}_k B^D_0 \le 1$, *then* A^D *has a finite universal SAGBI basis.*

Proof. For each $f \in A^D$ and $\preceq \in \Omega$, there exists a Γ-homogeneous element $f' \in A^D$ such that $v_\preceq(f) = v_\preceq(f')$ by assumption. We will show the existence of a finite number of elements $f_1, \dots, f_t \in A^D$ such that, for any Γ-homogeneous element $f \in A^D \setminus \{0\}$ and $\preceq \in \Omega$, there exist $a_1, \dots, a_t \in \mathbf{Z}_{\ge 0}$ such that $v_\preceq(f) = a_1 v_\preceq(f_1) + \cdots + a_t v_\preceq(f_t)$. Then, the remark after the definition of universal SAGBI bases in Section 1 implies that $\{f_1, \dots, f_t\}$ is a universal SAGBI basis for A^D.

The assumption $\operatorname{trans.deg}_k B^D_0 \le 1$ implies that the field B^D_0 is a simple extension of k. Actually, if $\operatorname{trans.deg}_k B^D_0 = 1$, then B^D_0 is a rational function field of one variable over k by Lüroth's theorem, while $B^D_0 = k$ otherwise. Let u_0, $u_1 \in A \setminus \{0\}$ be Γ-homogeneous elements with $B^D_0 = k(u_0/u_1)$. Then, we may find a finite subset $\Sigma_1 \subset \mathbf{P}^1_k$ of closed points such that, for any finite subset $\Sigma \subset \mathbf{P}^1_k$ of closed points containing Σ_1, the Newton polytopes of $\alpha u_0 - \beta u_1$ are the same for any $(\alpha, \beta) \in \overline{k}^2 \setminus \{0\}$ with $(\alpha : \beta) \notin \Sigma$. If this is the case, then it follows that

$$v_\preceq(\alpha u_0 - \beta u_1) = v_\preceq(u_e) \tag{2.3}$$

for all $(\alpha : \beta) \in \mathbf{P}^1_k \setminus \Sigma$ for some $e \in \{0, 1\}$ for each $\preceq \in \Omega$.

Similarly to the argument after [12, Lemma 2.1], we may find Γ-homogeneous elements $g_1, \dots, g_r \in B^D \setminus \{0\}$ such that, for each $\gamma \in \Gamma$, there exist $i_1, \dots, i_r \in \mathbf{Z}$ such that $A^D_\gamma = B^D_0 g_1^{i_1} \cdots g_r^{i_r} \cap A$. Since $A^D = \bigoplus_{\gamma\in\Gamma} A^D_\gamma$, we get

$$A^D = \sum_{i_1,\dots,i_r \in \mathbf{Z}} \left(B^D_0 g_1^{i_1} \cdots g_r^{i_r} \cap A \right).$$

By Lemma 2.1, there exist a finite subset $\Sigma \subset \mathbf{P}^1_k$ of closed points containing Σ_1, and a finite number of elements $f'_1, \dots, f'_s \in A^D \otimes_k \bar{k}$ which satisfy the following property. Let $f \in A^D \setminus \{0\}$ be a Γ-homogeneous element. Then, there exists $h \in \bar{k}[u_0, u_1] \setminus \{0\}$ of the form $h = \prod_{j=1}^{q} (\alpha_j u_0 - \beta_j u_1)^{m_j}$ with $(\alpha_j : \beta_j) \in \mathbf{P}^1_{\bar{k}} \setminus \Sigma$ and $m_j \in \mathbf{Z}_{\geq 0}$ for $j = 1, \dots, q$ such that $u_0^i u_1^{m-i} f/h$ is equal to a product of powers of $f'_1, \dots, f'_s$ multiplied by an element of $\bar{k} \setminus \{0\}$ for $0 \leq i \leq m$, where $m = \sum_{j=1}^{q} m_j$. Note that there exist a finite number of elements $f_1, \dots, f_t \in A^D$ such that, for each $1 \leq j \leq s$ and $\preceq \in \Omega$, we have $v_{\preceq}(f'_j) = v_{\preceq}(f_l)$ for some $1 \leq l \leq t$. We show that $f_1, \dots, f_t$ are what we are looking for. Take any $\preceq \in \Omega$. Then, it follows that

$$\begin{aligned} v_{\preceq}(f) &= v_{\preceq}\left(\frac{f}{h} \prod_{j=1}^{q} (\alpha_j u_0 - \beta_j u_1)^{m_j} \right) \\ &= v_{\preceq}\left(\frac{f}{h} \right) + \sum_{j=1}^{q} m_j v_{\preceq}(\alpha_j u_0 - \beta_j u_1) \\ &= v_{\preceq}\left(\frac{f}{h} \right) + \sum_{j=1}^{q} m_j v_{\preceq}(u_e) \\ &= v_{\preceq}\left(\frac{u_e^m f}{h} \right) \end{aligned}$$

for some $e \in \{0, 1\}$ by (2.3). Choose $a'_1, \dots, a'_s \in \mathbf{Z}_{\geq 0}$ such that $u_e^m f/h$ is equal to $(f'_1)^{a'_1} \cdots (f'_s)^{a'_s}$ multiplied by an element in $\bar{k} \setminus \{0\}$. Then, $v_{\preceq}(f) = \sum_{i=1}^{s} a'_i v_{\preceq}(f'_i)$. By the choice of $f_1, \dots, f_t$, we have $\sum_{i=1}^{s} a'_i v_{\preceq}(f'_i) = \sum_{i=1}^{t} a_i v_{\preceq}(f_i)$ for some $a_1, \dots, a_t \in \mathbf{Z}_{\geq 0}$. Thus, $v_{\preceq}(f) = \sum_{i=1}^{t} a_i v_{\preceq}(f_i)$. Therefore, the proof is completed. □

Let D be a k-derivation on $k[\mathbf{x}]$. For each $\delta \in \operatorname{supp}(D)$, we define

$$D_\delta = \mathbf{x}^\delta \left(\kappa_{\delta,1} x_1 \frac{\partial}{\partial x_1} + \cdots + \kappa_{\delta,n} x_n \frac{\partial}{\partial x_n} \right). \tag{2.4}$$

Then, it follows that

$$D_\delta(\mathbf{x}^a) = \lambda_\delta(a) \mathbf{x}^{a+\delta} \tag{2.5}$$

for any $a \in \mathbf{Z}^n$. For a subset S of $\operatorname{supp}(D)$, we define $D_S = \sum_{\delta \in S} D_\delta$. Of course, $D_{\operatorname{supp}(D)} = D$.

Proposition 2.3. *Assume that D is a k-derivation on $k[\mathbf{x}]$, $\delta \in \operatorname{supp}(D)$ and $\preceq \in \Omega$. If $\delta' \preceq \delta$ for any $\delta' \in \operatorname{supp}(D)$, then $v_{\preceq}(f)$ is in $\ker \lambda_\delta$ for each $f \in k[\mathbf{x}]^D \setminus \{0\}$. In particular, each vertex of the Newton polytope of $f \in k[\mathbf{x}]^D \setminus \{0\}$ is in $\ker \lambda_\delta$ for some vertex δ of* New(D).

Proof. It suffices to show the former part. Actually, each vertex of $\mathrm{New}(f)$ is equal to $v_{\preceq}(f)$ for some $\preceq \in \Omega$, and the maximum of $\mathrm{supp}(D)$ for $\preceq$ is a vertex of $\mathrm{New}(D)$. Suppose that $v_{\preceq}(f)$ is not in $\ker \lambda_\delta$. Then, $D_\delta(\mathrm{in}_{\preceq}(f)) \neq 0$ by (2.5). Since $D(f) = 0$, the term $D_\delta(\mathrm{in}_{\preceq}(f))$ is eliminated in the expression

$$D(f) = D_\delta(\mathrm{in}_{\preceq}(f)) + D_\delta(f - \mathrm{in}_{\preceq}(f)) + D_{\mathrm{supp}(D)\setminus\{\delta\}}(f).$$

Since $\mathrm{supp}(D(f))$ is contained in $\mathrm{supp}(D) + \mathrm{supp}(f)$, there exist $\delta' \in \mathrm{supp}(D)$ and $a' \in \mathrm{supp}(f)$ such that $\delta' + a' = \delta + v_{\preceq}(f)$ and $\delta' \neq \delta$ or $a' \neq v_{\preceq}(f)$. Since $\delta' \preceq \delta$ and $a' \preceq v_{\preceq}(f)$, this is a contradiction. Thus, $v_{\preceq}(f)$ is in $\ker \lambda_\delta$. □

We define M_D to be the submodule of $\mathbf{Z}^n$ generated by $\delta - \delta'$ for $\delta, \delta' \in \mathrm{supp}(D)$, and set $\Gamma_D = \mathbf{Z}^n / M_D$. Then, the Γ_D-grading $k[\mathbf{x}] = \bigoplus_{\gamma \in \Gamma_D} k[\mathbf{x}]_\gamma$ is defined by setting $k[\mathbf{x}]_\gamma$ to be the k-vector space generated by $\mathbf{x}^a$ with $a \in (\mathbf{Z}_{\geq 0})^n$ whose image in Γ_D is equal to γ for each $\gamma \in \Gamma$. Note that we have

$$(2.6) \qquad k[\mathbf{x}]^D = \bigoplus_{\gamma \in \Gamma_D} k[\mathbf{x}]^D_\gamma \qquad \text{and} \qquad \mathrm{in}_{\preceq}(k[\mathbf{x}]^D) = \bigoplus_{\gamma \in \Gamma_D} \mathrm{in}_{\preceq}(k[\mathbf{x}]^D_\gamma) \qquad (\preceq \in \Omega).$$

To show Theorem 1.1, we need the following lemma.

Lemma 2.4. *Assume that D is a k-derivation on $k[\mathbf{x}]$. We set*

$$\mathcal{S} = \{a \in (\mathbf{Z}_{\geq 0})^n \mid a \in \ker \lambda_\delta \text{ for all } \delta \in \mathrm{supp}(D) \setminus \mathrm{supp}^\circ(D)\}.$$

Then, it follows that $k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]^{D^\circ}$, where $D^\circ = D_{\mathrm{supp}^\circ(D)}$.

Proof. We use induction on the number of elements of $\mathrm{supp}(D)$. Put $S = \mathrm{supp}(D)$ and $S^\circ = \mathrm{supp}^\circ(D)$. If $S \neq S^\circ$, then there exists a vertex δ of $\mathrm{New}(D)$ such that $\delta \in S \setminus S^\circ$ and $S + \{-\delta\} \subset \ker \lambda_\delta$. Then, it suffices to show that

$$(2.7) \qquad k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in (\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_\delta\}]^{D_{S\setminus\{\delta\}}}$$

by the following reason. Note that the right hand side of (2.7) is equal to

$$k[\{\mathbf{x}^a \mid a \in (\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_\delta\}] \cap k[\mathbf{x}]^{D_{S\setminus\{\delta\}}}.$$

Since $\mathrm{supp}^\circ(D_{S\setminus\{\delta\}}) = \mathrm{supp}^\circ(D)$, we get $k[\mathbf{x}]^{D_{S\setminus\{\delta\}}} = k[\{\mathbf{x}^a \mid a \in \mathcal{S}'\}]^{D^\circ}$ by induction assumption, where

$$\mathcal{S}' = \{a \in (\mathbf{Z}_{\geq 0})^n \mid a \in \ker \lambda_{\delta'} \text{ for all } \delta' \in S \setminus (\{\delta\} \cup S^\circ)\}.$$

On the other hand, we have

$$k[\{\mathbf{x}^a \mid a \in (\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_\delta\}] \cap k[\{\mathbf{x}^a \mid a \in \mathcal{S}'\}]^{D^\circ} = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]^{D^\circ}.$$

Therefore, (2.7) implies $k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]^{D^\circ}$.

First, we show that every $f \in k[\mathbf{x}]^D$ is contained in the right hand side of (2.7). Without loss of generality, we may assume that f is Γ_D-homogeneous. Since δ is a vertex of the convex hull of S in $\mathbf{R}^n$, there exists $\preceq \in \Omega$ such that δ is the maximum among S for $\preceq$. Then, $v_\preceq(f)$ is in $\ker \lambda_\delta$ by Proposition 2.3. Since $S + \{-\delta\} \subset \ker \lambda_\delta$, we have $M_D \subset \ker \lambda_\delta$. Moreover, $\mathrm{supp}(f) + \{-v_\preceq(f)\} \subset M_D$, since f is Γ_D-homogeneous. Thus, $\mathrm{supp}(f) \subset \ker \lambda_\delta$, so f is in $k[\{\mathbf{x}^a \mid a \in (\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_\delta\}]$. Furthermore, $D_{S\setminus\{\delta\}}(f) = 0$. Actually, we have

$$D_{S\setminus\{\delta\}}(f) = D_{S\setminus\{\delta\}}(f) + D_\delta(f) = D(f), \tag{2.8}$$

since $D_\delta(f) = 0$ by (2.5). Thus, f is in the right hand side of (2.7). Conversely, if f is in the right hand side of (2.7), then the equality (2.8) holds. Hence, f is in $k[\mathbf{x}]^D$. Therefore, we get (2.7), and the proof is completed. □

Proof of Theorem 1.1. We set $A = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]$. Then, A is a finitely generated normal k-subalgebra of $k[\mathbf{x}]$, since $\mathcal{S}$ is a finitely generated normal subsemigroup of $(\mathbf{Z}_{\geq 0})^n$. Here, we say that a subsemigroup $\mathcal{S}$ of $\mathbf{Z}^n$ is *normal* if $\mathcal{S} = (\sum_{s\in\mathcal{S}} \mathbf{Z}s) \cap (\sum_{s\in\mathcal{S}} \mathbf{R}_{\geq 0}s)$, where $\mathbf{R}_{\geq 0}$ is the set of nonnegative real numbers.

We set Γ to be the image of the submodule

$$\mathcal{M} = \{a \in \mathbf{Z}^n \mid a \in \ker \lambda_\delta \text{ for all } \delta \in \mathrm{supp}(D) \setminus \mathrm{supp}^\circ(D)\}$$

of $\mathbf{Z}^n$ in Γ_{D°. Then, $\mathbf{x}^a$ is in $\bigoplus_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma$ if and only if a is in $\mathcal{M} + M_{D^\circ}$ for $a \in (\mathbf{Z}_{\geq 0})^n$. Since $M_{D^\circ} \subset \mathcal{M}$ and $\mathcal{S} = \mathcal{M} \cap (\mathbf{Z}_{\geq 0})^n$, it is equivalent to $a \in \mathcal{S}$. Thus, $A = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma$. In particular, we have $A^{D^\circ} = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma^{D^\circ}$ and $\mathrm{in}_\preceq(A^{D^\circ}) = \bigoplus_{\gamma\in\Gamma} \mathrm{in}_\preceq(k[\mathbf{x}]_\gamma^{D^\circ})$ for any $\preceq \in \Omega$ by (2.6).

Let us denote by $B = \bigoplus_{\gamma\in\Gamma} B_\gamma$ the localization of A by $\bigcup_{\gamma\in\Gamma} k[\mathbf{x}]_\gamma \setminus \{0\}$, and by $k(M_{D^\circ})$ the subfield of $k(\mathbf{x})$ generated by $\{\mathbf{x}^a \mid a \in M_{D^\circ}\}$ over k. Then, $B_0 \subset k(M_{D^\circ})$. Since the dimension of $\mathrm{supp}^\circ(D)$ is at most two, the rank of M_{D° is at most two. This implies that $\mathrm{trans.deg}_k\, k(M_{D^\circ}) \leq 2$. First, assume that $\mathrm{trans.deg}_k\, k(M_{D^\circ})^{D^\circ} = 2$. Take $\delta \in \mathrm{supp}^\circ(D)$, and define a k-derivation D' on $k(\mathbf{x})$ by $D'(f) = \mathbf{x}^{-\delta} D^\circ(f)$ for each f. Then, it induces a k-derivation on $k(M_{D^\circ})$. Moreover, $k(M_{D^\circ})^{D'} = k(M_{D^\circ})^{D^\circ}$. Since $k(M_{D^\circ})$ is a separable algebraic extension of $k(M_{D^\circ})^{D^\circ}$, it implies that D' is zero on $k(M_{D^\circ})$ (cf. [14, Chapter X, Proposition 7]), so $k(M_{D^\circ}) = k(M_{D^\circ})^{D^\circ}$. Hence, by [12, Lemma 3.2] and its proof, we have $k[\mathbf{x}]^{D^\circ} = k[\{\mathbf{x}^a \mid a \in \mathcal{S}_0\}]$ for some finitely generated subsemigroup $\mathcal{S}_0$ of $(\mathbf{Z}_{\geq 0})^n$. Then, $A^{D^\circ} = A \cap k[\mathbf{x}]^{D^\circ} = k[\{\mathbf{x}^a \mid a \in \mathcal{S} \cap \mathcal{S}_0\}]$. By Gordan's lemma [19, Proposition 1.1.(ii)], the semigroup $\mathcal{S} \cap \mathcal{S}_0$ is finitely generated. Hence, A^{D° is generated by a finite set of monomials over k. This set is clearly a universal SAGBI basis for A^{D°. Since $k[\mathbf{x}]^D = A^{D^\circ}$ by Lemma 2.4, the assertion of Theorem 1.1 is true in this case. If $\mathrm{trans.deg}_k\, k(M_{D^\circ})^{D^\circ} \leq 1$, then $\mathrm{trans.deg}_k\, B_0^{D^\circ} \leq 1$. Hence, $A^{D^\circ} = k[\mathbf{x}]^D$ has a finite universal SAGBI basis by Theorem 2.2. We have thus proved Theorem 1.1. □

As mentioned in Section 1, there exist various k-derivations D on $k[\mathbf{x}]$ such that the dimension of $\mathrm{supp}(D)$ is greater than two but that of $\mathrm{supp}^\circ(D)$ is at most two. Let us consider the k-derivation

$$D = x_2^2\frac{\partial}{\partial x_1} + (x_1^2x_3x_4 + 2x_2^2x_4^2)\frac{\partial}{\partial x_2} + (x_1x_2^4x_4 + 5x_2x_3x_4^2)\frac{\partial}{\partial x_3} + x_2x_4^3\frac{\partial}{\partial x_4} \tag{2.9}$$

on $k[\mathbf{x}]$ for $n \geq 4$. Since

$$\begin{aligned} x_1^{-1}D(x_1) &= x_1^{-1}x_2^2 \\ x_2^{-1}D(x_2) &= x_1^2x_2^{-1}x_3x_4 + 2x_2x_4^2 \\ x_3^{-1}D(x_3) &= x_1x_2^4x_3^{-1}x_4 + 5x_2x_4^2 \\ x_4^{-1}D(x_4) &= x_2x_4^2 \end{aligned}$$

and $x_i^{-1}D(x_i) = 0$ for $i \geq 5$, we have $\mathrm{supp}(D) = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, where

$$\delta_1 = (-1, 2, 0, 0, 0, \dots, 0),\ \delta_2 = (2, -1, 1, 1, 0, \dots, 0),\ \delta_3 = (1, 4, -1, 1, 0, \dots, 0),$$
$$\delta_4 = (0, 1, 0, 2, 0, \dots, 0).$$

We see easily that the dimension of $\mathrm{supp}(D)$ is three. Furthermore,

$$\lambda_{\delta_i}((a_1, \dots, a_n)) = a_i \quad (i = 1, 2, 3), \qquad \lambda_{\delta_4}((a_1, \dots, a_n)) = 2a_2 + 5a_3 + a_4$$

for $(a_1, \dots, a_n) \in \mathbf{Z}^n$. We show that $\mathrm{supp}^\circ(D) = \{\delta_1, \delta_2, \delta_3\}$. Since $\lambda_{\delta_4}(\delta_i - \delta_4) = 0$ for any i, we have $\delta_4 \notin S_1$. On the other hand, $\lambda_{\delta_i}(\delta_j - \delta_i) \neq 0$ for any $i, j \in \{1, 2, 3\}$ with $i \neq j$. Hence, $S_i = \{\delta_1, \delta_2, \delta_3\}$ for $i \geq 1$ and so $\bigcap_{i=0}^\infty S_i = \{\delta_1, \delta_2, \delta_3\}$. Moreover, the intersection of $\mathrm{supp}(D)$ and the convex hull of $\{\delta_1, \delta_2, \delta_3\}$ in $\mathbf{R}^n$ is equal to $\{\delta_1, \delta_2, \delta_3\}$. Therefore, $\mathrm{supp}^\circ(D) = \{\delta_1, \delta_2, \delta_3\}$, whose dimension is two. Thus, $k[\mathbf{x}]^D$ has a finite universal SAGBI basis by Theorem 1.1.

The following is an example of D which is not zero but $\mathrm{supp}^\circ(D) = \emptyset$. Let D be a k-derivation on $k[\mathbf{x}]$ defined by

$$D(x_i) = \frac{x_i}{i}\left(x_i^i + x_{i+1}^{i+1} + x_{i+2}^{i+2} + \cdots + x_n^n\right) \tag{2.10}$$

for $i = 1, \dots, n$. We set $\delta_i = i\mathbf{e}_i$ for each i, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the coordinate unite vectors of $\mathbf{Z}^n$. Then, $\mathrm{supp}(D) = \{\delta_i \mid i = 1, \dots, n\}$. Hence, the dimension of $\mathrm{supp}(D)$ is $n - 1$. Furthermore,

$$\lambda_{\delta_i}((a_1, \dots, a_n)) = a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_i}{i}$$

for $i = 1, \dots, n$. We show that $S_i = \{\delta_1, \dots, \delta_{n-i}\}$ by induction on i. If $i = 0$, then the assersion is clear. Assume that $i > 0$. Then, $S_{i-1} = \{\delta_1, \dots, \delta_{n-i+1}\}$ by induction assumption. Since $\lambda_{\delta_{n-i+1}}(\delta_j - \delta_{n-i+1}) = 0$ for $j = 1, \dots, n-i+1$, the vector δ_{n-i+1} is not contained in S_i. On the other hand, $\lambda_{\delta_l}(\delta_{n-i+1} - \delta_l) = -1$ for $l = 1, \dots, n-i$. Hence, we get $S_i = \{\delta_1, \dots, \delta_{n-i}\}$. Therefore, $\bigcap_{i=0}^{\infty} S_i = \emptyset$, and hence $\operatorname{supp}^\circ(D) = \emptyset$.

A k-derivation D on a k-algebra R is said to be *locally nilpotent* if, for each $f \in R$, there exists $r \in \mathbf{Z}_{\geq 0}$ such that $D^r(f) = 0$. We see easily that a triangular derivation is a locally nilpotent derivation on $k[\mathbf{x}]$. We note that a locally nilpotent derivation D on R defines an action $\sigma\colon R \to R[s]$ of the one-dimensional additive group scheme $G_a = \operatorname{Spec} k[s]$ by

$$\sigma(f) = \sum_{p=0}^{\infty} \frac{s^p}{p!} D^p(f) \tag{2.11}$$

for each $f \in R$. Since D is locally nilpotent, $\sigma(f) = \sum_{p=0}^{N_f} s^p D^p(f)/p!$ for some $N_f > 0$. The invariant subring R^{G_a} of R for this action of G_a is equal to R^D (cf. [16]).

The vertices of the Newton polytope of a locally nilpotent derivation have the following property.

Lemma 2.5. *Assume that D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$. Then, exactly one component of each vertex of* $\operatorname{New}(D)$ *is equal to* -1.

Proof. Let δ be a vertex of $\operatorname{New}(D)$ and suppose that it is in $(\mathbf{Z}_{\geq 0})^n$. We set a to be an element of $(\mathbf{Z}_{\geq 0})^n \setminus \ker \lambda_\delta$ if $\lambda_\delta(\delta) = 0$, while $a = \delta$ if $\lambda_\delta(\delta) \neq 0$. Then, it follows that $\lambda_\delta(a + j\delta) \neq 0$ for any $j \in \mathbf{Z}_{\geq 0}$. By a repeated use of (2.5), we get

$$D^l(\mathbf{x}^a) = \sum_{\delta_1 \in \operatorname{supp}(D)} \cdots \sum_{\delta_l \in \operatorname{supp}(D)} \left(\prod_{t=1}^{l} \lambda_{\delta_t}\left(a + \sum_{j=1}^{t-1} \delta_j\right)\right) \mathbf{x}^{a+\delta_1+\cdots+\delta_l}$$

for each l. Since δ is a vertex of $\operatorname{New}(D)$, we have $\delta_1 + \cdots + \delta_l = l\delta$ if and only if $\delta_1 = \cdots = \delta_l = \delta$ for $\delta_1, \dots, \delta_l \in \operatorname{supp}(D)$. Hence, the coefficient of $\mathbf{x}^{a+l\delta}$ in $D^l(\mathbf{x}^a)$ is equal to $\prod_{j=0}^{l-1} \lambda_\delta(a + j\delta)$. By the choice of a, it is not zero. This contradicts that $D^l(\mathbf{x}^a) = 0$ for sufficiently large l. Thus, $\delta \notin (\mathbf{Z}_{\geq 0})^n$. It implies that exactly one component of δ is equal to -1. □

By this lemma, Proposition 2.3 is considered as a generalization of [7, Theorem 3.2] which states that each vertex of $\operatorname{New}(f)$ for $f \in k[\mathbf{x}]^D \setminus \{0\}$ lies on a coordinate hyperplane if D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$. Actually, if the i-th component of δ is -1 for some i, then the i-th component of every element of $\ker \lambda_\delta$ is zero.

The dimension of $\operatorname{supp}^\circ(D)$ is one of the measure which shows the "complexity" of $k[\mathbf{x}]^D$. If it is -1, then $k[\mathbf{x}]^D$ is a semigroup ring of a finitely generated normal subsemigroup of $(\mathbf{Z}_{\geq 0})^n$. For a locally nilpotent derivation, we have the following.

Proposition 2.6. *Assume that D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$. Then,* $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$ *if and only if* $\mathrm{supp}^\circ(D) = \emptyset$. *If this is the case, then we have $D = f(\partial/\partial x_i)$ for some $1 \leq i \leq n$ and $f \in k[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n] \setminus \{0\}$.*

Proof. Since $D \neq 0$, it is clear that $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$ if $\mathrm{supp}^\circ(D) = \emptyset$. Assume that $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$. Then, $\mathrm{supp}(D) + \{-\delta\} \subset \ker \lambda_\delta$ for some vertex δ of $\mathrm{New}(D)$. By Lemma 2.5, the i-th component of δ is -1 for some i. Then, $\kappa_{\delta,i} \neq 0$ and $\kappa_{\delta,j} = 0$ for $j \neq i$. Since $\mathrm{supp}(D) + \{-\delta\} \subset \ker \lambda_\delta$, the i-th component of every element of $\mathrm{supp}(D)$ is -1. Thus, $D = f(\partial/\partial x_i)$ for some $f \in k[\mathbf{x}]$ which does not involve x_i. Moreover, $\lambda_{\delta_1}(\delta_2 - \delta_3) = 0$ for any δ_1, δ_2, $\delta_3 \in \mathrm{supp}(D)$. This implies that $\mathrm{supp}^\circ(D) = \emptyset$. □

We note that, if the dimension of $\mathrm{supp}^\circ(D)$ is greater than two, then $k[\mathbf{x}]^D$ does not always have finite SAGBI basis. Actually, there exists a k-derivation D on $k[\mathbf{x}]$ with the dimension of $\mathrm{supp}^\circ(D)$ greater than two whose kernel is not finitely generated. Consider the k-derivation

$$D = \mathbf{x}^{\eta_1}\frac{\partial}{\partial x_4} + \mathbf{x}^{\eta_2}\frac{\partial}{\partial x_5} + \mathbf{x}^{\eta_3}\frac{\partial}{\partial x_6} + \mathbf{x}^{\eta_4}\frac{\partial}{\partial x_7} + x_8\frac{\partial}{\partial x_8} \tag{2.12}$$

on $k[\mathbf{x}]$ for $n \geq 8$, where η_1, η_2, η_3, $\eta_4 \in (\mathbf{Z}_{\geq 0})^n$ whose last $n-3$ components are zero. We set $\delta_i = \eta_i - \mathbf{e}_{i+3}$ for $i = 1, 2, 3, 4$ and $\delta_5 = 0$. Then, $\mathrm{supp}(D) = \{\delta_1, \ldots, \delta_5\}$ and $\lambda_{\delta_i}((a_1, \ldots, a_n)) = a_{i+3}$ for $i = 1, \ldots, 5$. We may easily verify that $\mathrm{supp}^\circ(D) = \{\delta_1, \ldots, \delta_4\}$. We set $D^\circ = D - x_8(\partial/\partial x_8)$ and $\mathcal{S} = (\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_{\delta_5}$. Then, by Lemma 2.4, we have

$$k[\mathbf{x}]^D = k[\{\mathbf{x}^a \mid a \in \mathcal{S}\}]^{D^\circ} = k[x_1, \ldots, x_7, x_9, \ldots, x_n]^{D^\circ}. \tag{2.13}$$

Furthermore, [13, Theorem 1.4] says that there exist a large number of four-tuples $(\eta_1, \eta_2, \eta_3, \eta_4)$ of vectors such that the right hand side of (2.13) is not finitely generated.

3. A triangular derivation with two-dimensional support

Maubach [15] and Khoury [9] studied in respective papers the kernels of some triangular derivations on $k[\mathbf{x}]$. They showed the finite generation of them by giving generating sets explicitly. In this section, we consider the kernel $k[\mathbf{x}]^D$ of a triangular derivation D on $k[\mathbf{x}]$ with the dimension of $\mathrm{supp}^\circ(D)$ at most two. We will determine a universal SAGBI basis for it explicitly. This implies the results of both Maubach and Khoury as special cases.

Let D be a nonzero triangular derivation on $k[\mathbf{x}]$. We set N_D to be the number of indices $i \in \{1, \ldots, n\}$ such that $D(x_i) \neq 0$. Since a triangular derivation is locally nilpotent, $\mathrm{supp}^\circ(D) \neq \mathrm{supp}(D)$ implies $N_D = 1$ by Proposition 2.6. In this case, $\{x_j \mid$

$j \neq i\}$ is a universal SAGBI basis for $k[\mathbf{x}]^D$ for some i. In case of $N_D = 2$, we will determine a universal SAGBI basis for $k[\mathbf{x}]^D$ with $n-1$ elements explicitly in Corollary 3.5 below, as a consequence of a fact on the kernel of a locally nilpotent derivation. Our main result of this section is for the case where $N_D \geq 3$.

Lemma 3.1. *Assume that $n \geq 3$, and D is a nonzero triangular derivation on $k[\mathbf{x}]$. If the dimension of* supp(D) *is at most two, then N_D is at most three. If N_D is three, then, by a change of indices of the variables, we may write D as*

$$(3.1)\qquad \begin{aligned} D = {} & \kappa_0 \mathbf{x}^{\delta_0} \frac{\partial}{\partial x_{n-2}} + \kappa_1 \mathbf{x}^{\delta_1} x_{n-2}^{u_1 - 1} \frac{\partial}{\partial x_{n-1}} \\ & + \mathbf{x}^{\delta_2} x_{n-2}^{u_2-1} x_{n-1}^{v} \sum_{j=0}^{v} \kappa_{2,j} (\mathbf{x}^{\delta_1 - \delta_0} x_{n-2}^{u_1} x_{n-1}^{-1})^j \frac{\partial}{\partial x_n}, \end{aligned}$$

where $\delta_0, \delta_1, \delta_2 \in (\mathbf{Z}_{\geq 0})^n$ whose last three components are zero, $u_1, u_2, v \in \mathbf{Z}$ with $u_1, u_2 \geq 1$ and $v \geq 0$, and $\kappa_0, \kappa_1, \kappa_{2,j} \in k$ for $j = 1, \ldots, v$ with $\kappa_0, \kappa_1, \kappa_{2,0} \neq 0$.

Proof. First, we claim that we may change indices of the variables so that $D(x_i) = 0$ for $i \leq n - N_D$ and $D(x_i) \neq 0$ for $i > n - N_D$. We use induction on the number of indices $i \in \{1, \ldots, n\}$ such that $i < j$ and $D(x_i) \neq 0$, where j is the maximal index with $D(x_j) = 0$. Let i be the maximal index such that $i < j$ and $D(x_i) \neq 0$. Then, D remains triangular if we exchange i and j. Hence, by induction assumption, we may change indices as claimed.

Suppose that N_D is greater than three. Then, we may assume that $D(x_{n-i}) \neq 0$ for $0 \leq i \leq 3$. Take $a_i \in \operatorname{supp}(x_{n-i}^{-1} D(x_{n-i}))$ for each i. Since D is triangular, we have

$$\begin{pmatrix} a_2 - a_3 \\ a_1 - a_3 \\ a_0 - a_3 \end{pmatrix} = \begin{pmatrix} \ldots & -1 & 0 & 0 \\ \ldots\ldots & & -1 & 0 \\ \ldots\ldots\ldots & & & -1 \end{pmatrix}.$$

Hence, $a_2 - a_3$, $a_1 - a_3$ and $a_0 - a_3$ are linearly independent over $\mathbf{R}$. This contradicts that the dimension of supp(D) is at most two. Thus, N_D is at most three.

Assume that N_D is three. Then, we may assume that $D(x_{n-i}) \neq 0$ for $0 \leq i \leq 2$. We show that D is written as in (3.1). Take any $a_i \in \operatorname{supp}(x_{n-i}^{-1} D(x_{n-i}))$ for each i. Then, it suffices to show that $\operatorname{supp}(x_{n-i}^{-1} D(x_{n-i})) = \{a_i\}$ for $i = 1, 2$, and that $a_0 - a_0' \in \mathbf{Z}(a_1 - a_2)$ for every $a_0' \in \operatorname{supp}(x_n^{-1} D(x_n))$. First, suppose that there exists $a_2 \neq a_2' \in \operatorname{supp}(x_{n-2}^{-1} D(x_{n-2}))$. Then, since D is triangular, we have

$$\begin{pmatrix} a_2' - a_2 \\ a_1 - a_2 \\ a_0 - a_2 \end{pmatrix} = \begin{pmatrix} \cdots & 0 & 0 \\ \cdots & -1 & 0 \\ \ldots\ldots & & -1 \end{pmatrix}.$$

Hence, $a_2' - a_2$, $a_1 - a_2$ and $a_0 - a_2$ are linearly independent. This contradicts that the dimension of supp(D) is at most two. Hence, $\mathrm{supp}(x_{n-2}^{-1}D(x_{n-2})) = \{a_2\}$. In a similar way, we see that $\mathrm{supp}(x_{n-1}^{-1}D(x_{n-1})) = \{a_1\}$. Since $a_1 - a_2$ and $a_0 - a_2$ are linearly independent, supp(D) is contained in the $\mathbf{R}$-vector subspace of $\mathbf{R}^n$ generated by them. Hence, each $a_0' \in \mathrm{supp}(x_n^{-1}D(x_n))$ satisfies $a_0 - a_0' = \alpha(a_1 - a_2) + \beta(a_0 - a_2)$ for some $\alpha, \beta \in \mathbf{R}$. Note that the n-th components of $a_0 - a_0'$ and $a_1 - a_2$ are both zero, while that of $a_0 - a_2$ is -1. Hence, $\beta = 0$. Since the $(n-1)$-st component of $a_1 - a_2$ is -1, that of $a_0 - a_0'$ is equal to $-\alpha$. Thus, α is an integer. This completes the proof. □

Let $k[\mathbf{x}][\mathbf{y}] = k[\mathbf{x}][y_0, y_1, \dots, y_m]$ and $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] = k[\mathbf{x}, \mathbf{x}^{-1}][y_0, y_1, \dots, y_m]$ denote the polynomial rings in $m + 1$ variables over $k[\mathbf{x}]$ and $k[\mathbf{x}, \mathbf{x}^{-1}]$, respectively. We express monomials in $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ as $\mathbf{x}^a\mathbf{y}^b$ for $(a, b) \in \mathbf{Z}^n \times \mathbf{Z}^{m+1}$. For each $f \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] \setminus \{0\}$, we set $\mathbf{e}(f)$ to be the unique element of $\mathbf{Z}^n$ such that
(a) $\mathbf{x}^{\mathbf{e}(f)} f \in k[\mathbf{x}][\mathbf{y}]$.
(b) $\mathbf{x}^a f \in k[\mathbf{x}][\mathbf{y}]$ implies that $a - \mathbf{e}(f) \in (\mathbf{Z}_{\geq 0})^n$ for every $a \in \mathbf{Z}^n$.
Then, define $\rho(f) = \mathbf{x}^{\mathbf{e}(f)} f$.

In the situation of Lemma 3.1, we replace n by $n + 3$ and x_{n+1}, x_{n+2}, x_{n+3} by y_0, y_1, y_2, respectively. Then, the k-derivation (3.1) is described as the k-derivation

$$D = \kappa_0 \mathbf{x}^{\delta_0} \frac{\partial}{\partial y_0} + \kappa_1 \mathbf{x}^{\delta_1} y_0^{u_1 - 1} \frac{\partial}{\partial y_1} + \mathbf{x}^{\delta_2} y_0^{u_2-1} y_1^{v} \sum_{j=0}^{v} \kappa_{2,j} (\mathbf{x}^{\delta_1 - \delta_0} y_0^{u_1} y_1^{-1})^j \frac{\partial}{\partial y_2} \tag{3.2}$$

on $k[\mathbf{x}][\mathbf{y}]$ for $m = 2$, where $\delta_0, \delta_1, \delta_2 \in (\mathbf{Z}_{\geq 0})^n$, $u_1, u_2, v \in \mathbf{Z}$ with $u_1, u_2 \geq 1$ and $v \geq 0$, and $\kappa_0, \kappa_1, \kappa_{2,j} \in k$ for $j = 1, \dots, v$ with $\kappa_0, \kappa_1, \kappa_{2,0} \neq 0$. We note that D extends uniquely to a locally nilpotent derivation on $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$.

We set $\epsilon_{i,j} = \delta_i - \delta_j$ for i, j. Then, define two elements of $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ by

$$\tilde{F} = y_1 - \frac{\kappa_1}{\kappa_0 u_1} \mathbf{x}^{\epsilon_{1,0}} y_0^{u_1} \tag{3.3}$$

and

$$\tilde{G} = y_2 - \sum_{p=0}^{v} \left(\sum_{q=0}^{p} \phi(p, q) \right) \mathbf{x}^{p\epsilon_{1,0} + \epsilon_{2,0}} y_0^{pu_1 + u_2} y_1^{v-p}, \tag{3.4}$$

where

$$\phi(p, q) = \frac{(-\kappa_1)^{p-q} \kappa_{2,q}}{\kappa_0^{p-q+1}(v - q + 1)} \prod_{r=1}^{p-q} \frac{v - q - r + 1}{(q + r)u_1 + u_2} \tag{3.5}$$

for p, q. Then, it follows that $D(\tilde{F}) = D(\tilde{G}) = 0$. It is easily checked that $D(\tilde{F}) = 0$. We verify the equality $D(\tilde{G}) = 0$ only.

We set

$$P(p,q) = \left(\frac{(-\kappa_1)^{p-q}\kappa_{2,q}(v-p+1)}{\kappa_0^{p-q}(v-q+1)} \prod_{r=1}^{p-q-1} \frac{v-q-r+1}{(q+r)u_1+u_2} \right) \mathbf{x}^{p\epsilon_{1,0}+\delta_2} y_0^{pu_1+u_2-1} y_1^{v-p}$$

for p, q. Then, it follows that $D(y_2) = \sum_{p=0}^{v} P(p,p)$,

$$D(\phi(p,q)\mathbf{x}^{p\epsilon_{1,0}+\epsilon_{2,0}} y_0^{pu_1+u_2} y_1^{v-p}) = P(p,q) - P(p+1,q)$$

for $0 \leq q \leq p \leq v$, and $P(v+1,q) = 0$ for $0 \leq q \leq v$. Hence, we have

$$\begin{aligned} D(\tilde{G}) &= \sum_{p=0}^{v} P(p,p) - \sum_{p=0}^{v} \sum_{q=0}^{p} (P(p,q) - P(p+1,q)) \\ &= \sum_{q=0}^{v} \left(P(q,q) - \sum_{p=q}^{v} (P(p,q) - P(p+1,q)) \right) \\ &= \sum_{q=0}^{v} P(v+1,q) = 0. \end{aligned}$$

We set $\xi = \sum_{q=0}^{v} \phi(v,q)$, $u_i' = u_i / \gcd(u_1,u_2)$ for $i = 1, 2$ and

$$\eta = u_1'\epsilon_{2,0} - u_2'\epsilon_{1,0} \quad \text{and} \quad w = u_1'v + u_2'. \tag{3.6}$$

If $\xi \neq 0$, then set

$$\tilde{H} = \mathbf{x}^{\eta} \tilde{F}^{w} - (-1)^{w+u_1'} \frac{\kappa_1^{w}}{(\kappa_0 u_1)^{w} \xi^{u_1'}} \tilde{G}^{u_1'}. \tag{3.7}$$

We define $F = \rho(\tilde{F})$ and $G = \rho(\tilde{G})$. If $\xi \neq 0$, then define $H = \rho(\tilde{H})$, else set $H = 0$.

In the notation above, we have the following.

Theorem 3.2. *Assume that $m = 2$, and D is a k-derivation on $k[\mathbf{x}][\mathbf{y}]$ as in* (3.2). *Then, $\{x_1, \ldots, x_n, F, G, H\}$ is a universal SAGBI basis for $k[\mathbf{x}][\mathbf{y}]^D$. In particular, $k[\mathbf{x}][\mathbf{y}]^D$ is generated by at most $n+3$ elements over k.*

Before proving Theorem 3.2, we recall a fact on the kernel of a locally nilpotent derivation. Let R be a k-algebra, and D a locally nilpotent derivation on R. An element $s \in R$ is said to be a *slice* of D if $D(s) = 1$. Assume that D has a slice s. Then, for each $f \in R$, we define

$$\Psi_s(f) = \sum_{p=0}^{\infty} \frac{(-s)^p}{p!} D^p(f). \tag{3.8}$$

Since D is locally nilpotent, $\Psi_s(f)$ is in R. By definition, it follows that $\Psi_s(s) = 0$ and $\Psi_s(f) = f$ for any $f \in R^D$. The following fact is well-known (see [5, Corollary 1.3.23] for instance).

Lemma 3.3. *The map $R \ni f \mapsto \Psi_s(f) \in R$ is a homomorphism of k-algebras. Its image $\Psi_s(R)$ is equal to R^D. In particular, if S generates R over k, then $\{\Psi_s(f) \mid f \in S\}$ generates R^D over k.*

The following is a consequence of Lemma 3.3.

Corollary 3.4. *Assume that D is a locally nilpotent derivation on $k[\mathbf{x}]$ with $D(x_1) \in k \setminus \{0\}$. We set $s = x_1/D(x_1)$. Then, $\{\Psi_s(x_2), \ldots, \Psi_s(x_n)\}$ is a SAGBI basis for $k[\mathbf{x}]^D$ with respect to $\preceq \in \Omega$ satisfying $x_i = \mathrm{in}_\preceq(\Psi_s(x_i))$ for $i = 2, \ldots, n$.*

REMARK. Assume that D is triangular and $D(x_1) \neq 0$. Then, $D(x_1)$ is in $k \setminus \{0\}$. Moreover, the lexicographic order $\preceq$ on $k[\mathbf{x}]$ with $x_1 \prec \cdots \prec x_n$ satisfies that $x_i = \mathrm{in}_\preceq(\Psi_s(x_i))$ for $i = 2, \ldots, n$, where $s = x_1/D(x_1)$.

Proof. By Lemma 3.3, $\{\Psi_s(x_2), \ldots, \Psi_s(x_n)\}$ generates $k[\mathbf{x}]^D$ over k, since $\Psi_s(x_1) = 0$. So, it suffices to show that $\mathrm{in}_\preceq(k[\mathbf{x}]^D) = k[x_2, \ldots, x_n]$.

First, we prove that

$$\text{trans.deg}_k\, \mathrm{in}_\preceq(A) \leq \text{trans.deg}_k\, A \tag{3.9}$$

for any k-subalgebra A of $k[\mathbf{x}]$. Take $f_1, \ldots, f_r \in A$ so that their initial terms form a transcendence basis of $\mathrm{in}_\preceq(A)$ over k. Suppose that there exists a nontrivial algebraic relation

$$\sum_{(i_1,\ldots,i_r)\in(\mathbf{Z}_{\geq 0})^r} \alpha_{i_1,\ldots,i_r} f_1^{i_1} \cdots f_r^{i_r} = 0 \qquad (\alpha_{i_1,\ldots,i_r} \in k). \tag{3.10}$$

Choose $(i_1, \ldots, i_r) \in (\mathbf{Z}_{\geq 0})^r$ with $\alpha_{i_1,\ldots,i_r} \neq 0$ such that $v_\preceq(f_1^{i_1} \cdots f_r^{i_r})$ is the maximum among $v_\preceq(f_1^{i'_1} \cdots f_r^{i'_r})$ for $(i'_1, \ldots, i'_r) \in (\mathbf{Z}_{\geq 0})^r$ with $\alpha_{i'_1,\ldots,i'_r} \neq 0$. Then, there exists $(i_1, \ldots, i_r) \neq (j_1, \ldots, j_r) \in (\mathbf{Z}_{\geq 0})^r$ such that $v_\preceq(f_1^{i_1} \cdots f_r^{i_r}) = v_\preceq(f_1^{j_1} \cdots f_r^{j_r})$. Actually, if such $(j_1, \ldots, j_r)$ did not exist, then the initial term of the left hand side of (3.10) would be $\alpha_{i_1,\ldots,i_r} \mathrm{in}_\preceq(f_1^{i_1} \cdots f_r^{i_r}) \neq 0$. This is a contradiction. However, the existence of such $(j_1, \ldots, j_r)$ implies the algebraic dependence of $\mathrm{in}_\preceq(f_1), \ldots, \mathrm{in}_\preceq(f_r)$ over k. This contradicts the choice of $f_1, \ldots, f_r$. Thus, we get (3.9).

Since $D \neq 0$ and k is of characteristic zero, the transcendence degree of $k[\mathbf{x}]^D$ is less than n (cf. [14, Chapter X, Proposition 7]). Hence, that of $\mathrm{in}_\preceq(k[\mathbf{x}]^D)$ is less than n by (3.9). On the other hand, $\mathrm{in}_\preceq(k[\mathbf{x}]^D) \supset k[x_2, \ldots, x_n]$ by the choice of $\preceq$. Hence, no element in $\mathrm{in}_\preceq(k[\mathbf{x}]^D)$ involves x_1. Therefore, $\mathrm{in}_\preceq(k[\mathbf{x}]^D) = k[x_2, \ldots, x_n]$. □

Assume that D is a triangular derivation on $k[\mathbf{x}]$ with $N_D = 2$. Then, $D(x_p)$, $D(x_q) \neq 0$ for some $1 \leq p < q \leq n$ and $D(x_i) = 0$ for any $i \neq p, q$. We set $s = x_p/D(x_p)$. Then, D extends uniquely to a locally nilpotent derivation on $k[\mathbf{x}][s]$. Write $\Psi_s(x_q) = h/h'$, where $h, h' \in k[\mathbf{x}]$ with $\gcd(h, h') = 1$.

Corollary 3.5. *Assume that D is a triangular derivation on $k[\mathbf{x}]$. If there exist $1 \leq p < q \leq n$ such that $D(x_p)$, $D(x_q) \neq 0$ and $D(x_i) = 0$ for any $i \neq p, q$, then*

$$\{x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_{q-1}, x_{q+1}, \ldots, x_n, h\} \tag{3.11}$$

is a universal SAGBI basis for $k[\mathbf{x}]^D$.

Proof. We set $k[\mathbf{x}'] = k[\{x_i \mid i \neq p, q\}]$. Then, $k[\mathbf{x}]^D \supset k[\mathbf{x}']$. Since $\Psi_s(x_p) = \Psi_s(s) = 0$, we have

$$k[\mathbf{x}]^D = k[\mathbf{x}][s]^D \cap k[\mathbf{x}] = \Psi_s(k[\mathbf{x}][s]) \cap k[\mathbf{x}] = k[\mathbf{x}'] \left[h/h'\right] \cap k[\mathbf{x}]$$

by Lemma 3.3. Note that h' is in $k[x_1, \ldots, x_{p-1}]$. Actually, $D(x_p)$ is in $k[x_1, \ldots, x_{p-1}]$ and h/h' is an irreducible fraction in $k[\mathbf{x}][x_p/D(x_p)]$. Since $D(h/h') = 0$, this implies that h is in $k[\mathbf{x}]^D$. We show that $k[\mathbf{x}]^D = k[\mathbf{x}'][h]$. Clearly, $k[\mathbf{x}]^D \supset k[\mathbf{x}'][h]$. Suppose that there exists $f \in k[\mathbf{x}]^D \setminus k[\mathbf{x}'][h]$. Then, we may write $f = f_0(h/h')^r + f_1(h/h')^{r-1} + \cdots + f_r$, where $f_i \in k[\mathbf{x}'][h]$ for each i. Assume that r is the minimum among such expressions. Then, r is positive. Moreover, h' does not divide f_0. Actually, if h' divides f_0, then $f_0 h/h' + f_1$ is in $k[\mathbf{x}'][h]$. Since $f = (f_0 h/h' + f_1)(h/h')^{r-1} + \cdots + f_r$, this contradicts the minimality of r. Thus, $f' = f_0 h^r + f_1 h^{r-1} h' + \cdots + f_r (h')^r$ is not divisible by h'. This contradicts that $f = f'/(h')^r$ is in $k[\mathbf{x}]$. Therefore, $k[\mathbf{x}]^D = k[\mathbf{x}'][h]$.

Now, we show that $\mathrm{in}_\preceq(k[\mathbf{x}]^D) = k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$ for any $\preceq \in \Omega$. It suffices to verify that $\mathrm{in}_\preceq(k[\mathbf{x}]^D) \subset k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$. Assume that f is in $k[\mathbf{x}]^D$. Then, $f = f_0 h^r + f_1 h^{r-1} + \cdots + f_r$ for some r and $f_j \in k[\mathbf{x}']$ for each j. We set a_i to be the i-th component of $v_\preceq(h)$ for $i = p, q$. Then, either a_p or a_q is not zero, since each monomial of h involves x_p or x_q. For $i = p, q$ and j with $f_j \neq 0$, the i-th component of $v_\preceq(f_j h^{r-j})$ is $(r - j)a_i$. Hence, $v_\preceq(f_i h^{r-i}) \neq v_\preceq(f_j h^{r-j})$ for any $i \neq j$ with $f_i, f_j \neq 0$. This implies that $\mathrm{in}_\preceq(f) = \mathrm{in}_\preceq(f_i h^{r-i})$ for some i. Since $\mathrm{in}_\preceq(f_i h^{r-i})$ is in $k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$, we have $\mathrm{in}_\preceq(f) \in k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$. Thus, $\mathrm{in}_\preceq(k[\mathbf{x}]^D) \subset k[\mathbf{x}'][\mathrm{in}_\preceq(h)]$. □

We will show Theorem 3.2 as a consequence of Theorem 3.6 below. Let M be a submodule of $\mathbf{Z}^n \times \mathbf{Z}^{m+1}$ of rank two which is not contained in

$$L = \{(a, (b_0, b_1, \ldots, b_m)) \in \mathbf{Z}^n \times \mathbf{Z}^{m+1} \mid b_0 = 0\}. \tag{3.12}$$

Let $\Psi\colon k[\mathbf{x}][y_1, \ldots, y_m] \to k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ be any homomorphism of $k[\mathbf{x}]$-algebras satisfying

$$\Psi(y_i) - y_i \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]y_0 \quad \text{and} \quad \mathrm{supp}(y_i^{-1}\Psi(y_i)) \subset M \quad (i = 1, \ldots, m). \tag{3.13}$$

Let $\Phi\colon k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] \to k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]$ be the homomorphism which substitutes zero for y_0. We consider the k-subalgebra

$$A = \Psi(k[\mathbf{x}][y_1, \dots, y_m]) \cap k[\mathbf{x}][\mathbf{y}] \tag{3.14}$$

of $k[\mathbf{x}][\mathbf{y}]$. Put $F_i = \rho(\Psi(y_i))$ for $i = 1, \dots, m$. Take $\overline{\eta} = (\overline{\eta}', \overline{\eta}'') \in \mathbf{Z}^n \times \mathbf{Z}^{m+1}$ such that $M \cap L = \mathbf{Z}\overline{\eta}$. Set $\overline{\eta}_1''$ to be the vector obtained from $\overline{\eta}''$ by replacing the negative components by zero and $\overline{\eta}_2'' = \overline{\eta}_1'' - \overline{\eta}''$. Define $\tilde{H}(\beta) = \mathbf{x}^{\overline{\eta}'}\Psi(\mathbf{y}^{\overline{\eta}_1''}) - \beta\Psi(\mathbf{y}^{\overline{\eta}_2''})$ and $H(\beta) = \rho(\tilde{H}(\beta))$ for each $\beta \in \overline{k}$. Then, there exist a finite number of elements $\mu_0, \mu_1, \dots, \mu_r \in k \setminus \{0\}$ such that

(i) $\mathrm{New}(\tilde{H}(\mu_i)) \neq \mathrm{New}(\tilde{H}(\mu_j))$ if $i \neq j$.

(ii) $\mathrm{New}(\tilde{H}(\mu_0))$ contains $\mathrm{supp}(\mathbf{x}^{\overline{\eta}'}\Psi(\mathbf{y}^{\overline{\eta}_1''}))$ and $\mathrm{supp}(\Psi(\mathbf{y}^{\overline{\eta}_2''}))$.

(iii) $\mathrm{New}(\tilde{H}(\beta)) = \mathrm{New}(\tilde{H}(\mu_0))$ for all $\beta \in \overline{k} \setminus \{0, \mu_1, \dots, \mu_r\}$.

In the notation above, we have the following.

Theorem 3.6. *The set $\{x_1, \dots, x_n, F_1, \dots, F_m, H(\mu_1), \dots, H(\mu_r)\}$ is a universal SAGBI basis for A.*

Proof. Note that $\Psi(\Phi(f)) = f$ for $f \in \Psi(k[\mathbf{x}][y_1, \dots, y_m])$. We set $\Gamma = (\mathbf{Z}^n \times \mathbf{Z}^{m+1})/M$, and define a Γ-grading $k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] = \bigoplus_{\gamma\in\Gamma} k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$ similarly to that explained before Lemma 2.4. We show that $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, where $A_\gamma = A \cap k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$ for $\gamma \in \Gamma$. Clearly, A contains $\bigoplus_{\gamma\in\Gamma} A_\gamma$. To show the reverse inclusion, take any $f \in A$. Then, it is written as $f = \sum_\gamma f_\gamma$, where $f_\gamma \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$ for each γ. Since the supports of f_γ and $f_{\gamma'}$ do not intersect if $\gamma \neq \gamma'$, we have $f_\gamma \in k[\mathbf{x}][\mathbf{y}]$ for each γ. Moreover, it follows that $f_\gamma = \Psi(\Phi(f_\gamma))$ for each γ, since $f = \Psi(\Phi(f)) = \sum_{\gamma\in\Gamma} \Psi(\Phi(f_\gamma))$ and $\Psi(\Phi(f_\gamma)) \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]_\gamma$. Hence, f_γ is in A_γ for each γ, and so f is in $\bigoplus_{\gamma\in\Gamma} A_\gamma$. Therefore, $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$.

Now, take any $\preceq \in \Omega$, and define S to be the subsemigroup of $\mathbf{Z}^n \times \mathbf{Z}^{m+1}$ generated by $(\mathbf{Z}_{\geq 0})^n \times \{0\}$, $v_\preceq(F_i)$ for $i = 1, \dots, m$ and $v_\preceq(H(\mu_i))$ for $i = 1, \dots, r$. To complete the proof, it suffices to show that $v_\preceq(f)$ is in S for any Γ-homogeneous element $f \in A \setminus \{0\}$. First, we show that $v_\preceq(H(\mu))$ is in S for any $\mu \in \overline{k} \setminus \{0\}$. It is true if $\mu = \mu_i$ for some $i = \{1, \dots, r\}$. For $\mu \in \overline{k} \setminus \{0, \mu_1, \dots, \mu_r\}$, we have $\mathrm{New}(H(\mu)) = \mathrm{New}(H(\mu_0))$ by (iii). Hence, $v_\preceq(H(\mu)) = v_\preceq(H(\mu_0))$. So, we will verify that $v_\preceq(H(\mu_0))$ is in S. By (ii), we get $v_\preceq(\tilde{H}(\mu_0)) = v_\preceq(h_j)$ for some $j \in \{1, 2\}$, where $h_1 = \mathbf{x}^{\overline{\eta}'}\Psi(\mathbf{y}^{\overline{\eta}_1''})$ and $h_2 = \Psi(\mathbf{y}^{\overline{\eta}_2''})$. Since $H(\mu_0) = \mathbf{x}^{\mathbf{e}(\tilde{H}(\mu_0))}\tilde{H}(\mu_0)$ and $h_j = \mathbf{x}^{-\mathbf{e}(h_j)}\rho(h_j)$, we have

$$v_\preceq(H(\mu_0)) = \big(\mathbf{e}(\tilde{H}(\mu_0)) - \mathbf{e}(h_j), 0\big) + v_\preceq(\rho(h_j)).$$

The condition (ii) also implies that $\mathbf{e}(\tilde{H}(\beta_j)) - \mathbf{e}(h_i)$ is in $(\mathbf{Z}_{\geq 0})^n$. Since $\rho(h_j)$ is a product of powers of $F_1, \dots, F_m$, we have $v_\preceq(\rho(h_j)) \in S$. Thus, $v_\preceq(H(\mu_0))$ is in S.

Now, let f be a Γ-homogeneous element of $A \setminus \{0\}$. Then, there exist $a \in \mathbf{Z}^n$,

$b_1, \dots, b_m$, $l \in \mathbf{Z}_{\geq 0}$ and $\alpha_i \in k$ for $i = 0, \dots, l$ with α_0, $\alpha_l \neq 0$ such that

$$\Phi(f) = \mathbf{x}^a y_1^{b_1} \cdots y_m^{b_m} \mathbf{y}^{l\overline{\eta}_2''} \sum_{i=0}^{l} \alpha_i (\mathbf{x}^{\overline{\eta}'} \mathbf{y}^{\overline{\eta}_1''} \mathbf{y}^{-\overline{\eta}_2''})^i \tag{3.15}$$

by the following reason. Since f is Γ-homogeneous, every c, $d \in \operatorname{supp}(\Phi(f))$ satisfy $c - d \in \mathbf{Z}\overline{\eta}$. Hence, $\Phi(f) = \mathbf{x}^a \mathbf{y}^{b'} \sum_{i=0}^{l} \alpha_i (\mathbf{x}^{\overline{\eta}'} \mathbf{y}^{\overline{\eta}_1'' - \overline{\eta}_2''})^i$ for some $a \in \mathbf{Z}^n$, $b' \in \mathbf{Z}^{m+1}$, $l \in \mathbf{Z}_{\geq 0}$ and $\alpha_i \in k$ for $i = 0, \dots, l$ with α_0, $\alpha_l \neq 0$. Since $\Phi(f)$ is in $k[\mathbf{x}][y_1, \dots, y_m]$, the first component of b' is zero and b', $b' + l(\overline{\eta}_1'' - \overline{\eta}_2'')$ are in $(\mathbf{Z}_{\geq 0})^{m+1}$. This last condition implies $b' - l\overline{\eta}_2'' \in (\mathbf{Z}_{\geq 0})^{m+1}$. Set $b_i \in \mathbf{Z}_{\geq 0}$ such that $b' - l\overline{\eta}_2'' = (b_0, b_1, \dots, b_m)$. Then, we get (3.15). Let $\beta_1, \dots, \beta_l \in \overline{k}$ be the solutions of the equation $\sum_{i=0}^{l} \alpha_i X^i = 0$ in X. Since α_0, $\alpha_l \neq 0$, we have $\beta_i \neq 0$ for any i. Then, we may write (3.15) as

$$\begin{aligned} \Phi(f) &= \alpha_0 \mathbf{x}^a y_1^{b_1} \cdots y_m^{b_m} \mathbf{y}^{l\overline{\eta}_2''} \prod_{i=1}^{l} (\mathbf{x}^{\overline{\eta}'} \mathbf{y}^{\overline{\eta}_1'' - \overline{\eta}_2''} - \beta_i) \\ &= \alpha_0 \mathbf{x}^a y_1^{b_1} \cdots y_m^{b_m} \prod_{i=1}^{l} (\mathbf{x}^{\overline{\eta}'} \mathbf{y}^{\overline{\eta}_1''} - \beta_i \mathbf{y}^{\overline{\eta}_2''}). \end{aligned}$$

Since $f = \Psi(\Phi(f))$, it follows that

$$\begin{aligned} f &= \alpha_0 \mathbf{x}^a \Psi(y_1)^{b_1} \cdots \Psi(y_m)^{b_m} \prod_{i=1}^{l} (\mathbf{x}^{\overline{\eta}'} \Psi(\mathbf{y}^{\overline{\eta}_1''}) - \beta_i \Psi(\mathbf{y}^{\overline{\eta}_2''})) \\ &= \alpha_0 \mathbf{x}^{a'} \left(\prod_{j=1}^{m} F_j^{b_j} \right) \left(\prod_{i=1}^{l} H(\beta_i) \right), \end{aligned}$$

where $a' = a - \sum_{j=1}^{m} b_j \mathbf{e}(\Psi(y_j)) - \sum_{i=1}^{l} \mathbf{e}(\tilde{H}(\beta_i))$. Hence, we have

$$v_{\preceq}(f) = (a', 0) + \sum_{j=1}^{m} b_j v_{\preceq}(F_j) + \sum_{i=1}^{l} v_{\preceq}(H(\beta_i)).$$

Clearly, $\sum_{j=1}^{m} b_j v_{\preceq}(F_j)$ is in S. As we showed in the preceding paragraph, $v_{\preceq}(H(\beta_i))$ is in S for each i. We show that $(a', 0)$ is in S. Suppose the contrary, that is, the j-th component of a' is negative for some j. Then, $(\prod_{j=1}^{m} F_j^{b_j})(\prod_{i=1}^{l} H(\beta_i))$ is divisible by x_j, since f is in $k[\mathbf{x}][\mathbf{y}]$. However, x_j does not divide $\rho(g)$ for any $g \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}] \setminus \{0\}$ by definition. This is a contradiction. Hence, $(a', 0)$ is in S. Therefore, $v_{\preceq}(f)$ is in S. This completes the proof. □

To prove Theorem 3.2, we need the following two lemmas. Assume that D is a k-derivation on $k[\mathbf{x}][\mathbf{y}]$ as in (3.2). Then, $s = y_0/(\kappa_0 \mathbf{x}^{\delta_0})$ is a slice of D. We set M

to be the submodule of $\mathbf{Z}^n \times \mathbf{Z}^3$ generated by $\big(\epsilon_{1,0}, (u_1, -1, 0)\big)$ and $\big(\epsilon_{2,0}, (u_2, v, -1)\big)$. Then, $M \cap L = \mathbf{Z}\big(\eta, (0, w, -u_1')\big)$.

Lemma 3.7. (i) $k[\mathbf{x}][\mathbf{y}]^D = \Psi_s(k[\mathbf{x}][y_1, y_2]) \cap k[\mathbf{x}][\mathbf{y}]$.
(ii) *The map* $k[\mathbf{x}][y_1, y_2] \ni f \mapsto \Psi_s(f) \in k[\mathbf{x}][\mathbf{y}][s]^D$ *is an isomorphism. Its inverse is* $k[\mathbf{x}][\mathbf{y}][s]^D \ni f \mapsto \Phi(f) \in k[\mathbf{x}][y_1, y_2]$.
(iii) $\Psi_s(y_1) = \tilde{F}$ *and* $\Psi_s(y_2) = \tilde{G}$.
(iv) $\Psi_s(y_i) - y_i \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]y_0$ *and* $\operatorname{supp}(y_i^{-1}\Psi_s(y_i)) \subset M$ *for* $i = 1, 2$.

Proof. (i) By Lemma 3.3, we get $k[\mathbf{x}][\mathbf{y}][s]^D = \Psi_s(k[\mathbf{x}][\mathbf{y}][s])$. Since $\Psi_s(y_0) = \Psi_s(s) = 0$, it is equal to $\Psi_s(k[\mathbf{x}][y_1, y_2])$. Therefore,

$$k[\mathbf{x}][\mathbf{y}]^D = k[\mathbf{x}][\mathbf{y}][s]^D \cap k[\mathbf{x}][\mathbf{y}] = \Psi_s(k[\mathbf{x}][y_1, y_2]) \cap k[\mathbf{x}][\mathbf{y}].$$

(ii) For $f \in k[\mathbf{x}][y_1, y_2]$, we have $\Psi_s(f) = f - s\sum_{p=1}^{\infty}(-s)^{p-1}D^p(f)/p!$. Hence, $\Phi(\Psi_s(f)) = f$. Moreover, $\Psi_s(k[\mathbf{x}][y_0, y_1]) = k[\mathbf{x}][\mathbf{y}][s]^D$ by Lemma 3.3.
(iii) Note that $\tilde{F}$, $\tilde{G}$ are in $k[\mathbf{x}][\mathbf{y}][s]^D$. Since $\Phi(\tilde{F}) = y_1$ and $\Phi(\tilde{G}) = y_2$, we have $\Psi_s(y_1) = \tilde{F}$ and $\Psi_s(y_2) = \tilde{G}$ by (ii).
(iv) Since $\tilde{F} - y_1$, $\tilde{G} - y_2 \in k[\mathbf{x}, \mathbf{x}^{-1}][\mathbf{y}]y_0$ and $\operatorname{supp}(y_1^{-1}\tilde{F})$, $\operatorname{supp}(y_2^{-1}\tilde{G}) \subset M$, the assertion follows from (iii). ☐

We set $\overline{\eta}_1'' = w$ and $\overline{\eta}_2'' = u_1'$. Then, define $\tilde{H}(\beta) = \mathbf{x}^{\eta}\Psi_s(y_1)^{\overline{\eta}_1''} - \beta\Psi_s(y_2)^{\overline{\eta}_2''}$ and $H(\beta) = \rho(\tilde{H}(\beta))$ for each $\beta \in \overline{k} \setminus \{0\}$. If $\xi \neq 0$, then put

$$\mu_1 = (-1)^{w+u_1'}\frac{\kappa_1^w}{(\kappa_0 u_1)^w \xi^{u_1'}}, \tag{3.16}$$

and set μ_0 to be any element of $k \setminus \{0, \mu_1\}$. If $\xi = 0$, then set μ_0 to be any element of $k \setminus \{0\}$.

Lemma 3.8. *Assume that* $\xi \neq 0$. *Then, we have*
(i) $\operatorname{New}(\tilde{H}(\mu_0)) \neq \operatorname{New}(\tilde{H}(\mu_1))$.
(ii) $\operatorname{New}(\tilde{H}(\mu_0))$ *contains* $\operatorname{supp}(\mathbf{x}^{\eta}\Psi(y_1)^{\overline{\eta}_1''})$ *and* $\operatorname{supp}(\Psi(y_2)^{\overline{\eta}_2''})$.
(iii) $\operatorname{New}(\tilde{H}(\beta)) = \operatorname{New}(\tilde{H}(\mu_0))$ *for all* $\beta \in \overline{k} \setminus \{0, \mu_1\}$.
Assume that $\xi = 0$. *Then, we have*
(iv) $\operatorname{New}(\tilde{H}(\mu_0))$ *contains* $\operatorname{supp}(\mathbf{x}^{\eta}\Psi(y_1)^{\overline{\eta}_1''})$ *and* $\operatorname{supp}(\Psi(y_2)^{\overline{\eta}_2''})$.
(v) $\operatorname{New}(\tilde{H}(\beta)) = \operatorname{New}(\tilde{H}(\mu_0))$ *for all* $\beta \in \overline{k} \setminus \{0\}$.

Proof. Note that $\mathbf{x}^{\eta}\Psi_s(y_1)^{\overline{\eta}_1''} = \mathbf{x}^{\eta}\tilde{F}^w$ and $\Psi_s(y_2)^{\overline{\eta}_2''} = \tilde{G}^{u_1'}$ by Lemma 3.7 (iii). Assume that $\xi \neq 0$. Then, the sets of the vertices of $\operatorname{New}(\mathbf{x}^{\eta}\Psi_s(y_1)^{\overline{\eta}_1''})$ and $\operatorname{New}(\Psi_s(y_2)^{\overline{\eta}_2''})$ are $\{a_1, a_2\}$ and $\{b_1, b_2, b_3\}$, respectively. Here, we set

$$a_1 = \big(\eta, (0, w, 0)\big) \qquad b_1 = \big(0, (0, 0, u_1')\big)$$

$$a_2 = \big(\eta + w\epsilon_{1,0}, (u_1 w, 0, 0)\big) \qquad b_2 = \big(u_1'(\epsilon_{2,0} + v\epsilon_{1,0}), (u_1'(u_1 v + u_2), 0, 0)\big)$$
$$b_3 = \big(u_1'\epsilon_{2,0}, (u_1'u_2, u_1'v, 0)\big).$$

Note that $a_2 = b_2$, since

$$\eta + w\epsilon_{1,0} = (u_1'\epsilon_{2,0} - u_2'\epsilon_{1,0}) + (u_1'v + u_2')\epsilon_{1,0} = u_1'(\epsilon_{2,0} + v\epsilon_{1,0})$$

and $u_1 w = u_1'(u_1 v + u_2)$. We show that a_1, a_2, b_1 and b_3 are in $\mathrm{New}(\tilde{H}(\beta))$ for any $\beta \in \overline{k} \setminus \{0, \mu_1\}$, and a_2 is not in $\mathrm{New}(\tilde{H}(\mu_1))$. The assertions (i), (ii) and (iii) follow from this. Take any $\beta \in \overline{k} \setminus \{0, \mu_1\}$. Then, a_1, b_1 are in $\mathrm{New}(\tilde{H}(\beta))$, since $a_1 \notin \mathrm{supp}(\Psi_s(y_2)^{\overline{\eta}_2''})$ and $b_1 \notin \mathrm{supp}(\mathbf{x}^{\eta}\Psi_s(y_1)^{\overline{\eta}_1''})$. The coefficients of $\mathbf{x}^{\eta+w\epsilon_{1,0}} y_0^{u_1 w}$ in $\mathbf{x}^{\eta}\Psi_s(y_1)^{\overline{\eta}_1''}$ and $\Psi_s(y_2)^{\overline{\eta}_2''}$ are $(-\kappa_1/(\kappa_0 u_1))^w$ and $(-\xi)^{u_1'}$, respectively. Hence, a_2 is in $\mathrm{supp}(\tilde{H}(\beta'))$ if and only if $\beta' = \mu_1$ for $\beta' \in \overline{k}$. So, a_2 is in $\mathrm{New}(\tilde{H}(\beta))$. Since $b_3 = (1 - u_2'/w)a_1 + (u_2'/w)a_2$, we get $b_3 \in \mathrm{New}(\tilde{H}(\beta))$. Therefore, a_1, a_2, b_1 and b_3 are in $\mathrm{New}(\tilde{H}(\beta))$ for any $\beta \in \overline{k} \setminus \{0, \mu_1\}$. If $v > 0$, then a_1, b_1 and b_3 are not equal to a_2, while $b_3 = a_2$ if $v = 0$. In each case, the first component $u_1 w$ of the second factor of a_2 is greater than the first component of the second factor of any element of $\{a_1, b_1, b_3\} \setminus \{a_2\}$. Hence, $u_1 w$ is greater than that of any element but a_2 of the convex hull of $\{a_1, a_2, b_1, b_3\}$ in $\mathbf{R}^n$. Since $\mathrm{supp}(\tilde{H}(\mu_1))$ is contained in this convex set and $a_2 \notin \mathrm{supp}(\tilde{H}(\mu_1))$, we conclude that a_2 is not in $\mathrm{New}(\tilde{H}(\mu_1))$. Therefore, the lemma is true when $\xi \neq 0$.

Assume that $\xi = 0$. Then, the coefficient of $\mathbf{x}^{\eta+w\epsilon_{1,0}} y_0^{u_1 w}$ in $\Psi_s(y_2)^{\overline{\eta}_2''}$ is zero, while that in $\mathbf{x}^{\eta}\Psi_s(y_1)^{\overline{\eta}_1''}$ is not zero. Hence, a_2 is in $\mathrm{supp}(\tilde{H}(\beta))$ for any $\beta \in \overline{k}$. In a similar way as above, we see that a_1, b_1 and b_3 are also in $\mathrm{New}(\tilde{H}(\beta))$. This implies (iv) and (v). We have thus proved the lemma. □

Let us complete the proof of Theorem 3.2. Assume that $\xi \neq 0$. Then, by Theorem 3.6, Lemma 3.7 (i), (iv) and Lemma 3.8, the set

$$\{x_1, \ldots, x_n, \rho(\Psi_s(y_1)), \rho(\Psi_s(y_2)), H(\mu_1)\} \tag{3.17}$$

is a universal SAGBI basis for $k[\mathbf{x}][\mathbf{y}]^D$. Since $\Psi_s(y_1) = \tilde{F}$ and $\Psi_s(y_2) = \tilde{G}$ by Lemma 3.7 (iii), we have $\rho(\Psi_s(y_1)) = F$ and $\rho(\Psi_s(y_2)) = G$. Moreover, $H(\mu_1) = H$ by definition. Thus, the theorem is true if $\xi \neq 0$. Similarly, we see that $\{x_1, \ldots, x_n, F, G\}$ is a universal SAGBI basis for $k[\mathbf{x}][\mathbf{y}]^D$ if $\xi = 0$. Therefore, the proof of Theorem 3.2 is completed.

Now, Theorem 1.2 is a consequence of the results above. Actually, the theorem follows from what we mentioned before Lemma 3.1, Corollary 3.5 and Theorem 3.2. In each case, we described the universal SAGBI basis explicitly.

In [15], Maubach studied the kernel of a triangular derivation D on $k[\mathbf{x}]$ for $n = 4$ such that $D(x_i)$ is a monomial multiplied by an element of k for each i. He showed that $k[\mathbf{x}]^D$ is generated by at most four elements by giving them explicitly. As a consequence of our result, we know further a SAGBI basis for $k[\mathbf{x}]^D$.

Corollary 3.9. *Assume that* $n = 4$. *Let* D *be a triangular derivation on* $k[\mathbf{x}]$ *such that* $D(x_i) = \kappa_i x_i \mathbf{x}^{d_i}$ *for some* $\kappa_i \in k$ *and* $d_i \in \mathbf{Z}^4$ *for each* i. *If* $\kappa_i = 0$ *for some* i, *then* $k[\mathbf{x}]^D$ *has a universal SAGBI basis with at most four elements. If* $\kappa_i \neq 0$ *for all* i, *then* $\{\Psi_s(x_2), \Psi_s(x_3), \Psi_s(x_4)\}$ *is a SAGBI basis for* $\preceq \in \Omega$ *with* $d_i \prec d_1$ *for* $i = 2, 3, 4$, *where* $s = x_1/D(x_1)$. *In particular, it is a SAGBI basis for the lexicographic order on* $k[\mathbf{x}]$ *with* $x_1 \prec \cdots \prec x_4$.

Proof. The former part follows from Theorem 1.2. Assume that $D(x_i) \neq 0$ for any i. Then, the condition that $d_1 \prec d_i$ for $i = 2, 3, 4$ implies that $x_i = \mathrm{in}_\preceq(\Psi_s(x_i))$ for $i = 2, 3, 4$. Actually, $\mathrm{supp}(x_i^{-1}\Psi_s(x_i))$ is contained in $\sum_{i=1}^{3} \mathbf{R}_{\geq 0}(d_i - d_1)$ for each i. Hence, $\{\Psi_s(x_2), \Psi_s(x_3), \Psi_s(x_4)\}$ is a SAGBI basis for $\preceq$ by Corollary 3.4. Since D is triangular, the lexicographic order as above satisfies that $x_i = \mathrm{in}_\preceq(\Psi_s(x_i))$ for each i, as we noted after Corollary 3.4. $\square$

Assume that $m = 2$, and consider the k-derivation on $k[\mathbf{x}][\mathbf{y}]$ of the form

$$D = \mathbf{x}^{\delta_0}\frac{\partial}{\partial y_0} + \mathbf{x}^{\delta_1}\frac{\partial}{\partial y_1} + \mathbf{x}^{\delta_2}\frac{\partial}{\partial y_2} \quad (\delta_0, \delta_1, \delta_2 \in (\mathbf{Z}_{\geq 0})^n). \tag{3.18}$$

For each i, j, we define $\epsilon_{i,j}^+$ to be the vector obtained from $\epsilon_{i,j} = \delta_i - \delta_j$ by replacing the negative components by zero, and set $L_{i,j} = \mathbf{x}^{\epsilon_{j,i}^+} y_i - \mathbf{x}^{\epsilon_{i,j}^+} y_j$. Khoury [9, Corollary 2.2] showed that $k[\mathbf{x}][\mathbf{y}]^D$ is generated by $L_{1,0}$, $L_{2,0}$ and $L_{2,1}$ over $k[\mathbf{x}]$. As a consequence of Theorem 3.2, we have further the following.

Corollary 3.10. *Assume that* $m = 2$ *and* D *is a* k*-derivation on* $k[\mathbf{x}][\mathbf{y}]$ *as in* (3.18). *Then,* $\{x_1, \ldots, x_n, L_{1,0}, L_{2,0}, L_{2,1}\}$ *is a universal SAGBI basis for* $k[\mathbf{x}][\mathbf{y}]^D$.

Proof. Note that (3.18) is a special case of (3.2) where $\kappa_0 = \kappa_1 = \kappa_{2,0} = 1$, $u_1 = u_2 = 1$, and $v = 0$. In this case, $\tilde{F} = y_1 - \mathbf{x}^{\epsilon_{1,0}} y_0$, $\tilde{G} = y_2 - \mathbf{x}^{\epsilon_{2,0}} y_0$, and $\eta = \delta_2 - \delta_1 = \epsilon_{2,1}$. Since $\epsilon_{2,1} + \epsilon_{1,0} = \epsilon_{2,0}$, we have

$$\tilde{H} = \mathbf{x}^{\epsilon_{2,1}}(y_1 - \mathbf{x}^{\epsilon_{1,0}} y_0) - (y_2 - \mathbf{x}^{\epsilon_{2,0}} y_0) = \mathbf{x}^{\epsilon_{2,1}} y_1 - y_2.$$

For i, j, it follows that $\rho(y_i - \mathbf{x}^{\epsilon_{i,j}} y_j) = \mathbf{x}^{\epsilon_{j,i}^+} y_i - \mathbf{x}^{\epsilon_{i,j}^+} y_j$. Therefore, the assertion follows from Theorem 3.2. $\square$

4. The number of initial algebras

In this section, we prove Theorem 1.3.

First, assume that the dimension of $\mathrm{supp}^\circ(D)$ is two. Let $v_1, \ldots, v_p$ be the vertices of the convex hull of $\mathrm{supp}^\circ(D)$ in $\mathbf{R}^n$, and $H = M_{D^\circ} \otimes_{\mathbf{Z}} \mathbf{R}$, where $D^\circ = D_{\mathrm{supp}^\circ(D)}$. For each i, we set $\lambda_{v_i} = \lambda_i$, $L_i = H \cap (\ker \lambda_i) \otimes_{\mathbf{Z}} \mathbf{R}$ and $l_i = \dim_{\mathbf{R}} L_i$. By the definition of $\mathrm{supp}^\circ(D)$, there exists $\delta \in \mathrm{supp}^\circ(D)$ such that $\delta - v_i \not\in \ker \lambda_i$. Hence, l_i is at most one. So, there exists $\eta_i \in \mathbf{R}^n$ such that $L_i = \mathbf{R}\eta_i$. For each i and $0 \leq j \leq l_i$, we define $\Omega_{i,j}$ to be the set of $\preceq \in \Omega$ such that $\delta \preceq v_i$ for any $\delta \in \operatorname{supp}^\circ(D)$, $0 \preceq \eta_i$ if $j = 0$, and $\eta_i \prec 0$ otherwise. Then, $\Omega = \coprod_{i=1}^{p} \coprod_{j=1}^{l_i} \Omega_{i,j}$. Recall the Γ-grading on $k[\mathbf{x}]^D$ defined in the proof of Theorem 1.1. We will show that $v_{\preceq_1}(f) = v_{\preceq_2}(f)$ for any Γ-homogeneous element $f \in k[\mathbf{x}]^D \setminus \{0\}$ and $\preceq_1, \preceq_2 \in \Omega_{i,j}$ for i, j. This implies that $\operatorname{in}_{\preceq_1}(k[\mathbf{x}]^D) = \operatorname{in}_{\preceq_2}(k[\mathbf{x}]^D)$ for any $\preceq_1, \preceq_2 \in \Omega_{i,j}$ for i, j, so the number of the initial algebras of $k[\mathbf{x}]^D$ is at most $2p$.

By Lemma 2.4 and the definition of Γ-grading, f is a Γ_{D°-homogeneous element of $k[\mathbf{x}]^{D^\circ}$. Hence, $\operatorname{supp}(f)$ is contained in $\{v_{\preceq_e}(f)\} + H$ for $e = 1, 2$. We set $S = \operatorname{supp}(f) \cap \ker \lambda_i$. Then, $v_{\preceq_e}(f)$ is in S by Proposition 2.3. Moreover,

$$S \subset (\{v_{\preceq_e}(f)\} + H) \cap \ker \lambda_i \subset \{v_{\preceq_e}(f)\} + L_i \qquad (e = 1, 2). \tag{4.1}$$

If $l_i = 0$, then $S = \{v_{\preceq_e}(f)\}$ for each e by (4.1). Hence, $v_{\preceq_1}(f) = v_{\preceq_2}(f)$. Assume that $l_i = 1$. If $j = 0$, then S is contained in $\{v_{\preceq_e}(f)\} + \mathbf{R}_{\geq 0}(-\eta_i)$ for each e by (4.1), since $0 \prec_e \eta_i$. This implies that $v_{\preceq_1}(f) = v_{\preceq_2}(f)$. Similarly, we get this equality when $j = 1$. Therefore, the theorem is true if the dimension of $\operatorname{supp}^\circ(D)$ is two.

Now, assume that the dimension of $\operatorname{supp}^\circ(D)$ is one. Then, there exists $\eta \in \mathbf{R}^n \setminus \{0\}$ such that $H = \mathbf{R}\eta$. Let Ω_0 and Ω_1 be the sets of $\preceq \in \Omega$ such that $0 \prec \eta$ and $\eta \prec 0$, respectively. Then, $\Omega = \Omega_0 \sqcup \Omega_1$. So, it suffices to show that $v_{\preceq_1}(f) = v_{\preceq_2}(f)$ for any Γ-homogeneous element $f \in k[\mathbf{x}]^D \setminus \{0\}$ and $\preceq_1, \preceq_2 \in \Omega_i$ for $i = 0, 1$. Similarly to the preceding case, this equality follows from $\operatorname{supp}(f) \subset \{v_{\preceq_e}(f)\} + H$ for $e = 1, 2$. We have thus proved Theorem 1.3.

Note that, if the dimension of $\operatorname{supp}^\circ(D)$ is -1, then $k[\mathbf{x}]^D = \operatorname{in}_{\preceq}(k[\mathbf{x}]^D)$ for any $\preceq \in \Omega$, since $k[\mathbf{x}]^D$ is generated by monomials. Thus, together with Theorem 1.3, we get an upper bound for the number of the initial algebras of $k[\mathbf{x}]^D$ in the case where the dimension of $\operatorname{supp}^\circ(D)$ is at most two.

For any k-subalgebra A of $k[\mathbf{x}]$, the cardinality of $\{\operatorname{in}_{\preceq}(A) \mid \preceq \in \Omega_0\}$ is finite if $\operatorname{in}_{\preceq}(A)$ is finitely generated for each $\preceq \in \Omega_0$ by [11, Lemma 1.7 and Proposition 1.8]. Hence, we can also deduce from Theorem 1.1 that, if the dimension of $\operatorname{supp}^\circ(D)$ is at most two, then $k[\mathbf{x}]^D$ has only finitely many initial algebras for Ω_0.

5. A finitely generated G_a-invariant ring without finite universal SAGBI bases

We showed in [11, Theorem 2.2] that the invariant subring of a polynomial ring for certain action of a finite group does not have finitely generated initial algebras for any $\preceq \in \Omega$. However, it seems unknown whether there exists an invariant subring of a polynomial ring for an action of a connected affine algebraic group which is finitely generated but has infinitely generated initial algebras. In this section, we give an example of a locally nilpotent derivation on a polynomial ring which has a finitely generated kernel with both finitely generated and infinitely generated initial algebras. Since the kernel of a locally nilpotent derivation is equal to a G_a-invariant subring, this implies that a finitely generated invariant subring of a polynomial ring for an action of a connected affine algebraic group can have infinitely generated initial algebras.

Let D be a locally nilpotent derivation on $k[\mathbf{x}]$, and s an indeterminate over $k[\mathbf{x}]$. We define a k-derivation $\tilde{D}$ on $k[\mathbf{x}][s]$ by $\tilde{D}(x_i) = D(x_i)$ for $i = 1, \dots, n$ and $\tilde{D}(s) = -1$. Then, $\tilde{D}$ is locally nilpotent, and $-s$ is a slice of $\tilde{D}$. Hence, $k[\mathbf{x}][s]^{\tilde{D}}$ is generated by

$$\Psi_{-s}(x_i) = \sum_{p=0}^{\infty} \frac{s^p}{p!} D^p(x_i) \quad (i = 1, \dots, n) \tag{5.1}$$

over k by Lemma 3.3. Let $\preceq_1$ be an elimination order on $k[\mathbf{x}][s]$ with respect to s, i.e., a monomial order on $k[\mathbf{x}][s]$ such that $\mathrm{in}_{\preceq_1}(f) \in k[\mathbf{x}]$ implies $f \in k[\mathbf{x}]$ for each $f \in k[\mathbf{x}][s]$, and $\preceq_2$ a monomial order on $k[\mathbf{x}][s]$ such that $\mathrm{in}_{\preceq_2}(\Psi_{-s}(x_i)) = x_i$ for $i = 1, \dots, n$. An example of $\preceq_1$ is the lexicographic order on $k[\mathbf{x}][s]$ with $x_1 \prec_1 \cdots \prec_1 x_n \prec_1 s$. If the locally nilpotent derivation D is triangular, then the lexicographic order on $k[\mathbf{x}][s]$ with $s \prec_2 x_1 \prec_2 \cdots \prec_2 x_n$ satisfies $\mathrm{in}_{\preceq_2}(\Psi_{-s}(x_i)) = x_i$ for $i = 1, \dots, n$, as mentioned after Corollary 3.4.

Theorem 5.1. *Assume that D is a locally nilpotent derivation on $k[\mathbf{x}]$ whose kernel $k[\mathbf{x}]^D$ is not finitely generated over k. Then, $\mathrm{in}_{\preceq_1}(k[\mathbf{x}][s]^{\tilde{D}})$ is not finitely generated, while $\mathrm{in}_{\preceq_2}(k[\mathbf{x}][s]^{\tilde{D}}) = k[\mathbf{x}]$.*

To show Theorem 5.1, we use Vasconcelos' method [25, Section 7.4] of computing a generating set for a G_a-invariant subring of a polynomial ring using SAGBI bases as follows (see also [23]). Let $\sigma\colon k[\mathbf{x}] \to k[\mathbf{x}][s]$ be the G_a-action on $k[\mathbf{x}]$ defined by the locally nilpotent derivation D. We set $A = k[\sigma(x_1), \dots, \sigma(x_n)]$. Then, we have $k[\mathbf{x}]^D = k[\mathbf{x}]^{G_a} = A \cap k[\mathbf{x}]$. Assume that $\mathcal{S}'$ is a SAGBI basis for A with respect to $\preceq_1$. We set $\mathcal{S} = \{f \in \mathcal{S}' \mid \mathrm{in}_{\preceq_1}(f) \in k[\mathbf{x}]\}$. Then, since $\preceq_1$ is an elimination order, $\mathcal{S}$ is a SAGBI basis for $k[\mathbf{x}]^D$ with respect to $\preceq_1$. In particular, $\mathcal{S}$ is a generating set for $k[\mathbf{x}]^D$.

Now, we prove Theorem 5.1. First, we show that $\mathrm{in}_{\preceq_1}(k[\mathbf{x}][s]^{\tilde{D}})$ is not finitely generated. Since $\sigma(x_i) = \Phi_{-s}(x_i)$ for each i, we have $A = k[\mathbf{x}][s]^{\tilde{D}}$. Suppose that $\mathrm{in}_{\preceq_1}(k[\mathbf{x}][s]^{\tilde{D}})$ is finitely generated. Then, A has a finite SAGBI basis $\mathcal{S}'$ for $\preceq_1$. Hence, the cardinality of the set $\mathcal{S}$ of $f \in \mathcal{S}'$ such that $\mathrm{in}_{\preceq_1}(f) \in k[\mathbf{x}]$ is finite. This contradicts that $k[\mathbf{x}]^D$ is not finitely generated, since $\mathcal{S}$ generates $k[\mathbf{x}]^D$ over k. Thus, $\mathrm{in}_{\preceq_1}(k[\mathbf{x}][s]^{\tilde{D}})$ is not finitely generated. The equality $\mathrm{in}_{\preceq_2}(k[\mathbf{x}][s]^{\tilde{D}}) = k[\mathbf{x}]$ follows from Corollary 3.4. Therefore, Theorem 5.1 is proved.

Various triangular derivations with infinitely generated kernels have been constructed as counterexamples to the fourteenth problem of Hilbert (cf. [1], [6], [10], [13]). Hence, there actually exists a finitely generated G_a-invariant subring of a polynomial ring which does not have finite universal SAGBI basis by Theorem 5.1.

6. Construction of the kernel of a derivation

If D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$, then its kernel is expressed as

$$k[\mathbf{x}]^D = k[\Psi_s(x_1), \ldots, \Psi_s(x_n)] \cap k[\mathbf{x}] \tag{6.1}$$

for some $s \in k(\mathbf{x})$. Actually, for $g \in k[\mathbf{x}] \setminus k[\mathbf{x}]^D$, there exists $l \geq 1$ such that $D^l(g) \neq 0$ and $D^{l+1}(g) = 0$. Since $D(s) = 1$ for $s = D^{l-1}(g)/D^l(g)$, we get (6.1) by Lemma 3.3. However, if D is not locally nilpotent, then it is generally hard to describe its kernel. In this section, we investigate a method for doing this concretely.

Throughout this section, let k be a field of an arbitrary characteristic, and $\preceq$ an element of Ω. Consider the product $\prod_{a \in \mathbf{Z}^n} k\mathbf{x}^a$ of one-dimensional k-vector spaces $k\mathbf{x}^a$ for $a \in \mathbf{Z}^n$. It contains $k[\mathbf{x}, \mathbf{x}^{-1}]$ naturally. We define the support of each element of $\prod_{a \in \mathbf{Z}^n} k\mathbf{x}^a$ as in (1.1), which can be an infinite set. Let $k\langle\langle \mathbf{x}, \preceq \rangle\rangle$ denote the set of $f \in \prod_{a \in \mathbf{Z}^n} k\mathbf{x}^a$ such that $\operatorname{supp}(f)$ is reverse well-ordered, i.e., every subset of $\operatorname{supp}(f)$ has the maximum for $\preceq$. For each $f \in k\langle\langle \mathbf{x}, \preceq \rangle\rangle$, we define $v_{\preceq}(f)$ and $\operatorname{in}_{\preceq}(f)$ as in the case where f is a polynomial. We claim that the k-vector space $k\langle\langle \mathbf{x}, \preceq \rangle\rangle$ is a field with multiplication defined by

$$\left(\sum_{a \in \mathbf{Z}^n} \mu_a \mathbf{x}^a \right) \left(\sum_{b \in \mathbf{Z}^n} \nu_b \mathbf{x}^b \right) = \sum_{c \in \mathbf{Z}^n} \left(\sum_{a+b=c} \mu_a \nu_b \right) \mathbf{x}^c. \tag{6.2}$$

Before proving this, we notice some properties of reverse well-ordered sets.

Lemma 6.1. (i) *A subset of $\mathbf{Z}^n$ is reverse well-ordered if and only if it does not contain any infinite ascending chain.*
(ii) *A subset of a reverse well-ordered set is reverse well-ordered. The union of two reverse well-ordered sets is reverse well-ordered.*
(iii) *If S_1, $S_2 \subset \mathbf{Z}^n$ are reverse well-ordered, then $S_1 + S_2$ is reverse well-ordered. Moreover, the number of $(a_1, a_2) \in S_1 \times S_2$ such that $a_1 + a_2 = b$ is finite for each $b \in \mathbf{Z}^n$.*
(iv) *Assume that S is a reverse well-ordered subset of $\mathbf{Z}^n$ such that $a \prec 0$ for every $a \in S$. Then, $\bigcup_{i=0}^{\infty} iS$ is reverse well-ordered. Moreover, the number of $i \in \mathbf{Z}_{\geq 0}$ such that $a \in iS$ is finite for each $a \in \mathbf{Z}^n$.*

Proof. (i) and (ii) are clear. We show (iii) and (iv).

Suppose that $S_1 + S_2$ is not reverse well-ordered. Then, there exists an infinite ascending chain $(b_i)_i \subset S_1 + S_2$ such that $b_i = a_{1,i} + a_{2,i}$ with $a_{j,i} \in S_j$ for each i, j. Note that $a_{j,i} \prec a_{j,i+1}$ for some $j \in \{1, 2\}$ for each i. Hence, $(a_{1,i})_i$ or $(a_{2,i})_i$ contains an infinite ascending chain. This contradicts that S_1 and S_2 are reverse well-ordered. Thus, $S_1 + S_2$ is reverse well-ordered.

Suppose that there exist $b \in \mathbf{Z}^n$ and an infinite number of $(a_1, a_2) \in S_1 \times S_2$ such that $a_1 + a_2 = b$. Then, we may find an infinite descending chain $(a_{1,i})_i \subset S_1$

such that $a_{1,i} + a_{2,i} = b$ for some $a_{2,i} \in S_2$ for each i, since S_1 is reverse well-ordered. However, $(a_{2,i})_i$ is an infinite ascending chain of S_2. This contradicts that S_2 is reverse well-ordered. Therefore, (iii) is proved.

By [21, Theorem 2.5], there exist $1 \leq r \leq n$ and $\omega_1, \ldots, \omega_r \in \mathbf{R}^n$ such that $a \preceq b$ if and only if $\omega_j \cdot a < \omega_j \cdot b$ for the last j with $\omega_j \cdot a \neq \omega_j \cdot b$ for every a, $b \in \mathbf{Z}^n$. Suppose that $\bigcup_{i=0}^{\infty} iS$ is not reverse well-ordered. Then, there exist an integer $1 \leq s \leq r$ and an infinite ascending chain $(a_i)_{i=1}^{\infty} \subset \mathbf{Z}^n$ with $a_i = \sum_{j=1}^{l_i} a_{i,j}$ for some $l_i \in \mathbf{N}$ and $a_{i,j} \in S$ such that $\omega_t \cdot a_{i,j} = 0$ for any $s < t \leq r$ and $1 \leq j \leq l_i$ for each i. Actually, r satisfies this property for any infinite ascending chain of $\bigcup_{i=0}^{\infty} iS$. Take such s and $(a_i)_i$ so that s is the minimum among those. Since $a_{i,j} \prec 0$ and $\omega_t \cdot a_{i,j} = 0$ for every $s < t \leq r$, we have $\omega_s \cdot a_{i,j} \leq 0$ for any i, j. So, for each i, we assume that $\omega_s \cdot a_{i,j} < 0$ for $1 \leq j < m_i$ and $\omega_s \cdot a_{i,j} = 0$ for $m_i \leq j \leq l_i$ for some m_i. Since $a_i \prec a_{i+1}$ and $\omega_t \cdot a_i = \omega_t \cdot a_{i+1}$ for every $s < t \leq r$, we have $\omega_s \cdot a_i \leq \omega_s \cdot a_{i+1}$ for each i. On the other hand, $\omega_s \cdot a_i \leq -m_i \eta$ for each i, where $\eta = \min(\{|\omega_s \cdot a| \mid a \in \mathbf{Z}^n\} \setminus \{0\})$. Hence, there exists $m \in \mathbf{N}$ such that $m_i \leq m$ for each i. Put $a'_i = \sum_{j=1}^{m_i-1} a_{i,j}$ for each i. Then, $(a'_i)_i \subset \bigcup_{i=0}^{m} iS$. By (ii) and (iii), $\bigcup_{i=0}^{m} iS$ is reverse well-ordered. Hence, $(a'_i)_i$ does not contain any infinite ascending chain. This implies the existence of a subsequence $(b_i)_i$ of $(a'_i)_i$ with $b_{i+1} \preceq b_i$ for every i. By replacing $(a_i)_i$ with its subsequence, we may assume that $a'_{i+1} \preceq a'_i$ for every i. Put $a''_i = a_i - a'_i$ for each i. Then, $(a''_i)_i$ is an infinite ascending chain of $\mathbf{Z}^n$. Moreover, $a''_i = \sum_{j=m_i}^{l_i} a_{i,j}$ with $\omega_t \cdot a_{i,j} = 0$ for every $s - 1 < t \leq r$ and i, j. This contradicts the minimality of s. Therefore, $\bigcup_{i=0}^{\infty} iS$ is reverse well-ordered.

Suppose that there exist $a \in \mathbf{Z}^n$ and $(l_i)_{i=1}^{\infty} \subset \mathbf{N}$ with $l_i < l_{i+1}$ such that $a = \sum_{j=1}^{l_i} a_{i,j}$ for some $a_{i,j} \in S$ for each i. We claim that $\{a_{i,j} \mid i, j\}$ is an infinite set. Suppose the contrary. Then, there exists $\omega \in \mathbf{R}^n$ such that $\omega \cdot a_{i,j} < 0$ for any i, j, since $a_{i,j} \prec 0$. Then, $\omega \cdot a \leq l_i \eta'$ for each i, where $\eta' = \max\{\omega \cdot a_{i,j} \mid i, j\} < 0$. This is a contradiction, since $l_i \eta' < \omega \cdot a$ for sufficiently large i. Thus, $\{a_{i,j} \mid i, j\}$ is an infinite set. By replacing $(l_i)_i$ with its subsequence, we may assume that, for each i, there exists $1 \leq p_i \leq l_i$ such that $a_{i,p_i} \neq a_{i',j}$ for any $i' < i$ and $1 \leq j \leq l_{i'}$. Since S is reverse well-ordered, we may assume that $a_{i,p_i} \prec a_{i+1,p_{i+1}}$ for every i by replacing $(a_{i,p_i})_i$ with its subsequence. Put $b_i = a - a_{i,p_i}$ for each i. Then, $(b_i)_i$ is an infinite ascending chain. Since $b_i = \sum_{j \neq p_i} a_{i,j} \in (l_i - 1)S$, this contradicts that $\bigcup_{i=0}^{\infty} iS$ is reverse well-ordered. Thus, the number of i such that $a \in iS$ is finite for each $a \in \mathbf{Z}^n$. Therefore, (iv) is proved. □

Now, we verify that $k\langle\langle \mathbf{x}, \preceq \rangle\rangle$ is a field. By Lemma 6.1 (iii), we see easily that multiplication (6.2) is well-defined. We show that the inverse element of $f \neq 0$ is given by

$$\frac{1}{f} = \frac{1}{\mathrm{in}_{\preceq}(f)} \sum_{i=0}^{\infty} \left(1 - \frac{f}{\mathrm{in}_{\preceq}(f)}\right)^i . \tag{6.3}$$

Without loss of generality, we may assume that $\mathrm{in}_{\preceq}(f) = 1$. Put $S = \mathrm{supp}(1-f)$. Then, S is reverse well-ordered, and $a \prec 0$ for every $a \in S$. By Lemma 6.1 (iv), the number of i such that $a \in iS$ is finite for each $a \in \mathbf{Z}^n$, and $\bigcup_{i=0}^{\infty} iS$ is reverse well-ordered. Hence, $\sum_{i=0}^{\infty}(1-f)^i$ is in $k\langle\langle \mathbf{x}, \preceq\rangle\rangle$. Note that

$$f\sum_{i=0}^{\infty}(1-f)^i - 1 = f\sum_{i=N}^{\infty}(1-f)^i + f\sum_{i=0}^{N-1}(1-f)^i - 1 = f\sum_{i=N}^{\infty}(1-f)^i - (1-f)^N \tag{6.4}$$

for any $N > 0$. The support of the right hand side of (6.4) does not contain each $a \in \mathbf{Z}^n$ for sufficiently large N. Hence, (6.4) is zero. Thus, $f\sum_{i=0}^{\infty}(1-f)^i = 1$.

For example, if $\preceq$ is an element of Ω such that $a \preceq b$ if the last nonzero component of $b - a$ is negative for $a, b \in \mathbf{Z}^n$, then $k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ is equal to the field $k((x_1))\cdots((x_n))$ of multi-Laurent series.

Now, let D be a k-derivation on $k[\mathbf{x}]$, and δ_0 the maximum of $\mathrm{supp}(D)$ for $\preceq$. Since $k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ is transcendental over $k(\mathbf{x})$, we may extend D to a k-derivation on $k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ im many ways. We define an extension by

$$D\left(\sum_{a\in\mathbf{Z}^n}\mu_a\mathbf{x}^a\right) = \sum_{a\in\mathbf{Z}^n}\mu_a D(x^a). \tag{6.5}$$

Then, similarly to Proposition 2.3, $\upsilon_{\preceq}(f)$ is in $\ker\lambda_{\delta_0}$ for any $f \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle^D \setminus \{0\}$. Let $k\langle\langle \mathbf{x}, \preceq\rangle\rangle_{\delta_0}$ denote the set of $f \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ such that $\mathrm{supp}(f) \subset \ker\lambda_{\delta_0}$. It is a subfield of $k\langle\langle \mathbf{x}, \preceq\rangle\rangle$. We define a k-linear map $\phi_{\delta_0} : k\langle\langle \mathbf{x}, \preceq\rangle\rangle^D \to k\langle\langle \mathbf{x}, \preceq\rangle\rangle_{\delta_0}$ by

$$\sum_{a\in\mathbf{Z}^n}\mu_a\mathbf{x}^a \mapsto \sum_{a\in\mathbf{Z}^n}\nu_a\mathbf{x}^a, \quad \text{where} \quad \nu_a = \begin{cases}\mu_a & \text{if } \lambda_\delta(a) = 0\\ 0 & \text{otherwise.}\end{cases} \tag{6.6}$$

Then, ϕ_{δ_0} has the following property.

Proposition 6.2. *The k-linear map ϕ_{δ_0} is injective. Moreover,* $\mathrm{in}_{\preceq}(f) = \mathrm{in}_{\preceq}(\phi_{\delta_0}(f))$ *for each* $f \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle^D$.

Proof. Suppose that there exists $f \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle^D \setminus \{0\}$ such that $\phi_{\delta_0}(f) = 0$. Then, $\mathrm{supp}(f) \cap \ker\lambda_{\delta_0} = \emptyset$. This contradicts that $\upsilon_{\preceq}(f)$ is in $\ker\lambda_{\delta_0}$. Hence, ϕ_{δ_0} is injective. The rest of the assertion follows from the definitions of ϕ_{δ_0} and $\mathrm{in}_{\preceq}(f)$. ☐

We construct the inverse of $k\langle\langle \mathbf{x}, \preceq\rangle\rangle^D \ni f \mapsto \phi_{\delta_0}(f) \in \phi_{\delta_0}(k\langle\langle \mathbf{x}, \preceq\rangle\rangle^D)$ concretely. Set $S = \mathrm{supp}(D)$ and $S' = S \setminus \{\delta_0\}$. For each $\delta \in S'$, put $\epsilon_\delta = \delta - \delta_0$, and define a linear operator $E_\delta \in \mathrm{End}_k(k[\mathbf{x}, \mathbf{x}^{-1}])$ by

$$E_\delta(\mathbf{x}^a) = \frac{\lambda_\delta(a)}{\lambda_{\delta_0}(a+\epsilon_\delta)}\mathbf{x}^{a+\epsilon_\delta} \tag{6.7}$$

if $\lambda_{\delta_0}(a+\epsilon_\delta) \neq 0$, and $E_\delta(\mathbf{x}^a) = 0$ otherwise for $a \in \mathbf{Z}^n$. Set $E = \sum_{\delta \in S'} E_\delta$. Then, define $\psi_{\delta_0} \in \mathrm{End}_k(k\langle\langle \mathbf{x}, \preceq\rangle\rangle)$ by

$$\psi_{\delta_0}(\mathbf{x}^a) = \sum_{i=0}^{\infty}(-E)^i(\mathbf{x}^a) \tag{6.8}$$

for $a \in \mathbf{Z}^n$, and $\psi_{\delta_0}(g) = \sum_{a\in\mathbf{Z}^n} \mu_a \psi_{\delta_0}(\mathbf{x}^a)$ for $g = \sum_{a\in\mathbf{Z}^n} \mu_a \mathbf{x}^a \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle$. Since $\mathrm{supp}(E^i(\mathbf{x}^a)) \subset iS' + \{a\}$ for each i, we have $\psi_{\delta_0}(\mathbf{x}^a) \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ by Lemma 6.1 (iv). Since $\bigcup_{i=0}^{\infty} iS'$ and $\mathrm{supp}(g)$ are reverse well-ordered, $\psi_{\delta_0}(g)$ is in $k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ by Lemma 6.1 (iii).

Theorem 6.3. *It follows that $\psi_{\delta_0}(\phi_{\delta_0}(f)) = f$ for each $f \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle^D$.*

To show Theorem 6.3, we need the following lemma. Take any $g \in k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ and $a \in \mathbf{Z}^n \setminus \mathrm{supp}(g)$. For each $\delta \in \mathrm{supp}(D)$, we put $a_\delta = a - \epsilon_\delta$, and set u_δ to be the coefficient of $\mathbf{x}^{a_\delta}$ in $\psi_{\delta_0}(g)$.

Lemma 6.4. *In the notation above, it follows that*

$$u_{\delta_0}\mathbf{x}^a = -\sum_{\delta\in S'} u_\delta E_\delta(\mathbf{x}^{a_\delta}). \tag{6.9}$$

For $b \in \ker\lambda_{\delta_0}$, the coefficient of $\mathbf{x}^b$ in $\psi_{\delta_0}(g)$ is equal to that in g.

Proof. First, we show the last statement. Let β and β' be the coefficients of $\mathbf{x}^b$ in g and $\psi_{\delta_0}(g)$, respectively. Suppose that $\beta \neq \beta'$. Then, b is in $\mathrm{supp}(E^i(\mathbf{x}^c))$ for some $i > 0$ and $c \in \mathrm{supp}(g)$. Hence, there exists $\delta \in S'$ such that $E_\delta(\mathbf{x}^{b-\epsilon_\delta}) \neq 0$. This contradicts that $\lambda_{\delta_0}((b-\epsilon_\delta)+\epsilon_\delta) = \lambda_{\delta_0}(b) = 0$. Thus, $\beta = \beta'$.

Now, we verify (6.9). We may assume $g = \mathbf{x}^c$ for some $c \in \mathbf{Z}^n$ by the following reason. By Lemma 6.1 (iii), the number of $c \in \mathrm{supp}(g)$ such that $a_\delta \in \mathrm{supp}(\psi_{\delta_0}(\mathbf{x}^c))$ is finite for each $\delta \in S$. Actually, $a_\delta \in \mathrm{supp}(\psi_{\delta_0}(\mathbf{x}^c))$ implies that $s + c = a_\delta$ for some $s \in \bigcup_{i=0}^{\infty} iS'$. Hence, we may replace g by an element of $k[\mathbf{x}, \mathbf{x}^{-1}]$, say $g = \sum_{i=1}^m w_i \mathbf{x}^{c_i}$. Let $u_{\delta,i}$ be the coefficient of $\mathbf{x}^{a_\delta}$ in $\psi_{\delta_0}(\mathbf{x}^{c_i})$ for each δ and i. Then, $w_i u_{\delta_0,i}\mathbf{x}^a = -w_i \sum_{\delta\in S'} u_{\delta,i} E_\delta(\mathbf{x}^{a_\delta})$ by assumption. By adding each side of this equality for $i = 1, \ldots, m$, we get (6.9).

Let Σ be the set of sequences $(\delta_i)_{i=1}^r \subset S'$ such that $r \in \mathbf{Z}_{\geq 0}$ and $c + \sum_{i=1}^r \epsilon_{\delta_i} = a$, and Σ_δ the set of $(\delta_i)_{i=1}^r \in \Sigma$ such that $\delta_r = \delta$ for each $\delta \in S'$. Then, it follows that

$$u_{\delta_0}\mathbf{x}^a = \sum_{(\delta_i)_{i=1}^r \in \Sigma} (-E_{\delta_r}) \circ \cdots \circ (-E_{\delta_1})(\mathbf{x}^c)$$

and

$$u_\delta \mathbf{x}^{a_\delta} = \sum_{(\delta_i)_{i=1}^r \in \Sigma_\delta} (-E_{\delta_{r-1}}) \circ \cdots \circ (-E_{\delta_1})(\mathbf{x}^c)$$

for each $\delta \in S'$. Hence, we have

$$\begin{aligned} u_{\delta_0}\mathbf{x}^a &= \sum_{(\delta_i)_{i=1}^r \in \Sigma} (-E_{\delta_r}) \circ \cdots \circ (-E_{\delta_1})(\mathbf{x}^c) \\ &= -\sum_{\delta \in S'} E_\delta \left(\sum_{(\delta_i)_{i=1}^r \in \Sigma_\delta} (-E_{\delta_{r-1}}) \circ \cdots \circ (-E_{\delta_1})(\mathbf{x}^c) \right) \\ &= -\sum_{\delta \in S'} u_\delta E_\delta(\mathbf{x}^{a_\delta}). \end{aligned}$$

Therefore, the lemma is proved. □

Proof of Theorem 6.3. Take any $f \in k\langle\langle \mathbf{x}, \preceq \rangle\rangle^D \setminus \{0\}$, and put $h = \psi_{\delta_0}(\phi_{\delta_0}(f)) - f$. We show that $h = 0$. Suppose that $h \neq 0$. We set $a = v_\preceq(h)$ and, for each $\delta \in S$, put $a_\delta = a - \epsilon_\delta$ and let u_δ and u'_δ be the coefficients of $\mathbf{x}^{a_\delta}$ in f and $\psi_{\delta_0}(\phi_{\delta_0}(f))$, respectively. Then, $u_{\delta_0} \neq u'_{\delta_0}$, since a is in supp(h). Moreover, $\lambda_{\delta_0}(a) \neq 0$. Actually, if $\lambda_{\delta_0}(a) = 0$, then u'_{δ_0} is equal to the coefficient of $\mathbf{x}^a$ in $\phi_{\delta_0}(f)$ by Lemma 6.4. However, it is equal to that in f. This contradicts that $u_{\delta_0} \neq u'_{\delta_0}$. Hence, $\lambda_{\delta_0}(a) \neq 0$.

The coefficient of $\mathbf{x}^{a+\delta_0}$ in $D(f)$ is $\sum_{\delta \in S} u_\delta \lambda_\delta(a_\delta)$ by (2.5) and (6.5). It is equal to zero, since $D(f) = 0$. Hence, we get

$$u_{\delta_0} = -\sum_{\delta \in S'} u_\delta \frac{\lambda_\delta(a_\delta)}{\lambda_{\delta_0}(a)}. \tag{6.10}$$

Since $\operatorname{supp}(\phi_{\delta_0}(f)) \subset \ker \lambda_{\delta_0}$, we have $a \notin \operatorname{supp}(\phi_{\delta_0}(f))$. Hence,

$$u'_{\delta_0}\mathbf{x}^a = -\sum_{\delta \in S'} u'_\delta E_\delta(\mathbf{x}^{a_\delta}) = -\sum_{\delta \in S'} u'_\delta \frac{\lambda_\delta(a_\delta)}{\lambda_{\delta_0}(a)} \mathbf{x}^a \tag{6.11}$$

by Lemma 6.4. We have $a_\delta \notin \operatorname{supp}(h)$ for $\delta \in S'$, since $a \prec a_\delta$ and $a = v_\preceq(h)$. So, $u_\delta = u'_\delta$ for $\delta \in S'$. Thus, we get $u_{\delta_0} = u'_{\delta_0}$ by (6.10) and (6.11). This is a contradiction. Therefore, $\psi_{\delta_0}(\phi_{\delta_0}(f)) = f$. □

Lemma 6.5. *Assume that g is in $k\langle\langle \mathbf{x}, \preceq \rangle\rangle_{\delta_0}$. If $\lambda_{\delta_0}(a + \epsilon_\delta) \neq 0$ for each $a \in \operatorname{supp}(\psi_{\delta_0}(g)) \setminus \ker \lambda_\delta$ and $\delta \in S'$, then $\psi_{\delta_0}(g)$ is in $k\langle\langle \mathbf{x}, \preceq \rangle\rangle^D$.*

Proof. Suppose that $D(\psi_{\delta_0}(g)) \neq 0$. We put $b = v_\preceq(D(\psi_{\delta_0}(g)))$ and, for each $\delta \in S$, set $a_\delta = b - \delta$ and u_δ to be the coefficient of $\mathbf{x}^{a_\delta}$ in $\psi_{\delta_0}(g)$. Then, the coefficient of $\mathbf{x}^b$ in $D(\psi_{\delta_0}(g))$ is equal to $w = \sum_{\delta \in S} u_\delta \lambda_\delta(a_\delta) \neq 0$. First, assume that $\lambda_{\delta_0}(a_{\delta_0}) \neq 0$. Then, $a_{\delta_0} \in \mathbf{Z}^n \setminus \mathrm{supp}(g)$, since $\mathrm{supp}(g) \subset \ker \lambda_{\delta_0}$ by assumption. Note that $a_\delta = a_{\delta_0} - \epsilon_\delta$ for each $\delta \in S'$. Hence, we get

$$u_{\delta_0} \mathbf{x}^{a_{\delta_0}} = -\sum_{\delta \in S'} u_\delta E_\delta(\mathbf{x}^{a_\delta}) = -\sum_{\delta \in S'} u_\delta \frac{\lambda_\delta(a_\delta)}{\lambda_{\delta_0}(a_{\delta_0})} \mathbf{x}^{a_{\delta_0}}$$

by Lemma 6.4. This contradicts that $w \neq 0$. Now, assume that $\lambda_{\delta_0}(a_{\delta_0}) = 0$. We show that $u_\delta \lambda_\delta(a_\delta) = 0$ for each $\delta \in S'$. Suppose that $u_\delta \lambda_\delta(a_\delta) \neq 0$ for some $\delta \in S'$. Then, a_δ is in $\mathrm{supp}(\psi_{\delta_0}(g)) \setminus \ker \lambda_\delta$. However, $\lambda_{\delta_0}(a_\delta + \epsilon_\delta) = \lambda_{\delta_0}(a_{\delta_0}) = 0$. This contradicts the assumption. Hence, $u_\delta \lambda_\delta(a_\delta) = 0$ for $\delta \in S'$, and so $w = 0$. This is a contradiction. Therefore, $D(\psi_{\delta_0}(g)) = 0$. □

We set $k[\mathbf{x}]_{\delta_0} = k[\{\mathbf{x}^a \mid a \in (\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_{\delta_0}\}]$. Then, there exist a finite number of elements $v_1, \ldots, v_r \in (\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_{\delta_0}$ such that $k[\mathbf{x}]_{\delta_0} = k[\mathbf{x}^{v_1}, \ldots, \mathbf{x}^{v_r}]$. Actually, the semigroup $(\mathbf{Z}_{\geq 0})^n \cap \ker \lambda_{\delta_0}$ is finitely generated by Gordan's lemma [19, Proposition 1.1.(ii)]. We set

$$\mathcal{C} = \sum_{\delta \in S'} \mathbf{R}_{\geq 0} \epsilon_\delta \quad \text{and} \quad \mathcal{F} = \mathcal{C} \cap (\ker \lambda_{\delta_0}) \otimes_{\mathbf{Z}} \mathbf{R}. \tag{6.12}$$

Note that $\mathrm{supp}(\phi_{\delta_0}(g))$ is contained in $\mathcal{C} + \mathrm{supp}(g)$ for each g.

For a convex set $C \subset \mathbf{R}^n$, a subset $F \subset C$ is called a *face* of C if there exists $\omega \in \mathbf{R}^n$ such that

$$F = \{a \in C \mid \omega \cdot b \leq \omega \cdot a \text{ for all } b \in C\}. \tag{6.13}$$

Theorem 6.6. *Assume that $\mathcal{F}$ is a face of $\mathcal{C}$, and $\ker \lambda_{\delta_0} \subset \ker \lambda_\delta$ for each $\delta \in S'$ with $\lambda_{\delta_0}(\epsilon_\delta) = 0$. Then, $\phi_{\delta_0} : k\langle\langle \mathbf{x}, \preceq \rangle\rangle^D \to k\langle\langle \mathbf{x}, \preceq \rangle\rangle_{\delta_0}$ is an isomorphism of fields. In particular, we have*

$$k[\mathbf{x}]^D = k[\psi_{\delta_0}(\mathbf{x}^{v_1}), \ldots, \psi_{\delta_0}(\mathbf{x}^{v_r})] \cap k[\mathbf{x}]. \tag{6.14}$$

Proof. It suffices to show that $D(\psi_{\delta_0}(g)) = 0$ and $\psi_{\delta_0}(g_1 g_2) = \psi_{\delta_0}(g_1)\psi_{\delta_0}(g_2)$ for any g, g_1, $g_2 \in k\langle\langle \mathbf{x}, \preceq \rangle\rangle_{\delta_0}$.

Take any $a \in \mathrm{supp}(\psi_{\delta_0}(g))$ and $\delta \in S'$ such that $\lambda_{\delta_0}(a + \epsilon_\delta) = 0$. We show that $\lambda_\delta(a) = 0$. Then, $D(\psi_{\delta_0}(g)) = 0$ follows from Lemma 6.5. Note that $a = a' + b$ for some $a' \in \mathrm{supp}(g)$ and $b \in \mathcal{C}$. Since $\lambda_{\delta_0}(a') = 0$, we have $\lambda_{\delta_0}(b + \epsilon_\delta) = \lambda_{\delta_0}(a + \epsilon_\delta) = 0$. On the other hand, $b + \epsilon_\delta \in \mathcal{C}$, since b, $\epsilon_\delta \in \mathcal{C}$. Hence, $b + \epsilon_\delta \in \mathcal{F}$. This implies that b, $\epsilon_\delta \in \mathcal{F}$, since $\mathcal{F}$ is a face of $\mathcal{C}$. So, we have $\lambda_{\delta_0}(\epsilon_\delta) = 0$. Hence, $\lambda_{\delta_0}(a) = \lambda_{\delta_0}(a + \epsilon_\delta) = 0$ and, by assumption, $\ker \lambda_{\delta_0} \subset \ker \lambda_\delta$. Thus, $\lambda_\delta(a) = 0$. Therefore, we get $D(\psi_{\delta_0}(g)) = 0$.

Now, put $f = \psi_{\delta_0}(g_1 g_2) - \psi_{\delta_0}(g_1)\psi_{\delta_0}(g_2)$, and suppose that $f \neq 0$. Since f is in $k\langle\langle \mathbf{x}, \preceq \rangle\rangle^D \setminus \{0\}$, we have $v_\preceq(f) \in \ker \lambda_{\delta_0}$ as mentioned before Proposition 6.2. We note that f is expressed as

$$(\psi_{\delta_0}(g_1g_2) - g_1g_2) - (\psi_{\delta_0}(g_1) - g_1)g_2 - g_1(\psi_{\delta_0}(g_2) - g_2) - (\psi_{\delta_0}(g_1) - g_1)(\psi_{\delta_0}(g_2) - g_2).$$

Hence, $v_{\preceq}(f)$ is contained in one of

$$\operatorname{supp}(\psi_{\delta_0}(g_1g_2) - g_1g_2),\ \ S_1 + \operatorname{supp}(g_2),\ \ \operatorname{supp}(g_1) + S_2,\ \ S_1 + S_2,$$

where $S_i = \operatorname{supp}(\psi_{\delta_0}(g_i) - g_i)$ for $i = 1, 2$. By the last statement of Lemma 6.4, $\operatorname{supp}(\psi_{\delta_0}(g_1g_2) - g_1g_2)$, S_1 and S_2 do not contain any element of $\ker\lambda_{\delta_0}$, since $\operatorname{supp}(g_i) \subset \ker\lambda_{\delta_0}$. The same is true for $\operatorname{supp}(g_i) + S_i$ for $i = 1, 2$. Thus, $v_{\preceq}(f)$ is in $S_1 + S_2$. Take $a_i \in S_i$ for $i = 1, 2$ such that $v_{\preceq}(f) = a_1 + a_2$. Each a_i is written as $b_i + c_i$ for some $b_i \in \operatorname{supp}(g_i)$ and $c_i \in \mathcal{C} \setminus \mathcal{F}$. Then, it follows that

$$0 = \lambda_{\delta_0}(v_{\preceq}(f)) = \lambda_{\delta_0}(b_1 + b_2 + c_1 + c_2) = \lambda_{\delta_0}(c_1 + c_2).$$

Hence, $c_1 + c_2$ is in $\mathcal{F}$. Since $c_1, c_2 \in \mathcal{C}$ and $\mathcal{F}$ is a face of $\mathcal{C}$, we get $c_1, c_2 \in \mathcal{F}$. This is a contradiction. Therefore, $f = 0$. □

We remark on the case where k is of characteristic zero and D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$. By Lemma 2.5, the i-th component of δ_0 is -1 for some i. Then, $\ker\lambda_{\delta_0}$ is equal to the set of elements of $\mathbf{Z}^n$ whose i-th components are zero. Hence, $k\langle\langle \mathbf{x}, \preceq\rangle\rangle_{\delta_0}$ is equal to the set of elements of $k\langle\langle \mathbf{x}, \preceq\rangle\rangle$ which do not involve x_i. Moreover, we have the following.

Lemma 6.7. *Assume that k is of characteristic zero and D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$. Then, $\mathcal{F}$ is a face of $\mathcal{C}$. Moreover, $\lambda_{\delta_0}(\epsilon_\delta) = 0$ implies that $\ker\lambda_\delta = \ker\lambda_{\delta_0}$ for each $\delta \in S$.*

Proof. By Lemma 2.5, the i-th component of δ_0 is -1 for some i. Then, the i-th component of ϵ_δ is nonnegative for each $\delta \in S$. So, for $a \in \mathcal{C}$, the i-th component of a is zero if and only if $-\mathbf{e}_i \cdot b \leq -\mathbf{e}_i \cdot a$ for all $b \in \mathcal{C}$. Hence, $\mathcal{F}$ is a face of $\mathcal{C}$. If $\lambda_{\delta_0}(\epsilon_\delta) = 0$ for $\delta \in S$, then the i-th component of δ is -1. This implies that $\ker\lambda_\delta = \ker\lambda_{\delta_0}$. □

By Lemma 6.7, the assumption in Theorem 6.6 is satisfied if k is of characteristic zero and D is a nonzero locally nilpotent derivation on $k[\mathbf{x}]$.

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Mathematical Institute
Tohoku University
Sendai 980-8578
Japan

Current address:
Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502
Japan
e-mail: kuroda@kurims.kyoto-u.ac.jp