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ON CERTAIN HARDY SUMS AND THEIR $2m$-TH POWER MEAN

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1. Introduction

For a positive integer k and an arbitrary integer h, the classical Dedekind sums $s(h,k)$ is defined by

$$s(h,k)=\sum_{a=1}^{k}\left(\left(\frac{a}{k}\right)\right)\left(\left(\frac{ah}{k}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\dfrac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases}$$

The sum $s(h,k)$ plays an important role in the transformation theory of the Dedekind η function; See the Chapter 3 of [1]. There is an extensive literature about the Dedekind sums. H. Rademacher [8] wrote an introductory book on the subject.

Perhaps the most famous property of the Dedekind sums is the reciprocity formula

$$s(h,k)+s(k,h)=\frac{h^2+k^2+1}{12hk}-\frac{1}{4}$$

for positive coprime integers h and k. Some three term versions of this formula were discovered by H. Rademacher [8], R.R. Hall, M.N. Huxley [5] and J. Pommersheim [7].

J.B. Conrey, E. Fransen, R. Klein and C. Scott [4] studied the mean value of Dedekind sums and proved the following proposition.

Proposition 1. *Suppose that m is a given positive integer and k is any sufficiently large integer. Then*

$$\sum_{h=1}^{k}{}' s^{2m}(h,k)=f_m(k)\left(\frac{k}{12}\right)^{2m}+O\left(\left(k^{9/5}+k^{2m-1+1/(m+1)}\right)\log^3 k\right),$$

where $\sum_h'$ denotes the summation over all h such that $(h,k)=1$, and $f_m(k)$ is defined

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by the Dirichlet series

$$\sum_{k=1}^{\infty} \frac{f_m(k)}{k^s} = 2 \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \cdot \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \cdot \zeta(s),$$

where $\zeta(s)$ *is the Riemann zeta-function.*

In [3], J. Chaohua improved the error terms in Proposition 1. H. Walum [10] showed that for prime k,

$$\sum_{\substack{\chi \bmod k \\ \chi(-1)=-1}} |L(1,\chi)|^4 = \frac{\pi^4(k-1)}{k^2} \sum_{h=1}^{k} |s(h,k)|^2 .$$

In the spirit of [4] and [10], the second author [11] used an estimate for character sums to prove the following:

Proposition 2. *Suppose that* p *is any sufficiently large prime number and* n *is any positive integer. Then for* $k = p^n$*, we have*

$$\sum_{h=1}^{k}{}' |s(h,k)|^2 = \frac{5}{144} \cdot \frac{(p^2-1)^2}{p(p^3-1)} \cdot k^2 + O\left(k \exp\left(\frac{3\log k}{\log\log k}\right)\right),$$

where $\exp(y) = e^y$ *and the constant implied in the* O*-symbol is absolute.*

Also some interesting relations between Dedekind sums and Hurwitz zeta-function were established (see references [12], [13], [14] and [16]).

B.C. Berndt [2] gave an analogous transformation formula for the logarithm of the classical theta function

$$\theta(z) = \sum_{n=-\infty}^{+\infty} \exp(\pi i n^2 z), \quad \operatorname{Im} z > 0,$$

and showed that for $V = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the theta group

$$\log\theta(Vz) = \log\theta(z) + \frac{1}{2}\log(cz+d) - \frac{1}{4}\pi i + \frac{1}{4}\pi i S(d,c),$$

where

$$S(d,c) = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]}.$$

The sums $S(d,c)$ (and certain related ones) are sometimes called Hardy sums. They are closely connected with Dedekind sums [9]. Some arithmetical properties of $S(d,c)$

can be found in B.C. Berndt [2] and R. Sitaramachandra Rao [9]. In [15], the second author studied the $2m$-th power mean of $S(d,c)$, and proved the following:

Proposition 3. *Let p be an odd prime and m be a positive integer, then*

$$\sum_{h=1}^{p-1} |S(h,p)|^{2m} = p^{2m} \frac{\zeta^2(2m)\left(1-1/4^m\right)}{\zeta(4m)\left(1+1/4^m\right)} + O\left(p^{2m-1} \exp\left(\frac{6\ln p}{\ln\ln p}\right)\right).$$

In this paper, we use the important works of J.B. Conrey et al. [4] and J. Chaohua [3] to study the $2m$-th power mean of $S(h,k)$, and give a sharp asymptotic formula for $\sum'^{k}_{h=1} S^{2m}(h,k)$. That is, we shall prove the following theorem.

Theorem. *For any fixed integer $m \geq 2$ and any sufficiently large integer k, we have the asymptotic formula*

$$\sum_{h=1}^{k}{}' S^{2m}(h,k) = g_m(k)k^{2m} + O\left(k^{2m-1}\right),$$

where $g_m(k)$ is defined by the Dirichlet series

$$\sum_{k=1}^{\infty} \frac{g_m(k)}{k^s} = \frac{2^s\left(2^{s+4m}-2\right)\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)^2\left(2^{2m}+1\right)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)}\zeta(s) + \frac{2^s\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s+2m)}.$$

2. Some lemmas

To prove the Theorem, we need following lemmas. First we have

Lemma 1. *For any given positive integer k and any integer h with $(h,k)=1$ and any $P>1$, there exist a positive integer $q \leq P$ and an integer a with $(a,q)=1$ such that*

$$\left|\frac{h}{k} - \frac{a}{q}\right| < \frac{1}{qP}.$$

Proof. This is a well-known result; See Theorem 36 of [6]. ☐

Lemma 2. *Let a, b, c, d, h and k be positive integers with $ad-bc=1$ and $(h,k)=1$. Let*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

then we have

$$s(a,c)+s(h,k)-s(x,y)=\frac{c^2+k^2+y^2}{12cky}-\frac{1}{4}.$$

Proof. This is equation (26) of [5]. □

Lemma 3. *Let h and k denote relatively prime integers with $k>0$, then*

$$S(h,k)=\begin{cases}4s(h,k)-8s(h+k,2k), & \text{if } h+k \text{ is odd};\\ 0, & \text{if } h+k \text{ is even}.\end{cases}$$

Proof. This formula is an immediate consequence of (5.9) and (5.10) in [9]. □

Lemma 4. *For any positive integer q, we have*

$$\sum_{a=1}^{q}{}' |s(a,q)| \ll q\log^2 q.$$

Proof. This is Lemma 6 of [4]. □

Lemma 5. *Let k,h,a and q be positive integers with $(h,k)=(a,q)=1$, and set $z=qh-ak$. If $h+k$ is odd and $1\le |z|\le k/q$, then we have*

$$S(h,k)=\begin{cases}-\dfrac{k}{qz}+O\left(|s(a,q)|+\left|s\left(\dfrac{a+q}{2},q\right)\right|+|z|\right), & \text{if } a+q \text{ is an even number};\\ O\left(q+|z|\right), & \text{if } a+q \text{ is an odd number}.\end{cases}$$

Proof. Suppose that $a+q$ is even. Since $(a,q)=1$, a and q must be odd numbers.

First We consider the case that $z<0$. Since $(a,q)=1$, there exist positive integers b and d such that

$$ad-bq=1,\quad 1\le d<q.$$

Let $f=2dh-2bk$. Then we have

$$\begin{pmatrix}2d & -2b\\ -q & a\end{pmatrix}\begin{pmatrix}h\\ k\end{pmatrix}=\begin{pmatrix}f\\ -z\end{pmatrix}$$

and

$$\begin{pmatrix} \frac{a+q}{2} & b+d \\ q & 2d \end{pmatrix} \begin{pmatrix} f \\ -z \end{pmatrix} = \begin{pmatrix} h+k \\ 2k \end{pmatrix}.$$

The fact that $d < q$ and $z \geq -k/q$ yields

$$f = 2kd\left(\frac{h}{k} - \frac{b}{d}\right) = 2kd\left(\frac{h}{k} - \frac{a}{q} + \frac{1}{qd}\right) = 2kd\left(\frac{z}{qk} + \frac{1}{qd}\right) = \frac{2kd}{q}\left(\frac{z}{k} + \frac{1}{d}\right) > 0.$$

On the other hand, since $(h, k) = 1$ and z is odd, we get $(f, -z) = 1$. Then by Lemma 2,

$$s\left(\frac{a+q}{2}, q\right) + s(f, -z) - s(h+k, 2k) = -\frac{4k^2 + q^2 + z^2}{24kqz} - \frac{1}{4}.$$

That is,

$$s(h+k, 2k) = \frac{k}{6qz} + O\left(\left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right).$$

From Lemma 8 of [4] we also have

$$s(h, k) = \frac{k}{12qz} + O\left(|s\left(a, q\right)| + |z|\right).$$

Therefore by Lemma 3 we immediately have

$$S(h, k) = -\frac{k}{qz} + O\left(|s(a, q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right), \quad \text{if } z < 0.$$

For $z > 0$, we can find positive integers b and d satisfying

$$ad - bq = -1, \quad 1 \leq d < q.$$

Let $f = 2bk - 2dh$. Then we have

$$\begin{pmatrix} 2b & -2d \\ -a & q \end{pmatrix} \begin{pmatrix} k \\ h \end{pmatrix} = \begin{pmatrix} f \\ z \end{pmatrix}$$

and

$$\begin{pmatrix} q & 2d \\ \frac{a+q}{2} & b+d \end{pmatrix} \begin{pmatrix} f \\ z \end{pmatrix} = \begin{pmatrix} 2k \\ h+k \end{pmatrix}.$$

Similarly we can get $(f, z) = 1$ and

$$f = 2kd\left(\frac{b}{d} - \frac{h}{k}\right) = 2kd\left(\frac{1}{qd} + \frac{a}{q} - \frac{h}{k}\right) = 2kd\left(\frac{1}{qd} - \frac{z}{qk}\right) = \frac{2kd}{q}\left(\frac{1}{d} - \frac{z}{k}\right) > 0.$$

Then by Lemma 2,

$$s\left(q, \frac{a+q}{2}\right) + s(f, z) - s(2k, h+k) = \frac{\left((a+q)/2\right)^2 + z^2 + (h+k)^2}{12\left((a+q)/2\right) \cdot z \cdot (h+k)} - \frac{1}{4}.$$

Noting that

$$s\left(q, \frac{a+q}{2}\right) + s\left(\frac{a+q}{2}, q\right) = \frac{\left((a+q)/2\right)^2 + q^2 + 1}{12\left((a+q)/2\right) \cdot q} - \frac{1}{4}$$

and

$$s(2k, h+k) + s(h+k, 2k) = \frac{(h+k)^2 + (2k)^2 + 1}{12\,(h+k) \cdot 2k} - \frac{1}{4},$$

we have

$$s(h+k, 2k) = \frac{k}{6qz} + O\left(\left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right).$$

So from Lemma 8 of [4] and Lemma 3 we immediately have

$$S(h,k) = -\frac{k}{qz} + O\left(|s(a,q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right), \quad \text{for } z > 0.$$

This proves that

$$S(h,k) = -\frac{k}{qz} + O\left(|s(a,q)| + \left|s\left(\frac{a+q}{2}, q\right)\right| + |z|\right), \quad \text{if } a+q \text{ is an even number.}$$

On the other hand, if $a+q$ is an odd number, using the similar methods we can get

$$s(h+k, 2k) = \frac{k}{24qz} + O\left(q + |z|\right),$$

so we have

$$S(h,k) = 4s(h,k) - 8s(h+k, 2k) = O\left(q + |z|\right).$$

This completes the proof of Lemma 5. □

Lemma 6. *For any real $s > 1$, we have the identities*

$$\sum_{\substack{d=1 \\ 2|d}}^{\infty} \frac{\mu(d)}{d^s} = \frac{1}{(1-2^s)\zeta(s)}, \quad \sum_{\substack{d=1 \\ 2\nmid d}}^{\infty} \frac{\mu(d)}{d^s} = \frac{2^s}{(2^s-1)\zeta(s)};$$

$$\sum_{\substack{d=1 \\ 2\nmid d}}^{\infty} \sum_{e|d} \frac{\mu(e)(d/e)^{1-2m}}{d^{s+2m}} = \frac{2^{s+2m} - 2^{1-2m}}{2^{s+2m} - 1} \cdot \frac{\zeta(s+4m-1)}{\zeta(s+2m)}$$

and

$$\sum_{\substack{d=1 \\ 2|d}}^{\infty} \sum_{e|d} \frac{\mu(e)\left(d/e\right)^{1-2m}}{d^{s+2m}} = \frac{2^{1-2m} - 1}{2^{s+2m} - 1} \cdot \frac{\zeta(s+4m-1)}{\zeta(s+2m)}.$$

Proof. Using elementary methods we can easily deduce these identities. □

3. Proof of Theorem

We suppose that $m \geq 2$ and a sufficiently large number k are given. We set

$$Q = \left[k^{1/2}\right], \quad P = 2Q.$$

For integers a and q with $1 \leq q \leq Q$, let $I(a,q)$ be an open interval given by

$$I(a,q) = \left(\frac{a}{q} - \frac{1}{qP}, \frac{a}{q} + \frac{1}{qP}\right).$$

When $a/q \neq \acute{a}/\acute{q}$ and $q, \acute{q} \leq Q$, one has

$$\left|\frac{a}{q} - \frac{\acute{a}}{\acute{q}}\right| \geq \frac{1}{q\acute{q}} \geq \left(\frac{1}{qP} + \frac{1}{\acute{q}P}\right).$$

Thus the intervals $I(a,q)$ are pairwise disjoint.

If $1 \leq h \leq k$, $(h,k) = 1$ and $h+k$ is odd, then by Lemma 1, h/k falls into an interval $I(a,q)$ with $1 \leq q \leq P$, $0 \leq a \leq q$ and $(a,q) = 1$.

Let $z = qh - ak$. It is easy to see that $z \neq 0$ and

$$|z| = qk\left|\frac{h}{k} - \frac{a}{q}\right| \leq \frac{k}{P} \leq \frac{k}{q}.$$

If h/k falls into an interval $I(a,q)$ with $1 \leq q \leq P$, $0 \leq a \leq q$, $(a,q) = 1$ and $a+q$ is an odd number, then by Lemma 5, we have

$$S(h,k) = O\left(q + |z|\right) \ll P + \frac{k}{P} \ll k^{1/2}.$$

Thus,

$$\sum\nolimits^{*} S^{2m}(h,k) \ll k^{m+1} \ll k^{2m-1},$$

where the asterisk indicates summation over those integers h, $1 \le h \le k$, $(h,k)=1$ and $h+k$ is odd, for which h/k falls into an interval $I(a,q)$ with $1 \le q \le P$, $0 \le a \le q$, $(a,q)=1$ and $a+q$ is an odd number.

If h/k falls into an interval $I(a,q)$ with $Q \le q \le P$, $0 \le a \le q$, $(a,q)=1$ and $a+q$ is an even number, then by Lemma 5, we have

$$\begin{aligned} S(h,k) &= -\frac{k}{qz} + O\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right| + |z|\right) \\ &\ll \frac{k}{q} + q + \frac{k}{P} \ll \frac{k}{Q} + P + \frac{k}{P} \ll k^{1/2}. \end{aligned}$$

Thus,

$$\sum\nolimits^{*} S^{2m}(h,k) \ll k^{m+1} \ll k^{2m-1},$$

where the asterisk indicates summation over those integers $h, 1 \le h \le k, (h,k)=1$ and $h+k$ is odd, for which h/k falls into an interval $I(a,q)$ with $Q \le q \le P, 0 \le a \le q$, $(a,q)=1$ and $a+q$ is an even number.

Therefore

$$\sum_{h=1}^{k}{}' S^{2m}(h,k) = \sum_{\substack{h=1\\2\nmid h+k}}^{k}{}' S^{2m}(h,k) = \sum_{\substack{q=1\\2\nmid q}}^{Q} \sum_{\substack{a=1\\2\nmid a}}^{q}{}' \sum_{h/k\in I(a,q)}^{*} S^{2m}(h,k) + O\left(k^{2m-1}\right),$$

where the asterisk means that $1 \le h \le k$, $(h,k)=1$ and $h+k$ is odd.

Lemma 5 produces

$$S(h,k) = -\frac{k}{qz} + O\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right| + |z|\right), \quad \text{if } a \text{ and } q \text{ are odd numbers.}$$

Using the estimate

$$(A+B+C)^{2m} = A^{2m} + O\left(|A|^{2m-1}\left(|B|+|C|\right)\right) + O\left(B^{2m}+C^{2m}\right),$$

we obtain

$$\begin{aligned} S^{2m}(h,k) &= \left(\frac{k}{qz}\right)^{2m} + O\left(\left(\frac{k}{q|z|}\right)^{2m-1}\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right| + |z|\right)\right) \\ &\quad + O\left(\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right|\right)^{2m} + z^{2m}\right). \end{aligned}$$

Therefore

$$\sum_{\substack{q=1\\2\nmid q}}^{Q}\sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q}\sideset{}{^*}\sum_{h/k\in I(a,q)} S^{2m}(h,k) \equiv \Omega_1 + O(\Omega_2) + O(\Omega_3),$$

where

$$\begin{aligned}
\Omega_1 &= \sum_{\substack{q=1\\2\nmid q}}^{Q}\sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q}\sideset{}{^*}\sum_{h/k\in I(a,q)} \left(\frac{k}{qz}\right)^{2m},\\
\Omega_2 &= \sum_{\substack{q=1\\2\nmid q}}^{Q}\sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q}\sideset{}{^*}\sum_{h/k\in I(a,q)} \left(\frac{k}{q\,|z|}\right)^{2m-1}\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right| + |z|\right),\\
\Omega_3 &= \sum_{\substack{q=1\\2\nmid q}}^{Q}\sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q}\sideset{}{^*}\sum_{h/k\in I(a,q)} \left(\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right|\right)^{2m} + z^{2m}\right).
\end{aligned}$$

Noting that for the fixed a, q, k and z, the equation $z = qh - ak$ has at most one solution h. By Lemma 4, we have

$$\begin{aligned}
\Omega_2 &\ll k^{2m-1}\sum_{\substack{q=1\\2\nmid q}}^{Q}\sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q}\sideset{}{^*}\sum_{h/k\in I(a,q)} \frac{1}{q^{2m-1}}\cdot\frac{1}{z^{2m-2}}\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right| + 1\right)\\
&\ll k^{2m-1}\sum_{\substack{q=1\\2\nmid q}}^{Q}\frac{1}{q^{2m-1}}\sideset{}{'}\sum_{a=1}^{q}\left(|s(a,q)| + \left|s\left(\frac{a+q}{2},q\right)\right| + 1\right)\sum_{z\neq 0}\frac{1}{z^2}\\
&\ll k^{2m-1}\sum_{\substack{q=1\\2\nmid q}}^{Q}\frac{1}{q^{2m-1}}\cdot q\cdot\log^2(q+1) \ll k^{2m-1}\sum_{\substack{q=1\\2\nmid q}}^{Q}\frac{\log^2(q+1)}{q^2} \ll k^{2m-1}.
\end{aligned}$$

Moreover,

$$\Omega_3 \ll \sum_{\substack{q=1\\2\nmid q}}^{Q}\sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q}\sideset{}{^*}\sum_{h/k\in I(a,q)} \left(q^{2m} + \left(\frac{k}{P}\right)^{2m}\right) \ll k^m\sideset{}{'}\sum_{h=1}^{k} 1 \ll k^{m+1} \ll k^{2m-1}.$$

Combining these estimates, we obtain

$$\sideset{}{'}\sum_{h=1}^{k} S^{2m}(h,k) = \Omega_1 + O(k^{2m-1}),$$

where

$$\Omega_1 = k^{2m} \sum_{\substack{q=1 \\ 2\nmid q}}^{Q} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1 \\ 2\nmid a}}^{q} \sideset{}{^*}\sum_{h/k\in I(a,q)} \frac{1}{z^{2m}}.$$

It remains to obtain an asymptotic formula for Ω_1. Noting that if $1 \le h \le k$, then $h/k \notin I(a,q)$ if and only if $|z| \ge k/P$. Hence

$$\begin{aligned} k^{2m} \sum_{\substack{q=1 \\ 2\nmid q}}^{Q} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1 \\ 2\nmid a}}^{q} \sideset{}{^*}\sum_{h/k\notin I(a,q)} \frac{1}{z^{2m}} &\le k^{2m} \sum_{\substack{q=1 \\ 2\nmid q}}^{Q} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1 \\ 2\nmid a}}^{q} \sum_{|z|\ge k/P} \frac{1}{z^{2m}} \\ &\ll k^{2m} \left(\frac{P}{k}\right)^{2m-1} \sum_{\substack{q=1 \\ 2\nmid q}}^{Q} \frac{1}{q^{2m-1}} \\ &\ll kP^{2m-1} \ll k^{m+1/2} \ll k^{2m-1}. \end{aligned}$$

Thus

$$\Omega_1 = k^{2m} \sum_{\substack{q=1 \\ 2\nmid q}}^{Q} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1 \\ 2\nmid a}}^{q} \sideset{}{'}\sum_{\substack{h=1 \\ 2\nmid h+k}}^{k} \frac{1}{(qh-ak)^{2m}} + O\left(k^{2m-1}\right).$$

Using the estimate

$$\sum_{h\ge k+1} \frac{1}{(qh-ak)^{2m}} \le \int_k^{\infty} \frac{dx}{(qx-ak)^{2m}} = \int_{(q-a)k}^{\infty} \frac{dy}{qy^{2m}} \ll \frac{1}{qk^{2m-1}},$$

we get

$$k^{2m} \sum_{\substack{q=1 \\ 2\nmid q}}^{Q} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1 \\ 2\nmid a}}^{q} \sum_{h\ge k+1} \frac{1}{(qh-ak)^{2m}} \ll k^{2m} \sum_{\substack{q=1 \\ 2\nmid q}}^{Q} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1 \\ 2\nmid a}}^{q} \frac{1}{qk^{2m-1}} \ll k.$$

Since

$$\begin{aligned} \sum_{h\le 0} \frac{1}{(qh-ak)^{2m}} &\le \frac{1}{k^{2m}} + \sum_{r\ge 1} \frac{1}{(qr+ak)^{2m}} \le \frac{1}{k^{2m}} + \int_0^{\infty} \frac{dx}{(qx+ak)^{2m}} \\ &= \frac{1}{k^{2m}} + \int_{ak}^{\infty} \frac{dy}{qy^{2m}} \ll \frac{1}{k^{2m}} + \frac{1}{qk^{2m-1}} \ll \frac{1}{k^{2m-1}}, \end{aligned}$$

we have

$$k^{2m}\sum_{\substack{q=1\\2\nmid q}}^{Q}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'\sum_{h\le 0}\frac{1}{(qh-ak)^{2m}}\ll k^{2m}\sum_{\substack{q=1\\2\nmid q}}^{Q}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'\frac{1}{k^{2m-1}}\ll k.$$

Therefore

$$\Omega_1=k^{2m}\sum_{\substack{q=1\\2\nmid q}}^{Q}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'\sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h+k}}^{\infty}\frac{1}{(qh-ak)^{2m}}+O(k^{2m-1}).$$

Since

$$\begin{aligned}k^{2m}\sum_{\substack{q>Q\\2\nmid q}}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'\sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h+k}}^{\infty}\frac{1}{(qh-ak)^{2m}}&\ll k^{2m}\sum_{\substack{q>Q\\2\nmid q}}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'\sum_{z=-\infty}^{\infty}\frac{1}{z^{2m}}\\&\ll k^{2m}\sum_{\substack{q>Q\\2\nmid q}}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'1\ll k^{2m}\sum_{\substack{q>Q\\2\nmid q}}\frac{1}{q^{2m-1}}\\&\ll\frac{k^{2m}}{Q^{2m-2}}\ll k^{m+1}\ll k^{2m-1},\end{aligned}$$

we have

$$\Omega_1=k^{2m}\sum_{\substack{q=1\\2\nmid q}}^{\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'\sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h+k}}^{\infty}\frac{1}{(qh-ak)^{2m}}+O\left(k^{2m-1}\right).$$

Therefore

$$\sum_{h=1}^{k}{}'S^{2m}(h,k)=g_m(k)k^{2m}+O\left(k^{2m-1}\right),$$

where

$$g_m(k)=\sum_{\substack{q=1\\2\nmid q}}^{\infty}\frac{1}{q^{2m}}\sum_{\substack{a=1\\2\nmid a}}^{q}{}'\sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h+k}}^{\infty}\frac{1}{(qh-ak)^{2m}}.$$

Let

$$\begin{aligned}
U(s) &= \sum_{k=1}^{\infty} \frac{g_m(k)}{k^s} = \sum_{\substack{q=1\\2\nmid q}}^{\infty} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q} \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h+k}}^{\infty} \frac{1}{(qh-ak)^{2m}} \\
&= \sum_{\substack{q=1\\2\nmid q}}^{\infty} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q} \sum_{\substack{k=1\\2\mid k}}^{\infty} \frac{1}{k^s} \sum_{\substack{h=-\infty\\(h,k)=1\\2\nmid h}}^{\infty} \frac{1}{(qh-ak)^{2m}} + \sum_{\substack{q=1\\2\nmid q}}^{\infty} \frac{1}{q^{2m}} \sideset{}{'}\sum_{\substack{a=1\\2\nmid a}}^{q} \sum_{\substack{k=1\\2\nmid k}}^{\infty} \frac{1}{k^s} \sum_{\substack{h=-\infty\\(h,k)=1\\2\mid h}}^{\infty} \frac{1}{(qh-ak)^{2m}} \\
&\equiv U_1(s) + U_2(s).
\end{aligned}$$

We proceed to find an expression for $U_1(s)$. We remove the coprimality conditions by use of the Möbius relation

$$\sum_{d\mid n} \mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ 0, & \text{if } n \neq 1. \end{cases}$$

After rearranging the sums, we have

$$U_1(s) = \sum_{\substack{d=1\\2\nmid d}}^{\infty} \sum_{\substack{e=1\\2\nmid e}}^{\infty} \frac{\mu(d)}{d^{s+2m}} \frac{\mu(e)}{e^{4m}} \sum_{\substack{k=1\\2\mid k}}^{\infty} \frac{1}{k^s} \sum_{\substack{q=1\\2\nmid q}}^{\infty} \frac{1}{q^{2m}} \sum_{\substack{a=1\\2\nmid a}}^{q} \sum_{\substack{h=-\infty\\2\nmid h}}^{\infty} \frac{1}{(qh-ak)^{2m}}.$$

Let $g = (k, q)$, then the inner double sum is

$$\begin{aligned}
&= \frac{1}{g^{2m}} \sum_{\substack{a=1\\2\nmid a}}^{q} \sum_{\substack{n=-\infty\\2\nmid n\\n\equiv -a(k/g) \bmod q/g}}^{\infty} \frac{1}{n^{2m}} \\
&= \frac{1}{g^{2m}} \left[\sum_{\substack{a=1\\2\nmid a}}^{q} \sum_{\substack{n=1\\2\nmid n\\n\equiv -a(k/g) \bmod q/g}}^{\infty} \frac{1}{n^{2m}} + \sum_{\substack{a=1\\2\nmid a}}^{q} \sum_{\substack{n=1\\2\nmid n\\n\equiv -(q-a)(k/g) \bmod q/g}}^{\infty} \frac{1}{n^{2m}} \right] \\
&= \frac{1}{g^{2m}} \sum_{a=0}^{q} \sum_{\substack{n=1\\2\nmid n\\n\equiv -a(k/g) \bmod q/g}}^{\infty} \frac{1}{n^{2m}} = \frac{1}{g^{2m}} \left[\sum_{a=1}^{q} \sum_{\substack{n=1\\2\nmid n\\n\equiv -a(k/g) \bmod q/g}}^{\infty} \frac{1}{n^{2m}} + \sum_{\substack{n=1\\2\nmid n\\n\equiv 0 \bmod q/g}}^{\infty} \frac{1}{n^{2m}} \right] \\
&= \frac{1}{g^{2m}} \left[g \frac{\left(2^{2m}-1\right)\zeta(2m)}{2^{2m}} + \frac{g^{2m}}{q^{2m}} \frac{\left(2^{2m}-1\right)\zeta(2m)}{2^{2m}} \right] \\
&= \frac{\left(2^{2m}-1\right)\zeta(2m)}{2^{2m}} \left[g^{1-2m} + \frac{1}{q^{2m}} \right].
\end{aligned}$$

Thus by Lemma 6,

$$U_1(s) = \frac{2^{s+4m}}{\left(2^{s+2m}-1\right)\left(2^{2m}+1\right)} \frac{\zeta(2m)}{\zeta(s+2m)\zeta(4m)} \left[\sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2\nmid q}}^{\infty} \frac{g^{1-2m}}{k^s q^{2m}} + \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2\nmid q}}^{\infty} \frac{1}{k^s q^{4m}} \right].$$

By Lemma 6 we have

$$\begin{aligned} \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2\nmid q}}^{\infty} \frac{g^{1-2m}}{k^s q^{2m}} &= \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2\nmid q}}^{\infty} \frac{1}{k^s q^{2m}} \sum_{d|(k,q)} \sum_{e|d} \mu(e) \left(\frac{d}{e}\right)^{1-2m} \\ &= \sum_{\substack{d=1 \\ 2\nmid d}}^{\infty} \sum_{e|d} \frac{\mu(e)(d/e)^{1-2m}}{d^{s+2m}} \sum_{\substack{k=1 \\ 2|k}}^{\infty} \sum_{\substack{q=1 \\ 2\nmid q}}^{\infty} \frac{1}{k^s q^{2m}} \\ &= \frac{\left(2^{s+2m}-2^{1-2m}\right)\left(2^{2m}-1\right)}{2^{s+2m}\left(2^{s+2m}-1\right)} \cdot \frac{\zeta(s+4m-1)\zeta(s)\zeta(2m)}{\zeta(s+2m)}. \end{aligned}$$

Therefore

$$\begin{aligned} U_1(s) = {} & \frac{\left(2^{s+4m}-2\right)\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)^2\left(2^{2m}+1\right)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s) \\ & + \frac{\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s+2m)}. \end{aligned}$$

Using the same methods we can prove

$$\begin{aligned} U_2(s) = {} & \frac{(2^s-1)\left(2^{s+4m}-2\right)\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)^2\left(2^{2m}+1\right)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s) \\ & + \frac{(2^s-1)\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s+2m)}. \end{aligned}$$

So we have

$$\begin{aligned} U(s) = {} & \frac{2^s\left(2^{s+4m}-2\right)\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)^2\left(2^{2m}+1\right)} \cdot \frac{\zeta^2(2m)}{\zeta(4m)} \frac{\zeta(s+4m-1)}{\zeta^2(s+2m)} \zeta(s) \\ & + \frac{2^s\left(2^{2m}-1\right)}{\left(2^{s+2m}-1\right)} \cdot \frac{\zeta(s)\zeta(2m)}{\zeta(s+2m)}. \end{aligned}$$

This completes the proof of Theorem.

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References

[1] T.M. Apostol: Modular functions and Dirichlet, Series in Number Theory, Springer-Verlag, New York, 1976.

[2] B.C. Berndt: *Analytic eisentein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303**/**304** (1978), 332–365.

[3] Jia Chaohua: *On the mean values of Dedekind sums*, J. Number Theory **87** (2001), 173–188.

[4] J.B. Conrey, E. Fransen, R. Klein and C. Scott: *Mean values of Dedekind sums* J. Number Theory **56** (1996), 214–226.

[5] R.R. Hall and M.N. Huxley: *Dedekind sums and continued fractions*, Acta Arith. **63** (1993), 79–90.

[6] G.H. Hardy and E.M. Wright: An Introduction to the Theory of Numbers, Oxford University Press, Oxford, 1979.

[7] J. Pommersheim: *Toric varieties, lattice points, and Dedekind sums*, Math. Ann. **295** (1993), 1–24.

[8] H. Rademacher: Dedekind Sums, Carus Mathematical Monographs, Mathematical Association of America, Washington, D.C., 1972.

[9] R. Sitaramachandra Rao: *Dedekind and Hardy sums*, Acta Arith. **48** (1987), 325–340.

[10] H. Walum: *An exact formula for an average of L-series*, Illinois J. Math. **26** (1982), 1–3.

[11] Z. Wenpeng: *On the mean values of Dedekind sums*, J. The'or. Nombres Bordeaux **8** (1996), 429–442.

[12] Z. Wenpeng: *On the hybrid mean value of Dedekind sums and Hurwitz zeta-function*, Acta Arith. **92** (2000), 141–152.

[13] Z. Wenpeng: *A hybrid mean value formula of Dedekind sums and Hurwitz zeta-functions*, Analytic Number Theory, Kluwer, (2002) 395–408.

[14] Z. Wenpeng and Y. Yuan: *On the Dedekind sums and the Fibonacci numbers*, The Fibonacci Quarterly, **38** (2000), 223–226.

[15] Z. Wenpeng and Y. Yuan: *On the $2m$-th power mean of certain Hardy sums*, Soochow J. Math. **26** (2000), 73–84.

[16] H. Xiali and Z. Wenpeng: *On the mean value of the Dedekind sums with the weight of Hurwitz zeta-function*, J. Math. Anal. Appl. **240** (1999), 505–517.

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