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DEFORMATIONS OF RATIONAL DOUBLE POINTS AND SIMPLE ELLIPTIC SINGULARITIES IN CHARACTERISTIC *p*

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Introduction

Let X be a non-singular threefold defined over an algebraically closed field k of characteristic $p \geq 0$. We consider a proper morphism $f : X \to C$, where C is a non-singular curve and $f_*\mathcal{O}_X \cong \mathcal{O}_C$ is satisfied. If the characteristic of k is zero, it follows that a general fiber of f is non-singular, which is known as Sard's lemma. In positive characteristic it happens that f is not generically smooth, and our aim is to understand such phenomena explicitly. Our main concern is the cases where X is either a Calabi-Yau threefold or a Fano threefold. Examples of such fibrations can be found in [8] and [12].

In this article, we have two main results. One states that there is a tendency that Sard's lemma continues to hold except for some small $p > 0$.

Theorem 5.1. *Consider a fibration $f : X \to C$ from a non-singular threefold to a curve. We suppose that a general fiber of f is a normal surface. Then the following hold*:
i) *There does not appear a simple elliptic singularity on a general fiber if $p \geq 5$.*
ii) *Under the assumption that the anti-canonical divisor of a fiber is ample, a general fiber is non-singular if $p \geq 11$, i.e., it is a Del Pezzo surface.*
iii) *Under the assumption that a general fiber has a trivial dualizing sheaf and has only rational singularities, it is non-singular if $p \geq 23$, i.e., it is either an abelian surface or a $K3$ surface.*

The other concerns the local behavior of the fibration $f : X \to C$ along the singular locus of general fibers. We have Theorem 3.4 in which rational double points are treated. We use the notation of Artin in [3].

Theorem 3.4. *Suppose $p \geq 3$. Let $f : X \to C$ be a fibration from a non-singular threefold to a non-singular curve.*

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i) *The following are all the types of rational double points which can appear as singularities of a general fiber of* f:

$$\begin{array}{ll} E_8^0,\ E_6^0,\ A_{3^e-1} & \textit{in}\ p=3; \\ E_8^0,\ A_{5^e-1} & \textit{in}\ p=5; \\ A_{p^e-1} & \textit{in}\ p\geq 7, \end{array}$$

where e *is a positive integer*.
ii) *We assume that a general fiber of* f *has a rational double point. Let* t *be a parameter of the base curve at a general point. Then its pull-back* $f^*t \in \mathcal{O}_{X,x}$ *at the singular point in question* $x \in \operatorname{Sing} f^{-1}(t)$ *can be put into a normal form in the complete ring* $\hat{\mathcal{O}}_{X,x} \cong k[[x,y,z]]$ *as follows*;

$$\begin{array}{lll} E_8^0: & t=z^2+x^3+y^5, & p=3,5; \\ E_6^0: & t=z^2+x^3+y^4, & p=3,\ \textit{or} \\ & t=z^2+x^3+y^4+\psi y^3, & p=3,\ \psi\in xk[[x]],\ d\psi\neq 0; \\ A_{p^e-1}: & t=xy+z^{p^e}, & p\ \textit{any prime},\ e\geq 1. \end{array}$$

However, there remain questions we could not settle: One is to extend the results of Theorem 3.4 to the case where a general fiber of $f: X \to C$ has rational double points of type D, E in $p=2$. Another is, as in i), Theorem 5.1, whether a general fiber of $f: X \to C$ can have simple elliptic singularities in $p=2,3$. Also, when considering Calabi-Yau threefolds or Fano threefolds, one encounters the following situations which we did not treat in this article; i) $f: X \to C$ is a fibration from a non-singular threefold to a non-singular curve and the general fiber has either irrational singularities other than simple elliptic singularities or non-normal singularities, ii) $f: X \to Y$ is a fibration from a non-singular threefold to a normal surface and the general fiber is singular or non-reduced. We note that these questions are considered as a three dimensional generalization of what is known as quasi-elliptic fibrations of surfaces (cf. [4]).

1. Preliminaries

Let $(R,\mathfrak{m})$ be a two-dimensional local ring essentially of finite type over an algebraically closed field k of arbitrary characteristic. The embedding dimension and the multiplicity of $(R,\mathfrak{m})$ are defined as $\operatorname{emb.dim} R := \dim_k \mathfrak{m}/\mathfrak{m}^2$ and $e(\mathfrak{m},R) := \lim_{n\to\infty} 2/n^2 \dim_k R/\mathfrak{m}^n$ respectively. Assume furthermore that $(R,\mathfrak{m})$ is normal. The space of first order infinitesimal deformations of $(R,\mathfrak{m})$ is defined as $T^1 := \operatorname{Ext}^1_R\big(\Omega^1_{R/k}, R\big)$. If $(R,\mathfrak{m})$ has its embedding dimension three, we have the isomorphism

$$T^1 \cong k[[x,y,z]] \Big/ \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}, \varphi \right),$$

under an identification $\widehat{R} \cong k[[x,y,z]]/(\varphi)$ (cf. [2]).

We call a proper birational morphism $\pi\colon X \to \operatorname{Spec} R$ from a non-singular surface X a resolution of singularities if π induces an isomorphism $X \setminus E \cong \operatorname{Spec} R \setminus \mathfrak{m}$, where $E := \pi^{-1}(\mathfrak{m})$ is the exceptional set. A proof of the existence of resolutions of surface singularities can be found in [10]. If the exceptional set E contains no exceptional curves of the first kind, we say that the resolution of singularities π is minimal. The geometric genus of $(R, \mathfrak{m})$ is defined via a resolution of singularities as $p_g(R, \mathfrak{m}) := \dim_k R^1\pi_*\mathcal{O}_X$, which is independent of the choice of resolutions.

We call a proper morphism $f\colon X \to S$ from a non-singular variety X to a normal variety S a fibration when $f_*\mathcal{O}_X \cong \mathcal{O}_S$ is satisfied. If S has dimension one, any fiber X_t ($t \in S$) is a Cartier divisor in X and the embedding dimension of a singular point of X_t is equal to the dimension of X.

2. Two criteria

Let $f\colon X \to C$ be a fibration from a non-singular $(n+1)$-fold X to a non-singular curve C. We study the singularities of a general fiber of f.

Proposition 2.1. *Let $(R, \mathfrak{m})$ be an n-dimensional normal local ring essentially of finite type over an algebraically closed field k. We suppose that* $\operatorname{emb.dim} R = n+1$ *and $\mathfrak{m} \in \operatorname{Spec} R$ is an isolated singularity. Let $T^1 := \operatorname{Ext}^1_R\left(\Omega^1_{R/k}, R\right)$ be the space of first order infinitesimal deformations. If $(R, \mathfrak{m})$ is isomorphic to a singularity on a general fiber of f, the dimension of T^1 as a k-vector space is divisible by p.*

Proof. Let $f\colon X \to C$ be a fibration as above. We consider an ideal $I \subset \mathcal{O}_X$ defined as the image of a natural coupling $f^*\Omega^1_C \times T_X \to \mathcal{O}_X$, and denote by Δ the closed subvariety of X defined by I. By the assumption, there exists an irreducible curve Δ' in Δ such that the restricted morphism $f|_{\Delta'_{\mathrm{red}}}\colon \Delta'_{\mathrm{red}} \to C$ is surjective. Then $(\Delta'_{\mathrm{red}}, X_t)_q > 1$ at $q \in \Delta'_{\mathrm{red}} \cap X_t$ because $q \in \operatorname{Sing} X_t$. This indicates that $f|_{\Delta'_{\mathrm{red}}}$ is ramified at a general point, therefore $f|_{\Delta'_{\mathrm{red}}}$ is inseparable. It follows that $(\Delta'_{\mathrm{red}}, X_t)_q$ is divisible by p, and so is $(\Delta', X_t)_q$. On the other hand, by a local description, we know

$$(\Delta', X_t)_q = \dim_k \mathcal{O}_{X,q} \Big/ \left(\frac{\partial f^*t}{\partial x_1}, \frac{\partial f^*t}{\partial x_2}, \dots, \frac{\partial f^*t}{\partial x_n}, f^*t \right),$$

where $x_1, x_2, \dots, x_n$ (resp. t) are a regular system of parameters of $\mathcal{O}_{X,q}$ (resp. of $\mathcal{O}_{C,t}$). $\square$

Proposition 2.2. *Let $(R, \mathfrak{m})$ be an n-dimensional normal isolated singularity defined by a quasi-homogeneous equation $\varphi(x_0, x_1, \dots, x_n)$. Let w_i be the weight of x_i and m be the degree of φ with respect to these weights. If $(R, \mathfrak{m})$ is isomorphic to a singularity on a general fiber of f, the integer m is divisible by p.*

Proof. Assume that $(m, p) = 1$ and suppose that there exists a fibration $f: X \to C$ whose fiber X_t has a singular point $x \in X_t$ which is isomorphic to $(R, \mathfrak{m})$. Choose a local coordinate $t \in \mathcal{O}_{C,t}$ of the base curve C and pull it back by f. Then under an appropriate identification $\widehat{\mathcal{O}}_{X,x} \cong k[[x_0, x_1, \dots, x_n]]$ we have the expression $t = u\varphi(x_0, x_1, \dots, x_n)$, where $u \in \widehat{\mathcal{O}}_{X,x}$ is a unit. By the coordinate change $(x_0, x_1, \dots, x_n) \mapsto \left(u^{-w_0/m}x_0, u^{-w_1/m}x_1, \dots, u^{-w_n/m}x_n\right)$, we get rid of the unit u and have the equation $t = \varphi(x_0, x_1, \dots, x_n)$. Then consider the derivation $D := w_0x_0\partial/\partial x_0 + w_1x_1\partial/\partial x_1 + \cdots + w_nx_n\partial/\partial x_n$, so that we have $D(\varphi) = m\varphi$. Thus there is an isomorphism

$$\widehat{\mathcal{O}}_{X,x} \Big/ \left(\frac{\partial\varphi}{\partial x_0}, \frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_n}\right) \cong \widehat{\mathcal{O}}_{X,x} \Big/ \left(\frac{\partial\varphi}{\partial x_0}, \frac{\partial\varphi}{\partial x_1}, \dots, \frac{\partial\varphi}{\partial x_n}, \varphi\right).$$

This indicates that the singularity of the fiber $x \in X_t$ does not extend to singularities of other fibers. Thus we have proved the proposition. □

3. Rational double points

DEFINITION 3.1. Let $(R, \mathfrak{m})$ be a two-dimensional normal local ring essentially of finite type over an algebraically closed field k. We say that Spec$R$ has a rational double point at $\mathfrak{m}$ if $(R, \mathfrak{m})$ has multiplicity two, and $R^1\pi_*\mathcal{O}_X = 0$ holds for a resolution of singularities $\pi: X \to \operatorname{Spec} R$.

The classification of rational double points in positive characteristic is given by Lipman-Artin ([3], [9]). In characteristic 2, 3 and 5, the structure of the singularity is no longer determined uniquely from the configuration diagram on the minimal desingularization.

Theorem 3.2 (Artin [3]). *The following assertions hold*:
i) *In the classification of rational double points in characteristic p, the equation is uniquely determined from its configuration diagram except the following cases*:

$p = 2$	D_{2n}^0:	$z^2 + x^2y + xy^n$	$4n$	$n \geq 2$
	D_{2n}^r:	$z^2 + x^2y + xy^n + xy^{n-r}z$	$4n - 2r$	$r = 1, \dots, n-1$
	D_{2n+1}^0:	$z^2 + x^2y + y^nz$	$4n$	$n \geq 2$
	D_{2n+1}^r:	$z^2 + x^2y + y^nz + xy^{n-r}z$	$4n - 2r$	$r = 1, \dots, n-1$
	E_6^0:	$z^2 + x^3 + y^2z$	8	

	E_6^1:	$z^2+x^3+y^2z+xyz$	6
	E_7^0:	$z^2+x^3+xy^3$	14
	E_7^1:	$z^2+x^3+xy^3+x^2yz$	12
	E_7^2:	$z^2+x^3+xy^3+y^3z$	10
	E_7^3:	$z^2+x^3+xy^3+xyz$	8
	E_8^0:	$z^2+x^3+y^5$	16
	E_8^1:	$z^2+x^3+y^5+xy^3z$	14
	E_8^2:	$z^2+x^3+y^5+xy^2z$	12
	E_8^3:	$z^2+x^3+y^5+y^3z$	10
	E_8^4:	$z^2+x^3+y^5+xyz$	8
$p=3$	E_6^0:	$z^2+x^3+y^4$	9
	E_6^1:	$z^2+x^3+y^4+x^2y^2$	7
	E_7^0:	$z^2+x^3+xy^3$	9
	E_7^1:	$z^2+x^3+xy^3+x^2y^2$	7
	E_8^0:	$z^2+x^3+y^5$	12
	E_8^1:	$z^2+x^3+y^5+x^2y^3$	10
	E_8^2:	$z^2+x^3+y^5+x^2y^2$	8
$p=5$	E_8^0:	$z^2+x^3+y^5$	10
	E_8^1:	$z^2+x^3+y^5+xy^4$	8

The number to the right of each equation is the dimension of T^1.
ii) *In a family of the singularity* X_n^r ($X = A, D$ *or* E, *respectively*), *the index* n *is upper-semicontinuous, while the co-index* r *is lower semi-continuous.*

REMARK 3.3. The equations of rational double points other than the ones stated in the previous theorem are identical to the classical forms:

A_n:	$z^{n+1}+xy$	n	if $p \nmid (n+1)$,	$n \geq 1$
:	$z^{n+1}+xy$	$n+1$	if $p \mid (n+1)$,	$n \geq 1$
D_n:	$z^2+y(x^2+y^{n-2})$	n		$n \geq 4$
E_6:	$z^2+x^3+y^4$	6		
E_7:	$z^2+x^3+xy^3$	7		
E_8:	$z^2+x^3+y^5$	8		

The number to the right of each equation is the dimension of T^1.

We have our main theorem.

Theorem 3.4. *Suppose $p \geq 3$. Let $f\colon X \to C$ be a fibration from a non-singular threefold to a non-singular curve.*
i) *The following are all the types of rational double points which can appear as singularities of a general fiber of f:*

$$\begin{array}{ll} E_8^0,\ E_6^0,\ A_{3^e-1} & \textit{in } p = 3; \\ E_8^0,\ A_{5^e-1} & \textit{in } p = 5; \\ A_{p^e-1} & \textit{in } p \geq 7, \end{array}$$

where e is a positive integer.
ii) *We assume that a general fiber of f has a rational double point. Let t be a parameter of the base curve at a general point. Then its pull-back $f^*t \in \mathcal{O}_{X,x}$ at the singular point in question $x \in \operatorname{Sing} f^{-1}(t)$ can be put into a normal form in the complete ring $\hat{\mathcal{O}}_{X,x} \cong k[[x,y,z]]$ as follows;*

$$\begin{array}{lll} E_8^0\colon & t = z^2 + x^3 + y^5, & p = 3, 5; \\ E_6^0\colon & t = z^2 + x^3 + y^4, & p = 3, \ \textit{or} \\ & t = z^2 + x^3 + y^4 + \psi y^3, & p = 3, \psi \in xk[[x]], d\psi \neq 0; \\ A_{p^e-1}\colon & t = xy + z^{p^e}, & p \textit{ any prime}, \ e \geq 1. \end{array}$$

Proof. By using the criteria in Proposition 2.1 and Proposition 2.2, we see that the remaining cases are

$$\begin{array}{ll} A_n & p \mid (n+1), \\ E_8^0 & p = 3, 5, \\ E_7^0 & p = 3, \\ E_6^0 & p = 3. \end{array}$$

In order to determine the normal forms, we use the following conditions: i) There is an irreducible component $\Delta \subset X$ of the locus of singular points of fibers so that Δ_{red} is a non-singular curve at the point in question $x \in \operatorname{Sing} f^{-1}(t)$. ii) Consider the morphism $f|_{\Delta_{\text{red}}}\colon \Delta_{\text{red}} \to C$ and the extension of the function fields $k(\Delta_{\text{red}})/k(C)$. Then the normalization $C' \to C$ of C in the relative separable closure of $k(C)$ in $k(\Delta_{\text{red}})$ is not ramified at a general point. iii) For each singularity of the fiber $\mathcal{O}_{X,x}/(f^*t)$ we have $\dim_k T^1 = \dim_k k[[x,y,z]]/(\partial f^*t/\partial x, \partial f^*t/\partial y, \partial f^*t/\partial z, f^*t)$.

In case A_n with $p \mid (n+1)$, we start from $t = u_0\bigl(xy + z^{qp^e}\bigr)$ with some unit $u_0 \in k[[x,y,z]]$ and integers $q, e \geq 1$ with $n+1 = qp^e$ and $p \nmid q$. By appropriate coordinate changes, we have $t = \bigl(uz^{p^e}\bigr)^q + xy$, where $u \in k[[z]]^\times$. This one parameter deformation has a rational double point of type A_{qp^e-1} at $t = 0$. Its general fiber has the singularity of the same type if and only if $q = 1$ and $u \in k\bigl[\bigl[z^{p^e}\bigr]\bigr]^\times$. Indeed, the if part is obvious. For the only if part, Δ given above is defined by the ideal $\bigl(x, y, z^{qp^e} du/dz\bigr)$. Then condition i) implies that $du/dz = 0$. The morphism $f|_\Delta\colon \Delta \to C$ is given locally by $t = \left(uz^{p^e}\right)^q$. Then we have, from condition ii), that $q = 1$ and $u \in k\left[\left[z^{p^e}\right]\right]^\times$ as required. The normal form is given by $t = z^{p^e} + xy$.

In case E_8^0 in $p = 3, 5$, we start from $t = u_0(z^2 + x^3 + y^5)$ with $u_0 \in k[[x, y, z]]^\times$. After coordinate changes, we have $t = z^2 + (a_0 + a_1 y + a_2 y^2 + a_3 y^3)x^3 + y^5$ with $a_0 \in k[[x]]^\times$, $a_1, a_2, a_3 \in k[[x]]$ in $p = 3$; and $t = z^2 + x^3 + (b_0 + b_1 x)y^5$ with $b_0 \in k[[y]]^\times$, $b_1 \in k[[y]]$ in $p = 5$ respectively. Now the locus Δ is locally defined by the ideal $I := (z, (a_0' + a_1' y + a_2' y^2 + a_3' y^3)x^3, (a_1 + 2a_2 y)x^3 + 2y^4)$ in $p = 3$, and $I := (z, 3x^2 + b_1 y^5, (b_0' + b_1' x)y^5)$ in $p = 5$. From condition i), we know that there is an element $\phi \in (x, y) \setminus (x, y)^2$ such that $I \cong (z, \phi^n)$ with some $n > 0$. Then condition iii) implies that $n = 4$ in $p = 3$ and $n = 2$ in $p = 5$, from which it follows that $a_1 = a_2 = a_0' = a_3' = 0$ in $p = 3$ and $b_1 = b_0' = 0$ in $p = 5$. Then after a coordinate change, we obtain the normal form $t = z^2 + x^3 + y^5$.

In case E_6^0 in $p = 3$, we have $t = z^2 + (a_0 + a_1 y + a_2 y^2)x^3 + y^4$ with $a_0 \in k[[x]]^\times$, $a_1, a_2 \in k[[x]]$. Then the locus Δ as above is given by the ideal $I := (z, (a_0' + a_1' y + a_2' y^2)x^3, (a_1 + 2a_2 y)x^3 + y^3)$, which can be put, by condition i), as $I \cong (z, \phi^n)$ with some $\phi \in (x, y) \setminus (x, y)^2$ and $n > 0$. Then condition iii) implies that $n = 3$ and $a_2 = a_0' = a_1' = 0$. Then replacing y by $y - a_1^{1/3} x$ gives $t = z^2 + a_0 x^3 + y^4 - a_1^{1/3} x y^3$ and we put this as $t = z^2 + x^3 + y^4 + \psi y^3$ with $\psi \in xk[[x]]$. We divide this case into two according as $d\psi$ vanishes or not. The former corresponds to the normal form $t = z^2 + x^3 + y^4$, and the latter to $t = z^2 + x^3 + y^4 + \psi y^3$ with $\psi \in xk[[x]]$, $d\psi \neq 0$.

In case E_7^0 in $p = 3$, the one parameter family $t = u_0(z^2 + x^3 + xy^3)$ with $u_0 \in k[[x, y, z]]^\times$ can be transformed into $t = z^2 + (a_0 + a_1 y + a_2 y^2)x^3 + xy^3$ with $a_0 \in k[[x]]^\times$, $a_1, a_2 \in k[[x]]$. The locus Δ is defined by $I := (z, (a_0' + a_1' y + a_2' y^2)x^3 + y^3, (a_1 + 2a_2 y)x^3)$. Then condition i) implies that there exists $\phi \in (x, y) \setminus (x, y)^2$ such that $I = (z, \phi^n)$ for some $n > 0$. From condition iii) we have $n = 3$ and $a_0'' = a_1 = a_2 = 0$. Performing a coordinate change $y \mapsto y - \sqrt[3]{a_0'}\,x$, we may assume $a_0' = 0$, then replacing x by $1/\sqrt[3]{a_0}\,x$, we get $t = z^2 + x^3 + 1/\sqrt[3]{a_0}\,xy^3$. A general fiber has an E_6^0-singularity. □

When considering rational double points in characteristic two, we find the criteria in Proposition 2.1 and Proposition 2.2 useless except A_n-type.

Proposition 3.5. *Suppose $p = 2$. Let $f\colon X \to C$ be as in* Theorem 3.4. *Suppose that a general fiber has a rational double point of type A_n, then we have $n = 2^e - 1$ with an integer $e \geq 1$ and the normal form of f along this singularity is given as*

$$A_{2^e-1}\colon t = xy + z^{2^e}.$$

Since we know the local description of deformations, the following corollary can easily be verified.

Corollary 3.6. *Suppose that a fibration $f\colon X \to C$ from a non-singular threefold X to a non-singular curve C has a general fiber with rational double points in $p \geq 3$ (resp. rational double points of type A_n in $p = 2$), then there exists an integer $n_0 > 0$ such that the family of rational double points $X \times_C C^{+n} \to C^{+n}$ obtained by base change by the n-iterated Frobenius morphism $F\colon C^{+n} \to C$ is locally trivial for $n \geq n_0$. To be more precise, the integer n_0 is given as $n_0 = e$ for a rational double point of type A_{p^e-1}, $n_0 = 2$ for that of type E_6^0 in $p = 3$ and $n_0 = 1$ for E_8^0 in $p = 3, 5$.*

REMARK 3.7. i) We could not solve the question in rational double points of type D, E in characteristic $p = 2$. Direct calculation seems to fail in these cases.
ii) Suppose that a rational double point is given by a polynomial $\varphi(x, y, z) = 0$. Then we have the family

$$\operatorname{Spec} k[x, y, z, t_1, t_2, \dots, t_d] \Bigg/ \left(\varphi + \sum_{i=1}^{n} P_i t_i\right) \to \operatorname{Spec} k[t_1, t_2, \dots, t_d],$$

where $P_i \in k[x, y, z]$ induces a bases of the k-vector space $k[[x, y, z]]/(\partial\varphi/\partial x, \partial\varphi/\partial y, \partial\varphi/\partial z, \varphi)$. The formal completion of this family along the fiber at the origin gives a formal versal deformation of the given rational double point. The family whose fiber has still the same rational double point as the closed fiber is either the closed fiber only or given as:

$$\begin{array}{lll} p = 3 & E_8^0\colon & z^2 + x^3 + y^5 + t_2 y^3 + t_1 \\ & E_7^0\colon & z^2 + x^3 + xy^3 + t_1 x \\ & E_6^0\colon & z^2 + x^3 + y^4 + t_1 + t_2 y \\ & E_6^1\colon & z^2 + x^3 + t_1 y^3 + y^4 + x^2 y^2 \\ p = 5 & E_8^0\colon & z^2 + x^3 + y^5 + t_1 \\ p \text{ any} & A_{p^e m-1}\colon & xy + \left(z^{p^e} + t_1\right)^m \qquad (p, m) = 1. \end{array}$$

This is an analogue to the calculation for quasi-elliptic surfaces given in [4].
iii) We find the following theorem by Wahl interesting in connection with our main theorem. Here, S is a locally free rank two subsheaf in T_X, which we refer to his paper for definition.

Theorem (Wahl [15]). *Let $X \to \operatorname{Spec} R$ be the minimal resolution of an RDP. Then $H_E^1(S) = 0$, and in particular the resolution is equivariant, except in the following cases*:

$$\begin{array}{ll} A_n & p \mid n + 1, \\ D_n & p = 2, \end{array}$$

$$
\begin{array}{ll}
E_6 & p = 2, 3, \\
E_7 & p = 2, 3, \\
E_8 & p = 2, 3, 5.
\end{array}
$$

4. Simple elliptic singularities

DEFINITION 4.1 (K. Saito [11]). Let $(R, \mathfrak{m})$ be a two dimensional normal local ring, essentially of finite type over an algebraically closed field k. We say that $(R, \mathfrak{m})$ is a simple elliptic singularity if its minimal resolution $\pi\colon X \to \operatorname{Spec} R$ has a single non-singular elliptic curve E as its exceptional set.

A simple elliptic singularity whose exceptional curve has self-intersection number $E^2 = -3$, $E^2 = -2$, $E^2 = -1$ are called $\tilde{E}_6$-type, $\tilde{E}_7$-type, $\tilde{E}_8$-type, respectively. The following is known, if k is the field of complex numbers, as the Grauert's theorem. His assertion can be extended to arbitrary characteristic.

Theorem 4.2 (Grauert [5]). *We consider a normal surface singularity whose exceptional set of the minimal resolution consists of a non-singular curve E of genus g. If $E^2 < 4(1 - g)$, the singularity is defined by quasi-homogeneous equations.*

Proof. We denote the minimal resolution by $\pi\colon X \to \operatorname{Spec} R$ with the maximal ideal $\mathfrak{m} \subset R$ and the exceptional curve by E. Consider the exact sequence induced from the inclusion $\mathfrak{m}^n \subset R$ with $n \geq 1$

$$
\mathfrak{m}^n \otimes \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_X \otimes R/\mathfrak{m}^n \to 0.
$$

We have another exact sequence

$$
0 \to \mathcal{O}_X(-nE) \to \mathcal{O}_X \to \mathcal{O}_{nE} \to 0.
$$

First we show the equality

$$
\widehat{R} \cong \varprojlim_{n} H^0(X_n, \mathcal{O}_{X_n}) \cong \varprojlim_{n} H^0(E, \mathcal{O}_{nE}),
$$

where X_n is the fiber product $X \otimes_R R/\mathfrak{m}^n$. The first equality is a consequence of the formal function theorem. The second follows from the fact that, since their supports coincide, the inverse systems of sheaves $\{\mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X\}$ and $\{\mathcal{O}_X/\mathcal{O}_X(-nE)\}$ give the same limit.

Secondly, we consider the filtration of $\mathcal{O}_X$ by sheaves of ideals

$$
\mathcal{O}_X(-E) \subset \mathcal{O}_X(-2E) \subset \mathcal{O}_X(-3E) \subset \cdots \subset \mathcal{O}_X.
$$

The hypothesis $E^2 < 4(1-g)$ guarantees the following vanishing in any $i > 0$

$$H^1(E, T_E \otimes \mathcal{O}_X(-iE)) = H^1(E, \mathcal{O}_E \otimes \mathcal{O}_X(-iE)) = 0.$$

From the vanishing of the first term we deduce that $\mathcal{O}_{iE}$ has the structure of $\mathcal{O}_E$ algebra for any $i > 0$ (cf. [6, Chapter II, Exercise 8.6]). The second vanishing gives the splitting of the following exact sequence of $\mathcal{O}_E$-modules for $i > 0$

$$0 \to \mathcal{O}_E \otimes \mathcal{O}_X(-iE) \to \mathcal{O}_{(i+1)E} \to \mathcal{O}_{iE} \to 0.$$

So we have $\mathcal{O}_{(n+1)E} \cong \bigoplus_{i=0}^{n} \mathcal{O}_E(-iE)$ for $n \geq 0$, and

$$\varprojlim_n \mathcal{O}_X/\mathfrak{m}^n\mathcal{O}_X \cong \varprojlim_n \mathcal{O}_X/\mathcal{O}_X(-nE) \cong \varprojlim_n \bigoplus_{i\geq 0} \mathcal{O}_E(-iE) \Big/ \bigoplus_{i>n} \mathcal{O}_E(-iE).$$

Thus the $\mathfrak{m}$-adic completion of R is obtained as the completion of a finitely generated graded ring $\bigoplus_{i\geq 0} H^0(E, \mathcal{O}_E(-iE))$. □

Corollary 4.3. *A simple elliptic singularity of the embedding dimension three is given by one of the following*:

$$\begin{array}{lll}
p \geq 3 & \tilde{E}_6: & x(x-z)(x-\lambda z) - zy^2 = 0, \\
 & \tilde{E}_7: & zx(x-z)(x-\lambda z) - y^2 = 0, \\
 & \tilde{E}_8: & x(x-z^2)(x-\lambda z^2) - y^2 = 0, \qquad \textit{where} \quad \lambda \in k, \lambda \neq 0, 1. \\
p = 2 & \tilde{E}_6: & y^2z + a_1xyz + a_3yz^2 + x^3 + a_2x^2z + a_4xz^2 + a_6z^3 = 0, \\
 & \tilde{E}_7: & y^2 + a_1xyz + a_3yz^2 + x^3z + a_2x^2z^2 + a_4xz^3 + a_6z^4 = 0, \\
 & \tilde{E}_8: & y^2 + a_1xyz + a_3yz^3 + x^3 + a_2x^2z^2 + a_4xz^4 + a_6z^6 = 0, \\
 & & \textit{where} \quad a_i \in k, a_1^6a_6 + a_1^5a_3a_4 + a_1^4a_2a_3^2 + a_1^4a_4^2 + a_3^4 + a_1^3a_3^3 \neq 0.
\end{array}$$

Proof. The normal forms are determined as in the case of characteristic zero from the Riemann-Roch theorem; $\dim_k H^0(E, \mathcal{O}_E(-mE)) = m(-E^2)$ for $m > 0$. □

Theorem 4.4. *Simple elliptic singularities of the following types do not appear on a general fiber of a one-parameter deformation whose total space is non-singular.*

$$\begin{array}{ll}
\tilde{E}_8 & \textit{in } p \geq 3, \\
\tilde{E}_7 & \textit{in } p \geq 3, \\
\tilde{E}_6 & \textit{in } p \geq 5, \ p = 2.
\end{array}$$

Proof. The dimension of T^1 is calculated as follows

$$\begin{array}{lll} \tilde{E}_8 & \dim_k T^1 = 10, & \\ \tilde{E}_7 & \dim_k T^1 = 9, & (p \neq 2), \\ \tilde{E}_6 & \dim_k T^1 = 8, & (p \neq 3). \end{array}$$

The deviating cases are $\tilde{E}_7$ in $p = 2$ with $\dim_k T^1 = 10$, and $\tilde{E}_6$ in $p = 3$ with $\dim_k T^1 = 9$. By Proposition 2.1, and Proposition 2.2, we find $\tilde{E}_6$ in $p = 3$, $\tilde{E}_7$ in $p = 2$ and $\tilde{E}_8$ in $p = 2$ as the remaining cases. □

5. An application

In this section we consider applying our results to Calabi-Yau threefolds and Fano threefolds.

Theorem 5.1. *Consider a fibration $f: X \to C$ from a non-singular threefold to a curve. We suppose that a general fiber of f is a normal surface. Then the following hold*:

i) *There does not appear a simple elliptic singularity on a general fiber if $p \geq 5$.*

ii) *Under the assumption that the anti-canonical divisor of a fiber is ample, a general fiber is non-singular if $p \geq 11$, i.e., it is a Del Pezzo surface.*

iii) *Under the assumption that a general fiber has a trivial dualizing sheaf and has only rational singularities, it is non-singular if $p \geq 23$, i.e., it is either an abelian surface or a $K3$ surface.*

Proof. i) This assertion follows from Theorem 4.4.

ii) Let X_t be a general fiber of f and consider its minimal resolution of singularities $\pi: \tilde{X_t} \to X_t$. Then by [7, Theorem 2.2] and Theorem 4.4, we know that X_t is a rational surface and has only rational double points. Then by the Riemann-Roch theorem, we have the upper-bound of the Picard number $\rho(\tilde{X_t}) = 10 - K_{\tilde{X_t}}^2 \leq 9$. Then the assertion follows from Theorem 3.4.

iii) We also consider the minimal resolution of singularities of a general fiber $\pi: \tilde{X_t} \to X_t$. Since the singularities of X_t have the embedding dimension three, they are rational double points and we only need to consider the case where $\tilde{X_t}$ is a $K3$ surface. The Picard number satisfies $\rho(\tilde{X_t}) \leq 22$ (cf. [1]), so we have the assertion again by Theorem 3.4. □

REMARK 5.2. Umezu studied projective normal surfaces which have trivial dualizing sheaves ([13], [14]). Unlike normal Gorenstein surfaces with ample anti-canonical sheaves, irrational singularities other than simple elliptic singularities can appear on such surfaces.

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