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STABILITY OF EXTREMAL KÄHLER MANIFOLDS

Dedicated to Professor Shoshichi Kobayashi on his seventieth birthday

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1. Introduction

In Donaldson's study [10] of asymptotic stability for polarized algebraic manifolds (M, L), *critical metrics* originally defined by Zhang [39] (see also [22]) are referred to as balanced metrics and play a central role when the polarized algebraic manifolds admit Kähler metrics of constant scalar curvature. Let $T \cong (\mathbb{C}^*)^k$ be an algebraic torus in the identity component $\mathrm{Aut}^0(M)$ of the group of holomorphic automorphisms of M. In this paper, we define the concept of *critical metrics relative to* T, and as an application, choosing a suitable T, we shall show that a result in [26] on the asymptotic approximation of critical metrics (see [10], [39]) can be generalized to the case where (M, L) admits an extremal Kähler metric in the polarization class. Then in our forthcoming paper [27], we shall show that a slight modification of the concept of stability (see Theorem A below) allows us to obtain the asymptotic stability of extremal Kähler manifolds even when the obstruction as in [26] does not vanish. In particular, by an argument similar to [10], an extremal Kähler metric in a fixed integral Kähler class on a projective algebraic manifold M will be shown to be unique* up to the action of the group $\mathrm{Aut}^0(M)$.

2. Statement of results

Throughout this paper, we fix once for all an ample holomorphic line bundle L on a connected projective algebraic manifold M. Let H be the maximal connected linear algebraic subgroup of $\mathrm{Aut}^0(M)$, so that $\mathrm{Aut}^0(M)/H$ is an abelian variety. The corresponding Lie subalgebra of $H^0(M, \mathcal{O}(T^{1,0}M))$ will be denoted by $\mathfrak{h}$. For the complete linear system $|L^m|$, $m \gg 1$, we consider the Kodaira embedding

$$\Phi_m = \Phi_{|L^m|} : M \hookrightarrow \mathbb{P}^*(V_m), \qquad m \gg 1,$$

where $\mathbb{P}^*(V_m)$ denotes the set of all hyperplanes through the origin in $V_m := H^0(M, \mathcal{O}(L^m))$. Put $N_m := \dim V_m - 1$. Let n and d be respectively the dimension

* For this uniquness, we choose $Z^{\mathbb{C}}$ (cf. Section 2) as the algebraic torus T.

of M and the degree of the image $M_m := \Phi_m(M)$ in the projective space $\mathbb{P}^*(V_m)$. Put $W_m = \{\mathrm{Sym}^d(V_m)\}^{\otimes n+1}$. Then to the image M_m of M, we can associate a nonzero element $\hat{M}_m$ in W_m^* such that the corresponding element $[\hat{M}_m]$ in $\mathbb{P}^*(W_m)$ is the Chow point associated to the irreducible reduced algebraic cycle M_m on $\mathbb{P}^*(V_m)$. Replacing L by some positive integral multiple of L if necessary, we fix an H-linearization of L, i.e., a lift to L of the H-action on M such that H acts on L as bundle isomorphisms covering the H-action on M. For an algebraic torus T in H, this naturally induces a T-action on V_m for each m. Now for each character $\chi \in \mathrm{Hom}(T, \mathbb{C}^*)$, we set

$$V(\chi) := \{\, s \in V_m ; t \cdot s = \chi(t)\, s \quad \text{for all } t \in T \,\}.$$

Then we have mutually distinct characters $\chi_1, \chi_2, \ldots, \chi_{\nu_m} \in \mathrm{Hom}(T, \mathbb{C}^*)$ such that the vector space $V_m = H^0(M, \mathcal{O}(L^m))$ is uniquely written as a direct sum

$$V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_k). \tag{2.1}$$

Put $G_m := \prod_{k=1}^{\nu_m} \mathrm{SL}(V(\chi_k))$, and the associated Lie subalgebra of $\mathrm{sl}(V_m)$ will be denoted by $\mathfrak{g}_m$. More precisely, G_m and $\mathfrak{g}_m$ possibly depend on the choice of the algebraic torus T, and if necessary, we denote these by $G_m(T)$ and $\mathfrak{g}_m(T)$, respectively. The T-action on V_m is, more precisely, a right action, while we regard the G_m-action on V_m as a left action. Since T is Abelian, this T-action on V_m can be regarded also as a left action.

The group G_m acts diagonally on V_m in such a way that, for each k, the k-th factor $\mathrm{SL}(V(\chi_k))$ of G_m acts just on the k-th factor $V(\chi_k)$ of V_m. This induces a natural G_m-action on W_m and also on W_m^*.

DEFINITION 2.2. (a) The subvariety M_m of $\mathbb{P}^*(V_m)$ is said to be *stable relative to* T or *semistable relative to* T, according as the orbit $G_m \cdot \hat{M}_m$ is closed in W_m^* or the closure of $G_m \cdot \hat{M}_m$ in W_m^* does not contain the origin of W_m^*.
(b) Let $\mathfrak{t}_c$ denote the Lie subalgebra of the maximal compact subgroup T_c of T, and as a real Lie subalgebra of the complex Lie algebra $\mathfrak{t}$, we define $\mathfrak{t}_{\mathbb{R}} := \sqrt{-1}\,\mathfrak{t}_c$.

Take a Hermitian metric for V_m such that $V(\chi_k) \perp V(\chi_l)$ if $k \neq l$. Put $N_m := \dim V_m - 1$ and $n_k := \dim V(\chi_k)$. We then set

$$l(k,i) := (i-1) + \sum_{j=1}^{k-1} n_j, \qquad i = 1, 2, \ldots, n_k;\; k = 1, 2, \ldots, \nu_m,$$

where the right-hand side denotes $i-1$ in the special case $k = 1$. Let $\|\ \|$ denote the Hermitian norm for V_m induced by the Hermitian metric. Take a $\mathbb{C}$-basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for V_m.

DEFINITION 2.3. We say that $\{s_0, s_1, \dots, s_{N_m}\}$ is an *admissible normal basis* for V_m if there exist positive real constants b_k, $k = 1, 2, \dots, \nu_m$, and a $\mathbb{C}$-basis $\{s_{k,i}\,;\, i = 1, 2, \dots, n_k\}$ for $V(\chi_k)$, with $\sum_{k=1}^{\nu_m} n_k b_k = N_m + 1$, such that
(1) $s_{l(k,i)} = s_{k,i}, \qquad i = 1, 2, \dots, n_k;\ k = 1, 2, \dots, \nu_m;$
(2) $s_l \perp s_{l'}$ if $l \neq l'$;
(3) $\|s_{k,i}\|^2 = b_k, \qquad i = 1, 2, \dots, n_k;\ k = 1, 2, \dots, \nu_m.$
Then the real vector $b := (b_1, b_2, \dots, b_{\nu_m})$ is called the *index* of the admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m.

We now specify a Hermitian metric on V_m. For the maximal compact subgroup T_c of T above, let $\mathcal{S}$ be the set $(\neq \emptyset)$ of all T_c-invariant Kähler forms in the class $c_1(L)_{\mathbb{R}}$. Let $\omega \in \mathcal{S}$, and choose a Hermitian metric h for L such that $\omega = c_1(L; h)$. Define a Hermitian metric on V_m by

$$(s, s')_{L^2} := \int_M (s, s')_{h^m}\, \omega^n, \qquad s, s' \in V_m, \tag{2.4}$$

where $(s, s')_{h^m}$ denotes the function on M obtained as the pointwise inner product of s, s' by the Hermitian metric h^m on L^m. Now, let us consider the situation that V_m has the Hermitian metric (2.4). Then

$$V(\chi_k) \perp V(\chi_l), \qquad k \neq l,$$

and define a maximal compact subgroup $(G_m)_c$ of G_m by $(G_m)_c := \prod_{k=1}^{\nu_m} \mathrm{SU}(V(\chi_k))$. Again by this Hermitian metric $(\ ,\ )_{L^2}$, let $\{s_0, s_1, \dots, s_{N_m}\}$ be an admissible normal basis for V_m of a given index b. Put

$$E_{\omega,b} := \sum_{i=0}^{N_m} |s_i|^2_{h^m}, \tag{2.5}$$

where $|s|_{h^m} := (s, s)_{h^m}$ for all $s \in V_m$. Then $E_{\omega,b}$ depends only on ω and b. Namely, once ω and b are fixed, $E_{\omega,b}$ is independent of the choice of an admissible normal basis for $V(\chi_k)$ of index b. Fix a positive integer m such that L^m is very ample.

DEFINITION 2.6. An element ω in $\mathcal{S}$ is called a *critial metric relative to T*, if there exists an admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m such that the associated function $E_{\omega,b}$ on M is constant for the index b of the admissible normal basis. This generalizes a *critical metric* of Zhang [39] (see also [5]) who treated the case $T = \{1\}$. If ω is a critical metric relative to T, then by integrating the equality (2.5) over M, we see that the constant $E_{\omega,b}$ is $(N_m + 1)/c_1(L)^n[M]$.

For the centralizer $Z_H(T)$ of T in H, let $Z_H(T)^0$ be its identity component. For m as above, the following generalization of a result in [39] is crucial to our study of stability:

Theorem A. *The subvariety M_m of $\mathbb{P}(V_m)$ is stable relative to T if and only if there exists a critical metric $\omega \in \mathcal{S}$ relative to T. Moreover, for a fixed index b, a critical metric ω in $\mathcal{S}$ relative to T with constant $E_{\omega,b}$ is unique up to the action of $Z_H(T)^0$.*

We now fix a maximal compact connected subgroup K of H. The corresponding Lie subalgebra of $\mathfrak{h}$ is denoted by $\mathfrak{k}$. Let $\mathcal{S}_K$ denote the set of all Kähler forms ω in the class $c_1(L)_{\mathbb{R}}$ such that the identity component of the group of the isometries of (M, ω) coincides with K. Then $\mathcal{S}_K \neq \emptyset$, and an extremal Kähler metric, if any, in the class $c_1(L)_{\mathbb{R}}$ is always in H-orbits of elements of $\mathcal{S}_K$. For each $\omega \in \mathcal{S}_K$, we write

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta} g_{\alpha\bar{\beta}}\, dz^{\alpha} \wedge dz^{\bar{\beta}}$$

in terms of a system $(z^1, \dots, z^n)$ of holomorphic local coordinates on M. let $\mathcal{K}_\omega$ be the space of all real-valued smooth functions u on M such that $\int_M u\omega^n = 0$ and that

$$\operatorname{grad}^{\mathbb{C}}_{\omega} u := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g^{\bar{\beta}\alpha} \frac{\partial u}{\partial z^{\bar{\beta}}} \frac{\partial}{\partial z^{\alpha}}$$

is a holomorphic vector field on M. Then $\mathcal{K}_\omega$ forms a real Lie subalgebra of $\mathfrak{h}$ by the Poisson bracket for (M, ω). We then have the Lie algebra isomorphism

$$\mathcal{K}_\omega \cong \mathfrak{k}, \qquad u \leftrightarrow \operatorname{grad}^{\mathbb{C}}_{\omega} u.$$

For the space $C^\infty(M)_{\mathbb{R}}$ of real-valued smooth functions on M, we consider the inner product defined by $(u_1, u_2)_\omega := \int_M u_1 u_2\, \omega^n$ for $u_1, u_2 \in C^\infty(M)_{\mathbb{R}}$. Let $\mathrm{pr}\colon C^\infty(M)_{\mathbb{R}} \to \mathcal{K}_\omega$ be the orthogonal projection. Let $\mathfrak{z}$ be the center of $\mathfrak{k}$. Then the vector field

$$\mathcal{V} := \operatorname{grad}^{\mathbb{C}}_{\omega} \mathrm{pr}(\sigma_\omega) \in \mathfrak{z}$$

is callled the *extremal Kähler vector field* of (M, ω), where σ_ω denotes the scalar curvature of ω. Then $\mathcal{V}$ is independent of the choice of ω in $\mathcal{S}$, and satisfies $\exp(2\pi\gamma\mathcal{V}) = 1$ for some positive integer γ (cf. [13], [32]). Next, since we have an H-linearization of L, there exists a natural inclusion $H \subset \mathrm{GL}(V_m)$. By passing to the Lie algebras, we obtain

$$\mathfrak{h} \subset \mathfrak{gl}(V_m).$$

Take a Hermitian metric h for L such that the corresponding first Chern form $c_1(L; h)$ is ω. As in [23, (1.4.1)], the infinitesimal $\mathfrak{h}$-action on L induces an infinitesimal

$\mathfrak{h}$-action on the complexification $\mathcal{H}_m^{\mathbb{C}}$ of the space of all Hermitian metrics $\mathcal{H}_m$ on the line bundle L^m. The Futaki-Morita character $F\colon \mathfrak{h} \to \mathbb{C}$ is given by

$$F(\mathcal{Y}) := \frac{\sqrt{-1}}{2\pi} \int_M h^{-1}(\mathcal{Y}h)\omega^n,$$

which is independent of the choice of h (see for instance [15]). For the identity component Z of the center of K, we consider its complexification $Z^{\mathbb{C}}$ in H. Then the corresponding Lie algebra is just the complexification $\mathfrak{z}^{\mathbb{C}}$ of $\mathfrak{z}$ above. We now consider the set Δ of all algebraic tori in $Z^{\mathbb{C}}$. Let $T \in \Delta$. Put

$$q := \frac{1}{m}.$$

For $\omega = c_1(L; h) \in \mathcal{S}_K$, we consider the Hermitian metric (2.4) for V_m. We then choose an admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for V_m of index $(1, 1, \dots, 1)$. By the asymptotic expansion of Tian-Zelditch (cf. [33], [38]; see also [4]) for $m \gg 1$, there exist real-valued smooth functions $a_k(\omega)$, $k = 1, 2, \dots,$ on M such that

$$\frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|^2_{h^m} = 1 + a_1(\omega)q + a_2(\omega)q^2 + \cdots . \tag{2.7}$$

Then $a_1(\omega) = \sigma_\omega/2$ by a result of Lu [20]. Let $\mathcal{Y} \in \mathfrak{t}_{\mathbb{R}}$, and put $g := \exp^{\mathbb{C}} \mathcal{Y} \in T$, where the element $\exp(\mathcal{Y}/2)$ in T is written as $\exp^{\mathbb{C}} \mathcal{Y}$ by abuse of terminology. Recall that the T-action on V_m is a right action, though it can be viewed also as a left action. Put $h_g := h \cdot g$ for simplicity. Using the notation in Definition 2.3, we write $s_{k,i} = s_{l(k,i)}$, $k = 1, 2, \dots, \nu_m$; $i = 1, 2, \dots, n_k$. Then for a fixed k, $\int_M |s_{k,i}|^2_{h_g^m}\, g^*\omega^n = |\chi_k(\exp^{\mathbb{C}} \mathcal{Y})|^{-2}$ is independent of the choice of i. Put

$$Z(q, \omega; \mathcal{Y}) := \frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|^2_{h_g^m} = g^* \left\{ \frac{n!}{m^n} \sum_{k=1}^{\nu_m} |\chi_k(\exp^{\mathbb{C}} \mathcal{Y})|^{-2} \sum_{i=1}^{n_k} |s_{k,i}|^2_{h^m} \right\}, \quad \mathcal{Y} \in \mathfrak{t}_{\mathbb{R}}.$$

For extremal Kähler manifolds, the following generalization of [26] allows us to approximate arbitrarily some critical metrics relative to T:

Theorem B. *Let $\omega_0 = c_1(L; h_0)$ be an extremal Kähler metric in the class $c_1(L)_{\mathbb{R}}$ with extremal Kähler vector field $\mathcal{V}$. Then for some $T \in \Delta$, there exist a sequence of vector fields $\mathcal{Y}_k \in \mathfrak{t}_{\mathbb{R}}$, a formal power series C_q in q with real coefficients* (*cf.* Section 6), *and smooth real-valued functions φ_k, $k = 1, 2, \dots,$ on M such that*

$$Z(q, \omega(l); \mathcal{Y}(l)) = C_q + 0(q^{l+2}), \tag{2.8}$$

where $\mathcal{Y}(l) := (\sqrt{-1}\,\mathcal{V}/2)\, q^2 + \sum_{k=1}^{l} q^{k+2}\mathcal{Y}_k$, $h(l) := h_0 \exp(-\sum_{k=1}^{l} q^k \varphi_k)$, and $\omega(l) := c_1(L; h(l))$.

The equality (2.8) above means that there exists a positive real constant A_l independent of q such that $\|Z(q,\omega(l);\mathcal{Y}(l)) - C_q\|_{C^0(M)} \leq A_l q^{l+2}$ for all q with $0 \leq q \leq 1$. By [38], for every nonnegative integer j, a choice of a larger constant $A = A_{j,l} > 0$ keeps Theorem B still valid even if the $C^0(M)$-norm is replaced by the $C^j(M)$-norm.

3. A stability criterion

In this section, some stability criterion will be given as a preliminary. In a forthcoming paper [27], we actually use a stronger version of Theorem 3.2 which guarantees the stability only by checking the closedness of orbits through a point for special one-parameter subgroups "perpendicular" to the isotropy subgroup. Now, for a connected reductive algebraic group G, defined over $\mathbb{C}$, we consider a representation of G on an N-dimensional complex vector space W. We fix a maximal compact subgroup G_c of G. Moreover, let $\mathbb{C}^*$ be a one-dimensional algebraic torus with the maximal compact subgroup S^1.

DEFINITION 3.1. (a) An algebraic group homomorphism $\lambda\colon \mathbb{C}^* \to G$ is said to be a *special one-parameter subgroup* of G, if the image $\lambda(S^1)$ is contained in G_c.
(b) A point $w \neq 0$ in W is said to be *stable*, if the orbit $G \cdot w$ is closed in W.

Later, we apply the following stability criterion to the case where $W = W_m^*$ and $G = G_m$. Let $w \neq 0$ be a point in W.

Theorem 3.2. *A point w as above is stable if and only if there exists a point w' in the orbit $G \cdot w$ of w such that $\lambda(\mathbb{C}^*) \cdot w'$ is closed in W for every special one-parameter subgroup $\lambda\colon \mathbb{C}^* \to G$ of G.*

Proof. We prove this by induction on $\dim(G \cdot w)$. If $\dim(G \cdot w) = 0$, the statement of the above theorem is obviously true. Hence, fixing a positive integer k, assume that the statement is true for all $0 \neq w \in W$ such that $\dim(G \cdot w) < k$. Now, let $0 \neq w \in W$ be such that $\dim(G \cdot w) = k$, and the proof is reduced to showing the statement for such a point w. Let $\Sigma(G)$ be the set of all special one-parameter subgroups of G. Fix a G_c-invariant Hermitian metric $\| \ \|$ on W. The proof is divided into three steps:

STEP 1. First, we prove "only if" part of Theorem 3.2. Assume that w is stable. Since $G \cdot w$ is closed in W, the nonnegative function on $G \cdot w$ defined by

$$G \cdot w \ni g \cdot w \mapsto \|g \cdot w\| \in \mathbb{R}, \qquad g \in G, \tag{3.3}$$

has a critical point at some point w' in $G \cdot w$. Let $\lambda \in \Sigma(G)$, and it suffices to show the closedness of $\lambda(\mathbb{C}^*) \cdot w'$ in W. We may assume that $\dim \lambda(\mathbb{C}^*) \cdot w' > 0$. Then by using the coordinate system associated to an orthonormal basis for W, we can write

w' as $(w'_0, \dots, w'_r, 0, \dots, 0)$ in such a way that $w'_\alpha \neq 0$ for all $0 \le \alpha \le r$ and that

$$\lambda(e^t) \cdot w' = (e^{t\gamma_0} w'_0, \dots, e^{t\gamma_r} w'_r, 0, \dots, 0), \qquad t \in \mathbb{C},$$

where γ_α, $\alpha = 0, 1, \dots, r$, are integers independent of the choice of t in $\mathbb{C}$. Since the closed orbit $G \cdot w$ does not contain the origin of W, the inclusion $\lambda(\mathbb{C}^*) \cdot w' \subset G \cdot w$ shows that $r \ge 1$ and that the coincidence $\gamma_0 = \gamma_1 = \cdots = \gamma_r$ cannot occur. In particular,

$$f(t) := \log \|\lambda(e^t) \cdot w'\|^2 = \log\left(e^{2t\gamma_0}|w'_0|^2 + e^{2t\gamma_1}|w'_1|^2 + \cdots + e^{2t\gamma_r}|w'_r|^2\right), \quad t \in \mathbb{R},$$

satisfies $f''(t) > 0$ for all t. Moreover, since the function in (3.3) has a critical point at w', we have $f'(0) = 0$. It now follows that $\lim_{t\to+\infty} f(t) = +\infty$ and $\lim_{t\to-\infty} f(t) = +\infty$. Hence $\lambda(\mathbb{C}^*) \cdot w'$ is closed in W, as required.

STEP 2. To prove "if" part of Theorem 3.2, we may assume that $w = w'$ without loss of generality. Hence, suppose that $\lambda(\mathbb{C}^*) \cdot w$ is closed in W for every $\lambda \in \Sigma(G)$. It then suffices to show that $G \cdot w$ is closed in W. For contradiction, assume that $G \cdot w$ is not closed in W. Since the closure of $G \cdot w$ in W always contains a closed orbit O_1 in W, by $\dim O_1 < \dim(G \cdot w) = k$, the induction hypothesis shows that there exists a point $\hat{w} \in O_1$ such that

$$\lambda(\mathbb{C}^*) \cdot \hat{w} \text{ is closed in } W \text{ for every } \lambda \in \Sigma(G). \tag{3.4}$$

Moreover, there exist elements g_i, $i = 1, 2, \dots$, in G such that $g_i \cdot w$ converges to $\hat{w}$ in W. Then for each i, we can write $g_i = \kappa'_i \cdot \exp(2\pi A_i) \cdot \kappa_i$ for some κ_i, $\kappa'_i \in G_c$ and for some $A_i \in \mathfrak{a}$, where $2\pi\sqrt{-1}\,\mathfrak{a}$ is the Lie algebra of some maximal compact torus in G_c. Let $2\pi\sqrt{-1}\,\mathfrak{a}_{\mathbb{Z}}$ be the kernel of the exponential map of the Lie algebra $2\pi\sqrt{-1}\,\mathfrak{a}$, and put $\mathfrak{a}_{\mathbb{Q}} := \mathfrak{a}_{\mathbb{Z}} \otimes \mathbb{Q}$. Replacing $\{\kappa_i\}$ by its subsequence if necessary, we may assume that

$$\kappa_i \to \kappa_\infty \quad \text{and} \quad \{\exp(2\pi A_i) \cdot \kappa_i\} \cdot w \to w_\infty, \qquad \text{as } i \to \infty, \tag{3.5}$$

for some $\kappa_\infty \in G_c$ and $w_\infty \in G_c \cdot \hat{w}$. Then by (3.4), the orbit $\lambda(\mathbb{C}^*) \cdot w_\infty$ is also closed in W for every $\lambda \in \Sigma(G)$. Let $\mathfrak{a}_\infty$ denote the Lie subalgebra of $\mathfrak{a}$ consisting of all elements in $\mathfrak{a}$ whose associated vector fields on W vanish at $\kappa_\infty \cdot w$. For a Euclidean metric on $\mathfrak{a}$ induced from a suitable bilinear form on $\mathfrak{a}_{\mathbb{Q}}$ defined over $\mathbb{Q}$, we write $\mathfrak{a}$ as a direct sum $\mathfrak{a}_\infty^\perp \oplus \mathfrak{a}_\infty$, where $\mathfrak{a}_\infty^\perp$ is the orthogonal complement of $\mathfrak{a}_\infty$ in $\mathfrak{a}$. Let $\overline{A}_i$ be the image of A_i under the orthogonal projection

$$\mathrm{pr}_1 \colon \mathfrak{a}\,(= \mathfrak{a}_\infty^\perp \oplus \mathfrak{a}_\infty) \to \mathfrak{a}_\infty^\perp, \qquad A \mapsto \overline{A} := \mathrm{pr}_1(A).$$

Note that $\{\exp(2\pi A_i) \cdot \kappa_\infty\} \cdot w = \{\exp(2\pi \overline{A}_i) \cdot \kappa_\infty\} \cdot w$. Hence,

$$\limsup_{i\to\infty} \|\exp\left\{2\pi \mathrm{Ad}(\kappa_\infty^{-1})\overline{A}_i\right\} \cdot w\| = \limsup_{i\to\infty} \|\{\exp(2\pi A_i) \cdot \kappa_\infty\} \cdot w\| \tag{3.6}$$

$$\leq \lim_{i\to\infty} \| \{\exp(2\pi A_i)\cdot\kappa_i\}\cdot w\| = \|w_\infty\| < +\infty.$$

STEP 3. Since $\lambda(\mathbb{C}^*)\cdot w$ is closed in W for every $\lambda \in \Sigma(G)$, by the boundedness in (3.6), $\{\bar{A}_i\}$ is a bounded sequence in $\mathfrak{a}_\infty^\perp$ (see Remark 3.7 below). Hence, for some element A_∞ in $\mathfrak{a}_\infty^\perp$, replacing $\{\bar{A}_i\}$ by its subsequence if necessary, we may assume that $\bar{A}_i \to A_\infty$ as $i \to \infty$. Then by (3.5),

$$w_\infty = \lim_{i\to\infty}\{\exp(2\pi\bar{A}_i)\cdot\kappa_i\}\cdot w = \{\exp(2\pi\bar{A}_\infty)\cdot\kappa_\infty\}\cdot w.$$

Since we have $\exp(2\pi\bar{A}_\infty)\in G$, the point w_∞ in O_1 belongs to the orbit $G\cdot w$. This contradicts $O_1\cap(G\cdot w)=\emptyset$, as required. The proof of Lemma 3.2 is now complete. □

REMARK 3.7. The boundedness of the sequence $\{\bar{A}_i\}$ in $\mathfrak{a}_\infty^\perp$ in Step 3 above can be seen as follows: For contradiction, we assume that the sequence $\{\bar{A}_i\}$ is unbounded. Put $v := \kappa_\infty\cdot w$ for simplicity. Then by (3.6), we first observe that

$$\limsup_{i\to\infty}\|\exp(2\pi\bar{A}_i)\cdot v\| < +\infty. \tag{3.8}$$

Since $2\pi\sqrt{-1}\,\mathfrak{a}_\infty$ is the Lie algebra of the isotropy subgroup of the compact torus $\exp(2\pi\sqrt{-1}\,\mathfrak{a})$ at v, both $\mathfrak{a}_\infty$ and $\mathfrak{a}_\infty^\perp$ are defined over $\mathbb{Q}$ in $\mathfrak{a}$. By choosing a complex coordinate system of W, we can write v as $(v_0,\dots,v_r,0,\dots,0)$ for some integer r with $0\le r\le \dim W-1$ such that $v_\alpha\ne 0$ for all $0\le\alpha\le r$ and that

$$\exp(2\pi\bar{A})\cdot v = (e^{2\pi\chi_0(\bar{A})}v_0,\dots,e^{2\pi\chi_r(\bar{A})}v_r,0,\dots,0),\qquad \bar{A}\in\mathfrak{a}_\infty^\perp, \tag{3.9}$$

where $\chi_\alpha\colon \mathfrak{a}_\infty^\perp\to\mathbb{R}$, $\alpha=0,1,\dots,r$, are additive characters defined over $\mathbb{Q}$. Put $n := \dim_{\mathbb{R}}\mathfrak{a}_\infty^\perp$, and let $(\mathfrak{a}_\infty^\perp)_{\mathbb{Q}}$ denote the set of all rational points in $\mathfrak{a}_\infty^\perp$. Let us now identify

$$\mathfrak{a}_\infty^\perp = \mathbb{R}^n \qquad\text{and}\qquad (\mathfrak{a}_\infty^\perp)_{\mathbb{Q}} = \mathbb{Q}^n,$$

as vector spaces. Since the orbit $\lambda(\mathbb{C}^*)\cdot w$ is closed in W for all special one-parameter subgroups $\lambda\colon\mathbb{C}^*\to G$ of G, the same thing is true also for $\lambda(\mathbb{C}^*)\cdot v$. Hence,

$$\mathbb{Q}^n\setminus\{0\}\subset\bigcup_{\alpha,\beta=0}^{r}U_{\alpha\beta}, \tag{3.10}$$

where $U_{\alpha\beta} := \{\,A\in\mathfrak{a}\,;\,\chi_\alpha(A) > 0 > \chi_\beta(A)\,\}$. Note that the boundaries of the open sets $U_{\alpha\beta}$, $1\le\alpha\le r$, $1\le\beta\le r$, in $\mathbb{R}^n$ sit in the union of $\mathbb{Q}$-hyperplanes

$$H_\alpha := \{\,\chi_\alpha = 0\,\},\qquad \alpha = 0,1,\dots,r,$$

in $\mathbb{R}^r$. Since an intersection of any finite number of hyperplanes H_α, $\alpha = 0, 1, \dots, r$, has dense rational points, (3.10) above easily implies

$$\mathbb{R}^n \setminus \{0\} = \bigcup_{\alpha,\beta=0}^{r} U_{\alpha\beta}. \tag{3.11}$$

Replacing $\{\overline{A}_i\}$ by its suitable subsequence if necessary, we may assume that there exists an element A_∞ in $\mathfrak{a}_\infty^\perp$ $(= \mathbb{R}^n)$ with $\|A_\infty\|_{\mathfrak{a}} = 1$ such that

$$\lim_{i\to\infty} \frac{\overline{A}_i}{\|A_i\|_{\mathfrak{a}}} = A_\infty,$$

where $\|\ \|_{\mathfrak{a}}$ denotes the Euclidean norm for $\mathfrak{a}$ as in Step 2 in the proof of Theorem 3.2. By (3.11), there exist $\alpha, \beta \in \{0, 1, \dots, r\}$ such that $A_\infty \in U_{\alpha\beta}$, and in particular $\chi_\alpha(A_\infty) > 0$. On the other hand, $\limsup_{i\to\infty} \|\overline{A}_i\|_{\mathfrak{a}} = +\infty$ by our assumption. Thus,

$$\limsup_{i\to\infty} \chi_\alpha(\overline{A}_i) = \limsup_{i\to\infty} \{\, \|\overline{A}_i\|_{\mathfrak{a}} \cdot \chi_\alpha(\overline{A}_i/\|\overline{A}_i\|_{\mathfrak{a}})\,\} = (\limsup_{i\to\infty} \|\overline{A}_i\|_{\mathfrak{a}})\, \chi_\alpha(A_\infty) = +\infty,$$

in contradiction to (3.8) and (3.9), as required.

4. The Chow norm

Take an algebraic torus $T \subset \mathrm{Aut}^0(M)$, and let $\iota\colon \mathrm{SL}(V_m) \to \mathrm{PGL}(V_m)$ be the natural projection, where we regard $\mathrm{Aut}^0(M)$ as a subgroup of $\mathrm{PGL}(V_m)$ via the Kodaira embedding $\Phi_m\colon M \hookrightarrow \mathbb{P}^*(V_m)$, $m \gg 1$. In this section, we fix a $\tilde{T}_c$-invariant Hermitian metric ρ on V_m, where $\tilde{T}_c$ is the maximal compact subgroup of $\tilde{T} := \iota^{-1}(T)$. Obviously, in terms of this metric, $V(\chi_k) \perp V(\chi_l)$ if $k \neq l$. Using Deligne's pairings (cf. [8, 8.3]), Zhang ([39, 1.5]) defined a special type of norm on W_m^*, called the *Chow norm*, as a nonnegative real-valued function

$$W_m^* \ni w \longmapsto \|w\|_{\mathrm{CH}(\rho)} \in \mathbb{R}_{\geq 0}, \tag{4.1}$$

with very significant properties described below. First, this is a norm, so that it has the only zero at the origin satisfying the homogeneity condition

$$\|c\, w\|_{\mathrm{CH}(\rho)} = |c| \cdot \|w\|_{\mathrm{CH}(\rho)} \qquad \text{for all } (c, w) \in \mathbb{C} \times W_m^*.$$

For the group $\mathrm{SL}(V_m)$, we consider the maximal compact subgroup $\mathrm{SU}(V_m; \rho)$. For a special one-parameter subgroup

$$\lambda\colon \mathbb{C}^* \to \mathrm{SL}(V_m)$$

of $\mathrm{SL}(V_m)$, there exist integers γ_j, $j = 0, 1, \dots, N_m$, and an orthonormal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for (V_m, ρ) such that, for all j,

$$\lambda_z \cdot s_j = e^{z\gamma_j} s_j, \qquad z \in \mathbb{C}, \tag{4.2}$$

where $\lambda_z := \lambda(e^z)$. Recall that the subvariety M_m in $\mathbb{P}^*(V_m)$ is the image of the Kodaira embedding $\Phi_m \colon M \hookrightarrow \mathbb{P}^*(V_m)$ defined by

$$\Phi_m(p) = (s_0(p) : s_1(p) : \cdots : s_{N_m}(p)), \qquad p \in M, \tag{4.3}$$

where $\mathbb{P}^*(V_m)$ is identified with $\mathbb{P}^{N_m}(\mathbb{C}) = \{(z_0 : z_1 : \cdots : z_{N_m})\}$. Put $M_{m,t} := \lambda_t(M_m)$ for each $t \in \mathbb{R}$. As in Section 2, $\hat{M}_{m,t} := \lambda_t \cdot \hat{M}_m$ is the nonzero point of W_m^* sitting over the Chow point of the irreducible reduced cycle $M_{m,t}$ on $\mathbb{P}^*(V_m)$. Then (cf. [39, 1.4, 3.4.1])

$$\frac{d}{dt}\left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\right) = (n+1)\int_M \frac{\sum_{j=0}^{N_m} \gamma_j\, |\lambda_t \cdot s_j|^2}{\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2} (\Phi_m^* \lambda_t^* \omega_{\mathrm{FS}})^n, \tag{4.4}$$

where ω_{FS} is the Fubini-Study form $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{j=0}^{N_m} |z_j|^2)$ on $\mathbb{P}^*(V_m)$, and we regard λ_t as a linear transformation of $\mathbb{P}^*(V_m)$ induced by (4.2). Note that the term $\Phi_m^*\lambda_t^*\omega_{\mathrm{FS}}$ above is just $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2)$. Put $\Gamma := 2\pi\sqrt{-1}\,\mathbb{Z}$. By setting

$$\mathbb{C}/\Gamma = \{\, t + \sqrt{-1}\,\theta \,;\, t \in \mathbb{R}, \theta \in \mathbb{R}/(2\pi\mathbb{Z}) \,\},$$

we consider the complexified situation. Let $\eta \colon M \times \mathbb{C}/\Gamma \to \mathbb{P}^*(V_m)$ be the map sending each $(p, t + \sqrt{-1}\,\theta)$ in $M \times \mathbb{C}/\Gamma$ to $\lambda_{t+\sqrt{-1}\,\theta} \cdot \Phi_m(p)$ in $\mathbb{P}^*(V_m)$. For simplicity, we put

$$Q := \frac{\sum_{j=0}^{N_m} \gamma_j e^{2t\gamma_j} |s_j|^2}{\sum_{j=0}^{N_m} e^{2t\gamma_j} |s_j|^2} \left(= \frac{\sum_{j=0}^{N_m} \gamma_j\, |\lambda_t \cdot s_j|^2}{\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2} \right).$$

We further put $z := t + \sqrt{-1}\,\theta$. For the time being, on the total complex manifold $M \times \mathbb{C}/\Gamma$, the ∂-operator and the $\bar{\partial}$-operator will be written simply as ∂ and $\bar{\partial}$ respectively, while on M, they will be denoted by ∂_M and $\bar{\partial}_M$ respectively. Then

$$\eta^*\omega_{\mathrm{FS}} = \Phi_m^*\lambda_t^*\omega_{\mathrm{FS}} + \frac{\sqrt{-1}}{2\pi}(\partial_M Q \wedge d\bar{z} + dz \wedge \bar{\partial}_M Q) + \frac{\sqrt{-1}}{4\pi}\frac{\partial Q}{\partial t}\, dz \wedge d\bar{z}.$$

For $0 \neq r \in \mathbb{R}$, we consider the 1-chain $I_r := [0, r]$, where $[0, r]$ means the 1-chain $-[r, 0]$ if $r < 0$. Let $\mathrm{pr} \colon \mathbb{C}/\Gamma \to \mathbb{R}$ be the mapping sending each $t + \sqrt{-1}\,\theta$ to t. We now put $B_r := \mathrm{pr}^* I_r$. Then $\int_{M \times B_r} \eta^*\omega_{\mathrm{FS}}^{n+1}$ is nothing but

$$(n+1)\int_0^r dt \int_M \left(\frac{\partial Q}{\partial t}\, \Phi_m^*\lambda_t^*\omega_{\mathrm{FS}}^n + \frac{\sqrt{-1}}{\pi}\,\bar{\partial}_M Q \wedge \partial_M Q \wedge n\,\Phi_m^*\lambda_t^*\omega_{\mathrm{FS}}^{n-1} \right)$$

$$= \int_0^r \frac{d^2}{dt^2} \left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\right) dt = \frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\right) \Big|_{t=0}^{t=r},$$

and by assuming $r \geq 0$, we obtain the following convexity formula:

Theorem 4.5.
$$\frac{d}{dt} \left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\right) \Big|_{t=0}^{t=r} = \int_{M \times B_r} \eta^* \omega_{\mathrm{FS}}^{n+1} \geq 0.$$

REMARK 4.6. Besides special one-parameter subgroups of $\mathrm{SL}(V_m)$, we also consider a little more general smooth path λ_t, $t \in \mathbb{R}$, in $\mathrm{GL}(V_m)$ written explicitly by

$$\lambda_t \cdot s_j = e^{t\gamma_j + \delta_j} s_j, \qquad j = 0, 1, \ldots, N_m,$$

where γ_j, $\delta_j \in \mathbb{R}$ are not necessarily rational. In this case also, we easily see that the formula (4.4) and Theorem 4.5 are still valid.

5. Proof of Theorem A

The statement of Theorem A is divided into "if" part, "only if" part, and the uniqueness part. We shall prove these three parts separately.

Proof of "*if*" part. Let $\omega \in \mathcal{S}$ be a critical metric relative to T. Then by Definition 2.6, in terms of the Hermitian metric defined in (2.4), there exists an admissible normal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for V_m of index b such that the associated function $E_{\omega,b}$ has a constant value C on M. By operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on the identity $E_{\omega,b} = C$, we have

$$\Phi_m^* \omega_{\mathrm{FS}} = m\,\omega. \tag{5.1}$$

Besides the Hermitian metric defined in (2.4), we shall now define another Hermitian metric on V_m. By the identification $V_m \cong \mathbb{C}^{N_m}$ via the basis $\{s_0, s_1, \ldots, s_{N_m}\}$, the standard Hermitian metric on $\mathbb{C}^{N_m}$ induces a Hermitian metric ρ on V_m. As a maximal compact subgroup of G_m, we choose $(G_m)_c$ as in Section 2 by using the metric defined in (2.4). Then the Hermitian metric ρ is also preserved by the $(G_m)_c$-action on V_m. Let

$$\lambda \colon \mathbb{C}^* \to G_m$$

be a special one-parameter subgroup of G_m. By the notation $l(k,i)$ as in Definition 2.3, we put $s_{k,i} := s_{l(k,i)}$. If necessary, replacing $\{s_0, s_1, \ldots, s_{N_m}\}$ by another admissible normal basis for V_m of the same index b, we may assume without loss of generality that there exist integers $\gamma_{k,i}$, $i = 1, 2, \ldots, n_k$, satisfying

$$\lambda_t \cdot s_{k,i} = e^{t\gamma_{k,i}} s_{k,i}, \qquad t \in \mathbb{C}, \tag{5.2}$$

where $\lambda_t := \lambda(e^t)$ is as in (4.2), and the equality $\sum_{i=1}^{n_k} \gamma_{k,i} = 0$ is required to hold for every k. Put $\gamma_{k,i} = \gamma_{l(k,i)}$ for simplicity. Then by (4.4) and (5.1),

$$\frac{d}{dt}\left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\right)_{|t=0} = (n+1)\int_M \frac{\sum_{j=0}^{N_m} \gamma_j |s_j|^2}{\sum_{j=0}^{N_m} |s_j|^2} (\Phi_m^* \omega_{\mathrm{FS}})^n$$

$$= (n+1)\, m^n \int_M \frac{\sum_{j=0}^{N_m} \gamma_j |s_j|^2_{h^m}}{\sum_{j=0}^{N_m} |s_j|^2_{h^m}}\, \omega^n = (n+1)\, m^n \int_M \frac{\sum_{k=1}^{\nu_m} (\sum_{i=1}^{n_k} \gamma_{k,i} |s_i|^2_{h^m})}{E_{\omega,b}} \omega^n$$

$$= \frac{(n+1)\, m^n}{C} \int_M \sum_{k=1}^{\nu_m} \left(\sum_{i=1}^{n_k} \gamma_{k,i} |s_i|^2_{h^m} \right) \omega^n = \frac{(n+1)\, m^n}{C} \sum_{k=1}^{\nu_m} b_k \left(\sum_{i=1}^{n_k} \gamma_{k,i} \right) = 0.$$

Note also that, by Theorem 4.5, we have $c := (d^2/dt^2)(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)})_{|t=0} \geq 0$.

CASE 1. If c is positive, then $\lim_{t\to-\infty} \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)} = +\infty = \lim_{t\to+\infty} \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}$, and in particular $\lambda(\mathbb{C}^*) \cdot \hat{M}_m$ is closed.

CASE 2. If c is zero, then by applying Theorem 4.5 infinitesimally, we see that $\lambda(\mathbb{C}^*)$ preserves the subvariety M_m in $\mathbb{P}^*(V_m)$, and moreover by

$$\frac{d}{dt}\left(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\right)_{|t=0} = 0,$$

the isotropy representation of $\lambda(\mathbb{C}^*)$ on the complex line $\mathbb{C}\hat{M}_m$ is trivial. Hence, $\lambda(\mathbb{C}^*) \cdot \hat{M}_m$ is a single point, and in particular closed.

Thus, these two cases together with Theorem 3.2 show that the subvariety M_m of $\mathbb{P}^*(V_m)$ is stable relative to T, as required. □

REMARK 5.3. About the one-parameter subgroup $\{\lambda_t\,;\, t \in \mathbb{R}\}$ of G_m, we consider a more general situation that $\gamma_{k,i}$ in (5.2) are just real numbers which are not necessarily rational. The above computation together with Remark 4.6 shows that, even in this case, $(d/dt)_{t=0}(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)})$ vanishes.

Proof of "*only if*" part. Assume that the subvariety M_m in $\mathbb{P}^*(V_m)$ is stable relative to T. Take a Hermitian metric ρ for V_m such that $V(\chi_k) \perp V(\chi_l)$ for $k \neq l$. For this ρ, we consider the associated Chow norm. Since the orbit $G_m \cdot \hat{M}_m$ is closed in W_m, the Chow norm restricted to this orbit attains an abosolute minimum. Hence, for some $g_0 \in G_m$,

$$0 \neq \|g_0 \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} \leq \|g \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)}, \qquad \text{for all } g \in G_m.$$

By choosing an admissible normal basis $\{s_0, s_1, \dots, s_{N_m}\}$ for $(V_m; \rho)$ of index $(1, 1, \dots, 1)$, we identify V_m with $\mathbb{C}^{N_m} = \{(z_0, z_1, \dots, z_{N_m})\}$. Then $\mathrm{SL}(V_m)$ is identified with $\mathrm{SL}(N_m+1;\mathbb{C})$. Let $\mathfrak{g}_m$ be the Lie subalgebra of $\mathfrak{sl}(N_m+1;\mathbb{C})$ associated to the Lie subgroup G_m of $\mathrm{SL}(N_m+1;\mathbb{C})$. We can now write $g_0 = \kappa' \cdot \exp\{\mathrm{Ad}(\kappa)D\}$ for some κ, $\kappa' \in (G_m)_c$ and a real diagonal matrix D in $\mathfrak{g}_m$. By $\| \exp\{\mathrm{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} = \|g_0 \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)}$, we have

$$
(5.4)\quad \| \exp\{\mathrm{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)} \leq \| \exp\{t\, \mathrm{Ad}(\kappa)A\} \cdot \exp\{\mathrm{Ad}(\kappa)D\} \cdot \hat{M}_m\|_{\mathrm{CH}(\rho)}, \; t \in \mathbb{R},
$$

for every real diagonal matrix A in $\mathfrak{g}_m$. For $j = 0, 1, \dots, N_m$, we write the j-th diagonal element of A and D above as a_j and d_j, respectively. Put $c_j := \exp d_j$ and $s'_j := \kappa^{-1} \cdot s_j$. Then $\{s'_0, s'_1, \dots, s'_{N_m}\}$ is again an admissible normal basis for (V_m, ρ) of index $(1, 1, \dots, 1)$. By the notation in Definition 2.3, we rewrite s'_j, a_j, c_j, z_j as $s'_{k,i}$, $a_{k,i}$, $c_{k,i}$, $z_{k,i}$ by

$$
s'_{k,i} := s'_{l(k,i)}, \quad a_{k,i} := a_{l(k,i)}, \quad c_{k,i} := c_{l(k,i)}, \quad z_{k,i} := z_{l(k,i)},
$$

where $k = 1, 2, \dots, \nu_m$ and $i = 1, 2, \dots, n_k$. By (5.4), the derivative at $t = 0$ of the right-hand side of (5.4) vanishes. Hence by (4.4) together with Remark 4.6, fixing an arbitrary real diagonal matrix A in $\mathfrak{g}_m$, we have

$$
(5.5) \qquad \int_M \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} a_{k,i} c_{k,i}^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2} \Phi_m^*(\Theta^n) = 0,
$$

where we set $\Theta := (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} c_{k,i}^2 |z_{k,i}|^2)$. Let $k_0 \in \{1, 2, \dots, \nu_m\}$ and let i_1, $i_2 \in \{1, 2, \dots, n_k\}$ with $i_1 \neq i_2$. Using Kronecker's delta, we specify the real diagonal matrix A by setting

$$
a_{k,i} = \delta_{kk_0}(\delta_{ii_1} - \delta_{ii_2}), \qquad k = 1, 2, \dots, \nu_m; \; i = 1, 2, \dots, n_k.
$$

Apply (5.5) to this A, and let (i_1, i_2) run through the set of all pairs of two distinct elements in $\{1, 2, \dots, n_k\}$. Then there exists a positive constant $b_k > 0$ independent of the choice of i in $\{1, 2, \dots, n_k\}$ such that

$$
(5.6) \qquad \frac{N_m + 1}{m^n c_1(L)^n[M]} \int_M \frac{c_{k,i}^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2} \Phi_m^*(\Theta^n) = b_k, \qquad k = 1, 2, \dots, \nu_m.
$$

The following identity (5.7) allows us to define (cf. [39]) a Hermitian metric h_{FS} on L^m by

$$
(5.7) \qquad |s|^2_{h_{\mathrm{FS}}} := \frac{(N_m + 1)}{c_1(L)^n[M]} \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |(s, s'_{k,i})_\rho|^2 \, |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2}, \qquad s \in V_m.
$$

Then for this Hermitian metric, it is easily seen that

$$
(5.8) \qquad \sum_{j=0}^{N_m} |c_j s'_j|^2_{h_{\mathrm{FS}}} = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |c_{k,i} s'_{k,i}|^2_{h_{\mathrm{FS}}} = \frac{N_m + 1}{c_1(L)^n[M]}.
$$

By operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on both sides of (5.8), we obtain $\Phi_m^*\Theta = c_1(L^m; h_{\mathrm{FS}})$. We now set $h := (h_{\mathrm{FS}})^{1/m}$ and $\omega := c_1(L; h)$. Then

$$\omega = \frac{1}{m}\Phi_m^*\Theta.$$

Put $s''_{k,i} := c_{k,i}s'_{k,i}$, and as in Definition 2.3, we write $s''_{k,i}$ as $s''_{l(k,i)}$. Then by (5.8), we have the equality $\sum_{j=0}^{N_m} |s''_j|^2_{h^m} = (N_m+1)/c_1(L)^n[M]$. Moreover, in terms of the Hermitian metric defined in (2.4), the equality (5.6) is interpreted as

$$\|s''_{k,i}\|^2_{L^2} = b_k, \qquad k = 1, 2, \dots, \nu_m;\ i = 1, 2, \dots, n_k,$$

while by this together with (5.8) above, we obtain $\sum_{k=1}^{\nu_m} n_k b_k = N_m + 1$, as required. □

Proof of *uniqueness*. Let $\omega = c_1(L; h)$ and $\omega' = c_1(L; h')$ be critical metrics relative to T, and let $\{ s_j\,;\ j = 0, 1, \dots, N_m \}$ and $\{ s'_j\,;\ j = 0, 1, \dots, N_m \}$ be respectively the associated admissible normal bases for V_m of index b. We use the notation in Definition 2.3. Then

$$E_{\omega,b} := \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} |s_{k,i}|^2_{h^m} \quad \text{and} \quad E_{\omega',b} := \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} |s'_{k,i}|^2_{h'^m}$$

take the same constant value $C := (N_m + 1)/c_1(L)^n[M]$ on M. Note here that, by operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on both of these identities, we obtain

$$m\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} |s_{k,i}|^2\right) \quad \text{and} \quad m\omega' = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} |s'_{k,i}|^2\right).$$

If necessary, we replace each $s_{k,i}$ by $\zeta_k s_{k,i}$ for a suitable complex number ζ_k, independent of i, of absolute value 1. Then for each $k = 1, 2, \dots, \nu_m$, we may assume that there exists a matrix $g^{(k)} = (g^{(k)}_{i\,\hat{\imath}}) \in \mathrm{GL}(n_k; \mathbb{C})$ satisfying

$$s'_{k,\hat{\imath}} = \sum_{i=1}^{n_k} s_{k,i}\, g^{(k)}_{i\,\hat{\imath}},$$

where i and $\hat{\imath}$ always run through the integers in $\{1, 2, \dots, n_k\}$. Then the matrix $g^{(k)}$ above is written as $\kappa^{(k)} \cdot (\exp A^{(k)}) \cdot (\kappa'^{(k)})^{-1}$ for some real diagonal matrix $A^{(k)}$ and

$$\kappa^{(k)} = (\kappa^{(k)}_{i\,\hat{\imath}}) \quad \text{and} \quad \kappa'^{(k)} = (\kappa'^{(k)}_{i\,\hat{\imath}})$$

in $\mathrm{SU}(n_k)$. Let $a^{(k)}_i$ be the i-th diagonal element of $A^{(k)}$. For each $\hat{\imath}$, we put $\tilde{s}_{k,\hat{\imath}} := \sum_{i=1}^{n_k} s_{k,i}\, \kappa^{(k)}_{i\,\hat{\imath}}$ and $\tilde{s}'_{k,\hat{\imath}} := \sum_{i=1}^{n_k} s'_{k,i}\, \kappa'^{(k)}_{i\,\hat{\imath}}$. If necessary, we replace the bases

$\{s_{k,1}, s_{k,2}, \ldots, s_{k,n_k}\}$ and $\{s'_{k,1}, s'_{k,2}, \ldots, s'_{k,n_k}\}$ for $V(\chi_k)$ by the bases $\{\tilde{s}_{k,1}, \tilde{s}_{k,2}, \ldots, \tilde{s}_{k,n_k}\}$ and $\{\tilde{s}'_{k,1}, \tilde{s}'_{k,2}, \ldots, \tilde{s}'_{k,n_k}\}$, respectively. Then we may assume, from the beginning, that

$$s'_{k,i} = \{\exp\, a_i^{(k)}\}\, s_{k,i}, \qquad i = 1, 2, \ldots, n_k.$$

We now set $\tau_{k,i} := s_{k,i}/\sqrt{b_k}$, and the Hermitian metric for V_m defined in (2.4) will be denoted by ρ. Then $\{\tau_{k,i}\,;\, k = 1, 2, \ldots, \nu_m, i = 1, 2, \ldots, n_k\}$ is an admissible normal basis of index $(1, 1, \ldots, 1)$ for (V_m, ρ). Let $\{\lambda_t\,;\, t \in \mathbb{C}\}$ be the smooth one-parameter family of elements in $\mathrm{GL}(V_m)$ defined by

$$\lambda_t \cdot \tau_{k,i} = \{\exp(t\, a_i^{(k)})\}\sqrt{b_k}\, \tau_{k,i}, \qquad k = 1, 2, \ldots, \nu_m;\ i = 1, 2, \ldots, n_k.$$

Put $\hat{M}_{m,t} := \lambda_t \cdot \hat{M}_m$, $0 \le t \le 1$. Then by Remark 4.6 applied to the formula (4.4), the derivative $\mathfrak{d}(t) := (d/dt)(\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)})/(n+1)$ at $t \in [0, 1]$ is expressible as

$$\int_M \frac{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} a_i^{(k)} |\lambda_t \cdot \tau_{k,i}|^2}{\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} |\lambda_t \cdot \tau_{k,i}|^2} \left\{ \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} \log \left(\sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} |\lambda_t \cdot \tau_{k,i}|^2 \right) \right\}^n .$$

Hence at $t = 0$, we see that

$$\mathfrak{d}(0) = \int_M \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} \left\{ \frac{a_i^{(k)} |s_{k,i}|^2_{h^m}}{C} \right\} (m\omega)^n = \frac{m^n}{C} \sum_{k=1}^{\nu_m} \left\{ b_k \sum_{i=1}^{n_k} a_i^{(k)} \right\},$$

while at $t = 1$ also, we obtain

$$\mathfrak{d}(1) = \int_M \sum_{k=1}^{\nu_m}\sum_{i=1}^{n_k} \left\{ \frac{a_i^{(k)} |s'_{k,i}|^2_{h'^m}}{C} \right\} (m\omega')^n = \frac{m^n}{C} \sum_{k=1}^{\nu_m} \left\{ b_k \sum_{i=1}^{n_k} a_i^{(k)} \right\}.$$

Thus, $\mathfrak{d}(0)$ coincides with $\mathfrak{d}(1)$, while by Remark 4.6, we see from Theorem 4.5 that $(d^2/dt^2)\{\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\} \ge 0$ on $[0, 1]$. Hence, for all $t \in [0, 1]$,

$$\frac{d^2}{dt^2}\{\log \|\hat{M}_{m,t}\|_{\mathrm{CH}(\rho)}\} = 0, \qquad \text{on } M.$$

By Remark 4.6, the formula in Theorem 4.5 shows that λ_t, $t \in [0, 1]$, belong to H up to a positive scalar multiple. Since λ_1 commutes with T, the uniqueness follows, as required. □

6. Proof of Theorem B

Throughout this section, we assume that the first Chern class $c_1(L)_{\mathbb{R}}$ admits an extremal Kähler metric $\omega_0 = c_1(L; h_0)$. Then by a theorem of Calabi [3], the identity component K of the group of isometries of (M, ω_0) is a maximal compact connected subgroup of H, and we obtain $\omega_0 \in \mathcal{S}_K$ by the notation in the introduction.

DEFINITION 6.1. For a K-invariant Kähler metric $\omega \in \mathcal{S}_K$ on M in the class $c_1(L)_{\mathbb{R}}$, we choose a Hermitian metric h on L such that $\omega = c_1(L; h)$. Then the power series in q given by the right-hand side of (2.8) will be denoted by $\Psi(\omega, q)$. Given ω and q, the power series $\Psi(\omega, q)$ is independent of the choice of h.

Let $\mathcal{D}_0$ be the Lichnérowicz operator as defined in [3], (2.1), for the extremal Kähler manifold (M, ω_0). Then by $\mathcal{V} \in \mathfrak{k}$, the operator $\mathcal{D}_0$ preserves the space $\mathcal{F}$ of all real-valued smooth K-invariant functions φ such that $\int_M \varphi\omega_0^n = 0$. Hence, we regard $\mathcal{D}_0$ just as an operator $\mathcal{D}_0 \colon \mathcal{F} \to \mathcal{F}$, and the kernel in $\mathcal{F}$ of this restricted operator will be denoted simply by $\operatorname{Ker}\mathcal{D}_0$. Then $\operatorname{Ker}\mathcal{D}_0$ is a subspace of $\mathcal{K}_{\omega_0}$, and we have an isomorphism

$$(6.2) \qquad e_0 \colon \operatorname{Ker}\mathcal{D}_0 \cong \mathfrak{z}, \qquad \varphi \leftrightarrow e_0(\varphi) := \operatorname{grad}^{\mathbb{C}}_{\omega_0} \varphi.$$

By the inner product $(\ ,\ )_{\omega_0}$ defined in the introduction, we write $\mathcal{F}$ as an orthogonal direct sum $\operatorname{Ker}\mathcal{D}_0 \oplus \operatorname{Ker}\mathcal{D}_0^{\perp}$. We then consider the orthogonal projection

$$P \colon \mathcal{F}\,(= \operatorname{Ker}\mathcal{D}_0 \oplus \operatorname{Ker}\mathcal{D}_0^{\perp}) \to \operatorname{Ker}\mathcal{D}_0.$$

Now, starting from $\omega(0) := \omega_0$, we inductively define a Hermitian metric $h(k)$, a Kähler metric $\omega(k) := c_1(L; h(k)) \in \mathcal{S}_K$, and a vector field $\mathcal{Y}(k) \in \sqrt{-1}\,\mathfrak{z}$, $k = 1, 2, \dots$, by

$$(6.3) \qquad \begin{cases} h(k) := h(k-1)\exp(-q^k \varphi_k), \\ \omega(k) = \omega(k-1) + \dfrac{\sqrt{-1}}{2\pi}\, q^k\, \partial\bar{\partial}\varphi_k, \\ \mathcal{Y}(k) = \mathcal{Y}(k-1) + \sqrt{-1}\, q^{k+2} e_0(\zeta_k), \end{cases}$$

for appropriate $\varphi_k \in \operatorname{Ker}\mathcal{D}_0^{\perp}$ and $\zeta_k \in \operatorname{Ker}\mathcal{D}_0$, where $\omega(k)$ and $\mathcal{Y}(k)$ are required to satisfy the condition (2.8) with l replaced by k. We now set $g(k) := \exp^{\mathbb{C}} \mathcal{Y}(k)$. Then

$$\begin{aligned} &\{h(k)\cdot g(k)\}^{-m} h(k)^m \{Z(q, \omega(k); \mathcal{Y}(k)) - C_q\} \\ &= \frac{n!}{m^n}\left\{\sum_{j=0}^{N_m} |s_j|_{h(k)^m}\right\} - C_q\{g(k)\cdot h(k)^{-m}\} h(k)^m \\ &= \Psi(\omega(k), q) - C_q\, h(k)^m \{(\exp^{\mathbb{C}} \mathcal{Y}(k)) \cdot h(k)^{-m}\} \\ &= \Psi(\omega(k), q) - C_q \left\{1 + h(k)\frac{\mathcal{Y}(k)}{q}\cdot h(k)^{-1} + R(\mathcal{Y}(k); h(k))\right\}, \end{aligned}$$

where $C_q = 1 + \sum_{k=0}^{\infty} \alpha_k q^{k+1}$ is a power series in q with real coefficients α_k specified later, and the last term $R(\mathcal{Y}(k); h(k)) := h(k)^m \sum_{j=2}^{\infty} \{\mathcal{Y}(k)^j/j!\} \cdot h(k)^{-m}$ will be taken care of as a higher order term in q. Consider the truncated term $C_{q,l} = 1 + \sum_{k=0}^{l} \alpha_k q^{k+1}$. Put

$$\Xi(\omega(k), \mathcal{Y}(k), C_{q,k}) := \Psi(\omega(k), q) - C_{q,k} \left\{ 1 - \frac{\mathcal{Y}(k)}{q} \cdot \log h(k) + R(\mathcal{Y}(k); h(k)) \right\}$$

for each k. Then, in terms of $\omega(k)$, $\mathcal{Y}(k)$ and $C_{q,k}$, the condition (2.8) with l replaced by k is just the equivalence

$$\Xi(\omega(k), \mathcal{Y}(k), C_{q,k}) \equiv 0, \qquad \text{modulo } q^{k+2}. \tag{6.4}$$

We shall now define $\omega(k)$, $\mathcal{Y}(k)$ and $C_{q,k}$ inductively in such a way that the condition (6.4) is satisfied. If $k = 0$, then we set $\omega(0) = \omega_0$, $\mathcal{Y}(0) = \sqrt{-1}\, q^2 \mathcal{V}/2$ and $C_{q,0} = 1 + \alpha_0 q$, where we put $\alpha_0 := \{2c_1(L)^n[M]\}^{-1}\{\int_M \sigma_\omega \omega^n + 2\pi F(\mathcal{V})\}$ for $\omega \in \mathcal{S}_K$. This α_0 is obviously independent of the choice of ω in $\mathcal{S}_K$. Then, modulo q^2,

$$\begin{aligned}
&\Psi(\omega(k), q) - C_{q,0} \left\{1 - \frac{\mathcal{Y}(0)}{q} \cdot \log h(0) + R(\mathcal{Y}(0); h(0))\right\} \\
&\equiv \left(1 + \frac{\sigma_{\omega_0}}{2} q\right) - (1 + \alpha_0 q) \left\{1 - q\, h_0^{-1} \sqrt{-1}\, \frac{\mathcal{V}}{2} \cdot h_0\right\} \\
&\equiv \left(1 + \frac{\sigma_{\omega_0}}{2} q\right) - (1 + \alpha_0 q) \left\{1 + \left(\frac{\sigma_{\omega_0}}{2} - \alpha_0\right) q\right\} \equiv 0,
\end{aligned}$$

and we see that (6.4) is true for $k = 0$. Here, the equality $h_0^{-1}\sqrt{-1}\,(\mathcal{V}/2) \cdot h_0 = \alpha_0 - (\sigma_{\omega_0}/2)$ follows from a routine computation (see for instance [23]).

Hence, let $l \geq 1$ and assume (6.4) for $k = l - 1$. It then suffices to find φ_l, ζ_l and α_l satisfying (6.4) for $k = l$. Put $\mathcal{Y}_l := \sqrt{-1}\, e_0(\zeta_l)$. For each $(\varphi_l, \zeta_l, \alpha_l) \in \operatorname{Ker} \mathcal{D}_0^{\perp} \times \operatorname{Ker} \mathcal{D}_0 \times \mathbb{R}$, we consider

$$\begin{aligned}
\Phi(q; \varphi_l, \zeta_l, \alpha_l) := \Psi &\left(\omega(l-1) + \frac{\sqrt{-1}}{2\pi} q^l \partial\bar{\partial}\varphi_l, q\right) - \left(C_{q,l-1} + \alpha_l q^{l+1}\right) \\
&\times \left\{1 - \left(\frac{\mathcal{Y}(l-1)}{q} + q^{l+1}\mathcal{Y}_l\right) \cdot \log\left\{h(l-1)\exp\left(-q^l \varphi_l\right)\right\}\right. \\
&\left. + R\left(\frac{\mathcal{Y}(l-1)}{q} + q^{l+1}\mathcal{Y}_l;\ h(l-1)\exp\left(-q^l\varphi_l\right)\right)\right\}.
\end{aligned}$$

By the induction hypothesis, $\Xi(\omega(l-1), \mathcal{Y}(l-1), C_{q,l-1}) \equiv 0$ modulo q^{l+1}. Since $\Phi(q; 0, 0, 0) = \Xi(\omega(l-1), \mathcal{Y}(l-1), C_{q,l-1})$, we have

$$\Phi(q; 0, 0, 0) \equiv u_l q^{l+1}, \qquad \text{modulo } q^{l+2},$$

for some real-valued K-invariant smooth function u_l on M. Let $(\varphi_l, \zeta_l, \alpha_k) \in \operatorname{Ker} \mathcal{D}_0^{\perp} \times \operatorname{Ker} \mathcal{D}_0 \times \mathbb{R}$. Since φ_k is K-invariant, by $\mathcal{V} \in \mathfrak{k}$, we see that $\sqrt{-1}\, \mathcal{V} \varphi_k$ is a real-valued function on M. Note also that $\mathcal{Y}(0) = (\sqrt{-1}\,\mathcal{V}/2)\,q^2$. Then the variation formula for the scalar curvature (see for instance [3, (2.5)]) shows that, modulo q^{l+2},

$$\begin{aligned}
&\Phi(q;\varphi_l,\zeta_l,\alpha_l)\\
&\equiv \Phi(q;0,0,0) + \frac{q^{l+1}}{2}(-\mathcal{D}_0 + \sqrt{-1}\,\mathcal{V})\varphi_l - \alpha_l q^{l+1} + q^{l+1}h_0^{-1}(\mathcal{Y}_l\cdot h_0) - \frac{\sqrt{-1}}{2}\,\mathcal{V}\,\varphi_l\,q^{l+1}\\
&\equiv \left\{u_l - \mathcal{D}_0\frac{\varphi_l}{2} - \alpha_l - \hat{F}_m(\mathcal{Y}_l) + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_l)\right\} q^{l+1},
\end{aligned}$$

where we put $\hat{F}(\mathcal{Y}) := \{c_1(L)^n[M]\}^{-1}2\pi F(\sqrt{-1}\,\mathcal{Y})$ for each $\mathcal{Y}\in\sqrt{-1}\,\mathfrak{z}$. By setting $\mu_l := \{c_1(L)^n[M]\}^{-1}(\int_M u_l\omega_0^n)$, we write u_l as a sum

$$u_l = \mu_l + u_l' + u_l'',$$

where $u_l' := (1-P)(u_l-\mu_l)\in \operatorname{Ker}\mathcal{D}_0^{\perp}$ and $u_l'' := P(u_l-\mu_l)\in\operatorname{Ker}\mathcal{D}_0$. Now, let φ_l be the unique element of $\operatorname{Ker}\mathcal{D}_0^{\perp}$ such that $\mathcal{D}_0(\varphi_l/2) = u_l'$. Moreover, we put

$$\zeta_l := u_l'' \quad\text{ and }\quad \alpha_l := \mu_l - \hat{F}(\mathcal{Y}_l).$$

Then by $\mathcal{Y}_l = \sqrt{-1}\,e_0(\zeta_l) = \sqrt{-1}\,e_0(u_l'')$, we obtain

$$\begin{aligned}
\Phi(q;\varphi_l,\zeta_l,\alpha_l) &\equiv \left\{\mu_l + u_l' + u_l'' - \mathcal{D}_0\frac{\varphi_l}{2} - \alpha_l - \hat{F}_m(\mathcal{Y}_l) + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_l)\right\} q^{l+1}\\
&\equiv \{\,u_l'' + e_0^{-1}(\sqrt{-1}\,\mathcal{Y}_l)\,\}\,q^{l+1} \equiv 0, \qquad \bmod q^{l+2},
\end{aligned}$$

as required. Write $\sqrt{-1}\,\mathcal{V}/2$ as $\mathcal{Y}_0$ for simplicity. Now, for the real Lie subalgebra $\mathfrak{b}$ of $\mathfrak{z}$ generated by $\mathcal{Y}_k$, $k = 0,1,2,\dots$, its complexification $\mathfrak{b}^{\mathbb{C}}$ in $\mathfrak{z}^{\mathbb{C}}$ generates a complex Lie subgroup $B^{\mathbb{C}}$ of $Z^{\mathbb{C}}$. Then it is easy to check that the algebraic subtorus T of $Z^{\mathbb{C}}$ obtained as the closure of $B^{\mathbb{C}}$ in $Z^{\mathbb{C}}$ has the required properties.

REMARK 6.5. In Theorem B, assume that ω_0 is a Kähler metric of constant scalar curvature, and moreover that the actions $\rho_{m(\nu)}$, $\nu = 1,2,\dots,$ coincide (cf. [26, (2.3)]) for all sufficiently large ν. Then by [26], the trivial group $\{1\}$ can be chosen as the algebraic subtorus T above of $Z^{\mathbb{C}}$.

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