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ANTI-SELF-DUAL HERMITIAN METRICS AND PAINLEVÉ III

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0. Introduction

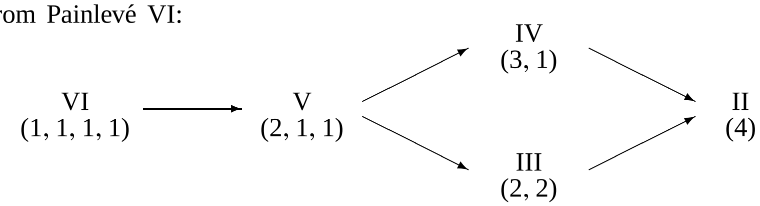
The aim of this paper is to study the $SU(2)$-invariant anti-self-dual metrics which is specified by the solutions of Painlevé III. We study not only the diagonal metrics, but also the non-diagonal metrics.

Hitchin [6] shows that the $SU(2)$-invariant anti-self-dual metric is generically specified by a solution of Painlevé VI with two complex parameters.

Painlevé VI is shown to be a deformation equation for a linear problem

$$\left(\frac{d}{dz} - B_1\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

where B_1 has four simple poles on $\mathbb{CP}^1$ [7]. And Painlevé V, IV, III, II are degenerated from Painlevé VI:



This is the confluence diagram of poles of B_1, where the Roman numerals represent the types of the Painlevé equation, and the parenthesized numbers represent the orders of poles of B_1. For example, Painlevé III is shown to be a deformation equation for a linear problem with two double poles.

Hitchin used the twistor correspondence [1, 11] to associate the anti-self-dual equation and the Painlevé equation. On the twistor space, the lifted action of $SU(2)$ determines a pre-homogeneous action of $SU(2)$, and it determines an isomonodromic family of connections on $\mathbb{CP}^1$, and then we obtain the Painlevé equation.

Due to the reality condition of the twistor space, the poles of B_1 makes two antipodal pairs. Therefore, the configuration of poles becomes the type of Painlevé III or VI. Generically, the anti-self-dual metric is specified by a solution of Painlevé VI.

In this framework, Hitchin [6] classified the diagonal anti-self-dual metrics, and Dancer [5] shows that the diagonal scalar-flat Kähler metric is specified by a solution of Painlevé III with a parameter $(0, 4, 4, -4)$, where *diagonal* metric is in the shape of (1) in Section 1. Since the anti-self-dual Einstein metrics are diagonal, the classification for diagonal metrics enough serves Hitchin's purpose. However, generically, the $SU(2)$-invariant metric is in the shape of (4) in Section 2. In this case, Hitchin shows that the metric is generically specified by a solution of Painlevé VI, but he dose not go into detail. In this paper, we study not only the diagonal metrics but also the non-diagonal metrics.

We show that the $SU(2)$-invariant anti-self-dual equations reduce to the following Painlevé equations:
(a) A family of Painlevé VI

$$\frac{d^2q}{dx^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-x}\right)\left(\frac{dq}{dx}\right)^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{q-x}\right)\frac{dq}{dx} + \frac{q(q-1)(q-x)}{x^2(x-1)^2}\left\{\alpha + \beta\frac{x}{q^2} + \gamma\frac{x-1}{(q-1)^2} + \delta\frac{x(x-1)}{(q-x)^2}\right\}$$

with two complex parameters,

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\overline{\theta}_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \overline{\theta}_1^2)\right).$$

If the metric is in the form (15), then $\theta_0 = \theta_1$ or $\theta_0, \theta_1 \in \mathbb{R}$.
(b) A family of Painlevé III

$$\frac{d^2q}{dx^2} = \frac{1}{q}\left(\frac{dq}{dx}\right)^2 - \frac{1}{x}\frac{dq}{dt} + \frac{1}{x}\left(\alpha q^2 + \beta\right) + \gamma q^3 + \frac{\delta}{q}.$$

with one complex parameter,

$$(\alpha, \beta, \gamma, \delta) = \left(4\theta, 4(1 + \overline{\theta}), 4, -4\right).$$

If the metric is in the form (15), then $\theta \in \mathbb{R}$.
The case (b) is a generalization of Dancer's result [5].

Generically, the $SU(2)$-invariant anti-self-dual metric is specified by a solution of Painlevé VI with a parameter above. The metric is specified by a solution of Painlevé III, if and only if there exists an $SU(2)$-invariant hermitian structure. With an appropriate conformal rescaling, the hermitian metric turns into a scalar-flat Kähler metric.

1. The diagonal anti-self-dual equations

In this section, we review the anti-self-dual equations on the $SU(2)$-invariant diagonal metrics.

The $SU(2)$-invariant diagonal metric is represented in the following form:

$$g = w_1 w_2 w_3\, dt^2 + \frac{w_2 w_3}{w_1}\sigma_1^2 + \frac{w_3 w_1}{w_2}\sigma_2^2 + \frac{w_1 w_2}{w_3}\sigma_3^2. \tag{1}$$

w_1, w_2 and w_3 are functions of t, and σ_1, σ_2, σ_3 are left invariant one-forms on each $SU(2)$-orbit satisfying

$$d\sigma_1 = \sigma_2 \wedge \sigma_3, \qquad d\sigma_2 = \sigma_3 \wedge \sigma_1, \qquad d\sigma_3 = \sigma_1 \wedge \sigma_2. \tag{2}$$

Tod [12] showed that the (scalar-flat) anti-self-dual equations on the $SU(2)$-invariant diagonal metric are given by the following system:

$$\begin{aligned} \dot{w}_1 &= -w_2 w_3 + w_1 (\alpha_2 + \alpha_3), \\ \dot{w}_2 &= -w_3 w_1 + w_2 (\alpha_3 + \alpha_1), \\ \dot{w}_3 &= -w_1 w_2 + w_3 (\alpha_1 + \alpha_2), \\ \dot{\alpha}_1 &= -\alpha_2 \alpha_3 + \alpha_1 (\alpha_2 + \alpha_3), \\ \dot{\alpha}_2 &= -\alpha_3 \alpha_1 + \alpha_2 (\alpha_3 + \alpha_1), \\ \dot{\alpha}_3 &= -\alpha_1 \alpha_2 + \alpha_3 (\alpha_1 + \alpha_2), \end{aligned} \tag{3}$$

where α_1, α_2, α_3 are auxiliary functions and the dots denote differentiation with respect to t. The anti-self-dual equation (3) has a first integral

$$k = \frac{\alpha_1(w_2^2 - w_3^2) + \alpha_2(w_3^2 - w_1^2) + \alpha_3(w_1^2 - w_2^2)}{8(\alpha_1 - \alpha_2)(\alpha_2 - \alpha_3)(\alpha_3 - \alpha_1)}.$$

Furthermore, if we set

$$\begin{aligned} x &= \frac{\alpha_2 - \alpha_1}{\alpha_2 - \alpha_3}, \\ q &= \frac{w_2(\alpha_1 - \alpha_2)(w_2(w_1^2 - w_3^2) + 2\sqrt{2k}\, w_1 w_3(\alpha_1 - \alpha_3))}{w_1^2(w_2^2 - w_3^2)\alpha_1 + w_2^2(w_3^2 - w_1^2)\alpha_2 + w_3^2(w_1^2 - w_2^2)\alpha_3}, \end{aligned}$$

then the system (3) generically reduces to a family of Painlevé VI with a special parameter

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{\left(\sqrt{2k} - 1\right)^2}{2}, k, -k, \frac{1 + 2k}{2} \right).$$

2. The non-diagonal anti-self-dual equations

We can express an $SU(2)$-invariant metric in the form

$$g = f(\tau)\, d\tau^2 + \sum_{l,m=1}^{3} h_{l\,m}(\tau)\, \sigma_l \sigma_m, \tag{4}$$

Using the Killing form, we can diagonalize the metric g on each $SU(2)$-orbit. Then we can express the metric as follows:

$$g = (abc)^2\, dt^2 + a^2 d\hat{\sigma}_1^2 + b^2 \hat{\sigma}_2^2 + c^2 \hat{\sigma}_3^2,$$

where $t = t(\tau)$, $a = a(t)$, $b = b(t)$, $c = c(t)$ and

$$\begin{pmatrix} \hat\sigma_1 \\ \hat\sigma_2 \\ \hat\sigma_3 \end{pmatrix} = R(t) \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix},$$

where $R(t)$ is an $SO(3)$-valued function.

Since $\dot{R}R^{-1} \in \mathfrak{so}(3)$, we obtain

$$\begin{aligned} d \begin{pmatrix} \hat\sigma_1 \\ \hat\sigma_2 \\ \hat\sigma_3 \end{pmatrix} &= R(t) \begin{pmatrix} \sigma_2 \wedge \sigma_3 \\ \sigma_3 \wedge \sigma_1 \\ \sigma_2 \wedge \sigma_2 \end{pmatrix} + \dot{R}\, dt \wedge \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \\ &= \begin{pmatrix} \hat\sigma_2 \wedge \hat\sigma_3 \\ \hat\sigma_3 \wedge \hat\sigma_1 \\ \hat\sigma_1 \wedge \hat\sigma_2 \end{pmatrix} + \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} dt \wedge \begin{pmatrix} \hat\sigma_1 \\ \hat\sigma_2 \\ \hat\sigma_3 \end{pmatrix}, \end{aligned}$$

for some $\xi_1 = \xi_1(t)$, $\xi_2 = \xi_2(t)$, $\xi_3 = \xi_3(t)$.

If $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, then the matrix (h_{lm}) can be chosen to be diagonal for all τ, and then we call that g has a diagonal form.

In the following, we mainly study the non-diagonal metrics.

To compute the curvature tensor, we choose a basis for $\bigwedge^2$

$$\{\Omega_1^+, \Omega_2^+, \Omega_3^+, \Omega_1^- \Omega_2^-, \Omega_3^-\},$$

where

$$\begin{aligned} \Omega_1^+ &= a^2bc\, dt \wedge \hat\sigma_1 + bc\, \hat\sigma_2 \wedge \hat\sigma_3, \\ \Omega_2^+ &= ab^2c\, dt \wedge \hat\sigma_2 + ca\, \hat\sigma_3 \wedge \hat\sigma_1, \\ \Omega_3^+ &= abc^2\, dt \wedge \hat\sigma_3 + ab\, \hat\sigma_1 \wedge \hat\sigma_2, \\ \Omega_1^- &= a^2bc\, dt \wedge \hat\sigma_1 - bc\, \hat\sigma_2 \wedge \hat\sigma_3, \\ \Omega_2^- &= ab^2c\, dt \wedge \hat\sigma_2 - ca\, \hat\sigma_3 \wedge \hat\sigma_1, \\ \Omega_3^- &= abc^2\, dt \wedge \hat\sigma_3 - ab\, \hat\sigma_1 \wedge \hat\sigma_2. \end{aligned}$$

With respect to this frame, the curvature tensor has the following block form [3]:

$$\begin{pmatrix} A & B \\ {}^tB & D \end{pmatrix},$$

where $s = 4\,\mathrm{trace}\, D$ is the scalar curvature, $W^+ = A - (1/12)\,s$ and $W^- = D - (1/12)\,s$ are the self-dual and anti-self-dual parts of the Weyl tensor, and B is the trace free parts of Ricci tensor.

We set $w_1 = bc$, $w_2 = ca$, $w_3 = ab$ and determine auxiliary functions α_1, α_2, α_3 by

$$
\begin{aligned}
\dot{w}_1 &= -w_2 w_3 + w_1(\alpha_2 + \alpha_3), \\
\dot{w}_2 &= -w_3 w_1 + w_2(\alpha_3 + \alpha_1), \\
\dot{w}_3 &= -w_1 w_2 + w_3(\alpha_1 + \alpha_2).
\end{aligned} \tag{5}
$$

Calculating the condition $A = 0$, we obtain the following theorem.

Theorem 2.1. *The metric is anti-self-dual with vanishing scalar curvature, if and only if* α_1, α_2, α_3 *and* ξ_1, ξ_2, ξ_3 *satisfy the following equations*:

$$
\begin{aligned}
\dot{\alpha}_1 = &- \alpha_2\alpha_3 + \alpha_1(\alpha_2 + \alpha_3) + \frac{1}{4}(w_2^2 - w_3^2)^2 \left(\frac{\xi_1}{w_2 w_3}\right)^2 \\
&+ \frac{1}{4}(w_3^2 - w_1^2)(3w_1^2 + w_3^2) \left(\frac{\xi_2}{w_3 w_1}\right)^2 \\
&+ \frac{1}{4}(w_2^2 - w_1^2)(3w_1^2 + w_2^2) \left(\frac{\xi_3}{w_1 w_2}\right)^2, \\
\dot{\alpha}_2 = &- \alpha_3\alpha_1 + \alpha_2(\alpha_3 + \alpha_1) + \frac{1}{4}(w_3^2 - w_1^2)^2 \left(\frac{\xi_2}{w_3 w_1}\right)^2 \\
&+ \frac{1}{4}(w_1^2 - w_2^2)(3w_2^2 + w_1^2) \left(\frac{\xi_3}{w_1 w_2}\right)^2 \\
&+ \frac{1}{4}(w_3^2 - w_2^2)(3w_2^2 + w_3^2) \left(\frac{\xi_1}{w_2 w_3}\right)^2, \\
\dot{\alpha}_3 = &- \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_2) + \frac{1}{4}(w_1^2 - w_2^2)^2 \left(\frac{\xi_3}{w_1 w_2}\right)^2 \\
&+ \frac{1}{4}(w_2^2 - w_3^2)(3w_3^2 + w_2^2) \left(\frac{\xi_1}{w_2 w_3}\right)^2 \\
&+ \frac{1}{4}(w_1^2 - w_3^2)(3w_3^2 + w_1^2) \left(\frac{\xi_2}{w_3 w_1}\right)^2,
\end{aligned} \tag{6}
$$

and

$$
(7)\quad
\begin{aligned}
(w_2^2 - w_3^2)\frac{d}{dt}\left(\frac{\xi_1}{w_2w_3}\right) &= \frac{\xi_2}{w_3w_1}\frac{\xi_3}{w_1w_2}(-2w_2^2w_3^2 + w_3^2w_1^2 + w_1^2w_2^2)\\
&\quad + \frac{\xi_1}{w_2w_3}(\alpha_2w_2^2 - \alpha_3w_3^2 + 3\alpha_2w_3^2 - 3\alpha_3w_2^2),\\
(w_3^2 - w_1^2)\frac{d}{dt}\left(\frac{\xi_2}{w_3w_1}\right) &= \frac{\xi_3}{w_1w_2}\frac{\xi_1}{w_2w_3}(-2w_3^2w_1^2 + w_1^2w_2^2 + w_2^2w_3^2)\\
&\quad + \frac{\xi_2}{w_3w_1}(\alpha_3w_3^2 - \alpha_1w_1^2 + 3\alpha_3w_1^2 - 3\alpha_1w_3^2),\\
(w_1^2 - w_2^2)\frac{d}{dt}\left(\frac{\xi_3}{w_1w_2}\right) &= \frac{\xi_1}{w_2w_3}\frac{\xi_2}{w_3w_1}(-2w_1^2w_2^2 + w_2^2w_3^2 + w_3^2w_1^2)\\
&\quad + \frac{\xi_3}{w_1w_2}(\alpha_1w_1^2 - \alpha_2w_2^2 + 3\alpha_1w_2^2 - 3\alpha_2w_1^2).
\end{aligned}
$$

REMARK 2.2. If we take a conformal rescaling g to $F(t)g$, then t turns into s that satisfies $ds/dt = 1/F$, and w_1, w_2, w_3 turn into Fw_1, Fw_2, Fw_3, and ξ_1, ξ_2, ξ_3 turn into $F\xi_1$, $F\xi_2$, $F\xi_3$ respectively. And then α_1, α_2, α_3 turn into

$$
\tilde{\alpha}_1 = \frac{1}{2}\frac{dF}{dt} + F\alpha_1, \quad \tilde{\alpha}_2 = \frac{1}{2}\frac{dF}{dt} + F\alpha_2, \quad \tilde{\alpha}_3 = \frac{1}{2}\frac{dF}{dt} + F\alpha_3.
$$

The equations (5), (6), (7) are invariant under a conformal rescaling g to Fg, if $2F\dot{F}^2 = \ddot{F}^2$.

REMARK 2.3. By the equation (5), (6), (7), we obtain $-2w_1^2w_2^2+w_2^2w_3^2+w_3^2w_1^2 \not\equiv 0$. Therefore, if $\xi_3 \equiv 0$, then $\xi_1 \equiv 0$ or $\xi_2 \equiv 0$. In the same way, if $\xi_1 \equiv 0$, then $\xi_2 \equiv 0$ or $\xi_3 \equiv 0$, and if $\xi_2 \equiv 0$, then $\xi_3 \equiv 0$ or $\xi_1 \equiv 0$.

REMARK 2.4. If $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$, then the equation (5), (6), (7) reduces to a sixth-order system (3) given by Tod [12]. Furthermore, if $\alpha_1 = w_1$, $\alpha_2 = w_2$, $\alpha_3 = w_3$, then (5), (6), (7) reduce to a third-order system, which determines Atiyah-Hitchin family [1]. And if $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, then (5), (6), (7) reduce to a third-order system, which determines BGPP family [4].

REMARK 2.5. If $w_2 = w_3$, then we can set $\xi_1 = 0$, $\xi_2 = 0$, $\xi_3 = 0$ by taking another frame. This is also in the diagonal case. Therefore we assume $(w_2-w_3)(w_3-w_1)(w_1-w_2) \neq 0$.

3. The isomonodromic deformations

Let (M, g) be an oriented Riemannian four manifold. We define a manifold Z to be a unit sphere bundle in the bundle of anti-self-dual two-forms, and let $\pi\colon Z \to M$

denote the projection. Each point z in the fiber over $\pi(z)$ defines a complex structure on the tangent space $T_{\pi(z)}M$, compatible with the metric and its orientation.

Using the Levi-Civita connection, we can split the tangent space T_zZ into horizontal and vertical spaces, and the projection π identifies the horizontal space with $T_{\pi(z)}M$. This space has a complex structure defined by z and the vertical space is the tangent space of the fiber $S^2 \cong \mathbb{CP}^1$ which has its natural complex structure.

The almost complex structure on Z is integrable, if and only if the metric is anti-self-dual [2, 11]. In this situation Z is called the twistor space of (M, g) and the fibers are called the real twistor lines.

The almost complex structure on Z can be determined by the following $(1, 0)$-forms:

$$
\begin{aligned}
\Theta_1 =& z\left(e^1 + \sqrt{-1}\, e^2\right) - \left(e^0 + \sqrt{-1}\, e^3\right),\\
\Theta_2 =& z\left(e^0 - \sqrt{-1}\, e^3\right) + \left(e^1 - \sqrt{-1}\, e^2\right),\\
\Theta_3 =& dz + \frac{1}{2}z^2\left(\omega^0_1 - \omega^2_3 + \sqrt{-1}\,(\omega^0_2 - \omega^3_1)\right)\\
&- \sqrt{-1}\, z\left(\omega^0_3 - \omega^1_2\right) + \frac{1}{2}\left(\omega^0_1 - \omega^2_3 - \sqrt{-1}\left(\omega^0_2 - \omega^3_1\right)\right),
\end{aligned}
\tag{8}
$$

where $\{e^0, e^1, e^2, e^3\}$ is an orthonormal frame, and ω^i_j are the connection forms determined by $de^i + \omega^i_j \wedge e^j = 0$ and $\omega^i_j + \omega^j_i = 0$. (M, g) is anti-self-dual, if and only if the Pfaffian system (or the distribution defined by the following system)

$$
\Theta_1 = 0, \qquad \Theta_2 = 0, \qquad \Theta_3 = 0 \tag{9}
$$

is integrable, that is to say

$$
d\Theta_1 \equiv 0, \qquad d\Theta_2 \equiv 0, \qquad d\Theta_3 \equiv 0 \qquad (\mathrm{mod}\ \Theta_1, \Theta_2, \Theta_3). \tag{10}
$$

REMARK 3.1. The Pfaffian system (9) is invariant under $z \mapsto (z+\sqrt{-1})/(z-\sqrt{-1})$ and permutation $1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1$ of suffixes of e^i and ω^i_j.

Theorem 3.2. *The Pfaffian system* (9) *is invariant under the conjugate action and* $z \mapsto -1/\overline{z}$ [2].

If the metric is $SU(2)$ invariant, we obtain

$$
\begin{pmatrix}\Theta_1\\ \Theta_2\\ \Theta_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 1\end{pmatrix} dz + \begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} dt + A\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix}, \tag{11}
$$

where $v_1 = v_1(z, t),\ v_2 = v_2(z, t),\ v_3 = v_3(z, t);\ A = \left(a_{i\,j}(z, t)\right)_{i,j=1,2,3}$.

If $\det A \equiv 0$, then the metric turns to be diagonal, and the metric is in the BGPP family [4].

If $\det A \neq 0$, then we obtain

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} \equiv -A^{-1}\left(\begin{pmatrix}0\\0\\1\end{pmatrix} dz + \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} dt\right), \quad (\mathrm{mod}\ \Theta_1, \Theta_2, \Theta_3).$$

If we set

$$\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix} = -A^{-1}\left(\begin{pmatrix}0\\0\\1\end{pmatrix} dz + \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} dt\right), \tag{12}$$

then

$$d\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix} \equiv \begin{pmatrix}s_2\wedge s_3\\ s_3\wedge s_1\\ s_1\wedge s_2\end{pmatrix}, \quad (\mathrm{mod}\ \Theta_1, \Theta_2, \Theta_3). \tag{13}$$

Since s_1, s_2, s_3 are one-forms on (z,t)-plane, the congruency equation (13) turns to be a plain equation:

$$d\begin{pmatrix}s_1\\ s_2\\ s_3\end{pmatrix} = \begin{pmatrix}s_2\wedge s_3\\ s_3\wedge s_1\\ s_1\wedge s_2\end{pmatrix}.$$

By Theorem 3.2, s_1, s_2, s_3 are invariant under the conjugate action and $z \mapsto -1/\overline{z}$.

If we set

$$\begin{aligned}\Sigma &= \frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{-1}\,s_2 & -s_1+\sqrt{-1}\,s_3\\ s_1+\sqrt{-1}\,s_3 & -\sqrt{-1}\,s_2\end{pmatrix}\\ &=: -B_1\,dz - B_2\,dt,\end{aligned}$$

then

$$d\Sigma + \Sigma\wedge\Sigma = 0.$$

This is the isomonodromic condition for the following linear problem [7]

$$\left(\frac{d}{dz} - B_1\right)\begin{pmatrix}y_1\\ y_2\end{pmatrix} = 0. \tag{14}$$

Lemma 3.3. *The components of B_1 are rational functions of z,*

$$B_1 = \frac{F(z)}{G(z)},$$

where $F(z)$ is degree 2 and $G(z)$ is degree 4. We must have $B_1 \mapsto -{}^t B_1$ under the conjugate action and $z \mapsto -1/\bar{z}$.

Proof. Since s_1, s_2, s_3 are invariant under the conjugate action and $z \mapsto -1/\bar{z}$, we obtain $\Sigma \mapsto -{}^t\Sigma$, and then $B_1 \mapsto -{}^t B_1$.

If we set

$$\begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{pmatrix} = R(t) \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix},$$

then we have

$$\begin{pmatrix} \hat{s}_1 \\ \hat{s}_2 \\ \hat{s}_3 \end{pmatrix} \equiv \begin{pmatrix} \hat{\sigma}_1 \\ \hat{\sigma}_2 \\ \hat{\sigma}_3 \end{pmatrix} \quad (\mathrm{mod}\ \Theta_1, \Theta_2, \Theta_3).$$

By a straightforward calculation, we obtain

$$\begin{aligned} \hat{s}_1\left(\frac{\partial}{\partial z}\right) &\equiv \frac{2\left(1+z^2\right) w_1}{G(z)}, \\ \hat{s}_2\left(\frac{\partial}{\partial z}\right) &\equiv \frac{2\sqrt{-1}\,\left(-1+z^2\right) w_2}{G(z)}, \\ \hat{s}_3\left(\frac{\partial}{\partial z}\right) &\equiv \frac{-4\sqrt{-1}\,z w_3}{G(z)}, \end{aligned}$$

where

$$\begin{aligned} G(z) = z^4 &\left((\alpha_1-\alpha_2) - \sqrt{-1}\,\frac{w_1^2-w_2^2}{w_1 w_2}\xi_3\right) - 2z^3\left(\frac{w_2^2-w_3^2}{w_2 w_3}\xi_1 - \sqrt{-1}\,\frac{w_3^2-w_1^2}{w_3 w_1}\xi_2\right) \\ &+ 2z^2(\alpha_1+\alpha_2-2\alpha_3) + 2z\left(\frac{w_2^2-w_3^2}{w_2 w_3}\xi_1 + \sqrt{-1}\,\frac{w_3^2-w_1^2}{w_3 w_1}\xi_2\right) \\ &+ \left((\alpha_1-\alpha_2) + \sqrt{-1}\,\frac{w_1^2-w_2^2}{w_1 w_2}\xi_3\right). \end{aligned}$$

Since $R(t)$ is independent of z, the components of B_1 are rational functions of z,

$$B_1 = \frac{F(z)}{G(z)},$$

where $F(z)$ is degree 2 and $G(z)$ is degree 4. □

For this lemma, generically B_1 has four simple poles. In this case, the deformation equation of (14) is Painlevé VI.

Theorem 3.4. *The $SU(2)$-invariant anti-self-dual metric is generically specified by the solution of Painlevé VI.*

The idea of Hitchin [6] is that the lifted action of $SU(2)$ on the twistor space Z gives a homomorphism of vector bundles $\alpha\colon Z \times \mathfrak{su}(2)^{\mathbb{C}} \to TZ$, and the inverse of α gives a flat meromorphic $SL(2,\mathbb{C})$-connection, which determines isomonodromic deformations. Since one-forms Θ_1, Θ_2, Θ_3 on Z can be considered as infinitesimal variations, we can identify Σ with α^{-1}.

By Lemma 3.3, the poles of B_1 make antipodal pairs ζ_0, $-1/\overline{\zeta_0}$, and ζ_1, $-1/\overline{\zeta_1}$ on $\mathbb{CP}^1$. Therefore we obtain two types of configuration of poles of B_1. In each case, we can calculate the local exponents at singularities. These local exponents corresponding to the parameter of the Painlevé equation (see [8]).

(a) B_1 has four simple poles ζ_0, $-1/\overline{\zeta_0}$, ζ_1, $-1/\overline{\zeta_1}$ on $\mathbb{CP}^1$.

$$B_1 = \frac{A_0}{z-\zeta_0} + \frac{-{}^t\overline{A_0}}{z+1/\zeta_0} + \frac{A_1}{z-\zeta_1} + \frac{-{}^t\overline{A_1}}{z+1/\zeta_1}.$$

The deformation equation is Painlevé VI with a parameter,

$$(\alpha,\beta,\gamma,\delta) = \left(\frac{1}{2}(\theta_0-1)^2, \frac{1}{2}\overline{\theta_0^2}, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1+\overline{\theta_1^2})\right),$$

where $\theta_0^2 = 2\operatorname{tr} A_0^2$, $\theta_1^2 = 2\operatorname{tr} A_1^2$.

(b) B_1 has two double poles ζ, $-1/\overline{\zeta}$ on $\mathbb{CP}^1$.

$$B_1 = \frac{A_2}{(z-\zeta)^2} + \frac{\sqrt{-1}\,C}{z-\zeta} + \frac{-\sqrt{-1}\,C}{z+1/\zeta} + \frac{-{}^t\overline{A_2}/\overline{\zeta}^2}{(z+1/\zeta)^2},$$

where $C = -{}^t\overline{C}$. The deformation equation is Painlevé III with a parameter,

$$(\alpha,\beta,\gamma,\delta) = \left(4\theta, 4(1+\overline{\theta}), 4, -4\right),$$

where $\theta^2 = 2(\operatorname{tr}(A_2C))^2/\operatorname{tr} C^2$.

Theorem 3.5. *The anti-self-dual equations reduce to the following Painlevé equations*:

(a) *A family of Painlevé VI with two complex parameters,*

$$(\alpha,\beta,\gamma,\delta) = \left(\frac{1}{2}(\theta_0-1)^2, \frac{1}{2}\overline{\theta_0^2}, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1+\overline{\theta_1^2})\right).$$

(b) *A family of Painlevé III with one complex parameter,*

$$(\alpha,\beta,\gamma,\delta) = \left(4\theta, 4(1+\overline{\theta}), 4, -4\right).$$

REMARK 3.6. It is known that the anti-self-dual equations reduce to Painlevé VI with the parameter as above ([9], [6]). Dancer [5] shows the diagonal scalar-flat Kähler metric is specified by a solution of Painlevé III with a parameter $(\alpha, \beta, \gamma, \delta) = (0, 4, 4, -4)$. Now, Theorem 3.5 (b) is a generalization of Dancer's result.

By Remark 2.3, if $\xi_1\xi_2\xi_3 = 0$, then at least two of ξ_1, ξ_2, ξ_3 must be zero. From now on this section, we assume $\xi_2 = \xi_3 = 0$, and then we obtain the metric in the form:

$$g = f(\tau)\, d\tau + h_{11}(\tau)\,\sigma_1^2 + h_{22}(\tau)\,\sigma_2^2 + h_{23}(\tau)\,\sigma_2\sigma_3 + h_{33}(\tau)\,\sigma_3^2. \tag{15}$$

Therefore, there exists an isometric action

$$(\sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_1, -\sigma_2, -\sigma_3), \tag{16}$$

which preserves each orbit. Since

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} \mapsto \begin{pmatrix} \overline{\Theta}_1 \\ \overline{\Theta}_2 \\ \overline{\Theta}_3 \end{pmatrix}$$

under the action (16) and $z \mapsto \bar{z}$, then we obtain

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dz + \overline{\left.\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right|_{z\mapsto\bar{z}}} dt + \overline{A|_{z\mapsto\bar{z}}} \begin{pmatrix} \sigma_1 \\ -\sigma_2 \\ -\sigma_3 \end{pmatrix}.$$

Therefore

$$\left.\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}\right|_{z\mapsto\bar{z}} = \begin{pmatrix} \bar{s}_1 \\ -\bar{s}_2 \\ -\bar{s}_3 \end{pmatrix},$$

and then we obtain $B_1|_{z\mapsto\bar{z}} = \overline{B}_1$. Therefore we obtain the following:

(a) If B_1 has four simple poles, then

$$\begin{aligned} B_1 &= \frac{A_0}{z-\zeta_0} + \frac{-{}^t\overline{A}_0}{z+1/\zeta_0} + \frac{A_1}{z-\zeta_1} + \frac{-{}^t\overline{A}_1}{z+1/\zeta_1} \\ &= \frac{\overline{A}_0}{z-\zeta_0} + \frac{-{}^tA_0}{z+1/\zeta_0} + \frac{\overline{A}_1}{z-\zeta_1} + \frac{-{}^tA_1}{z+1/\zeta_1}. \end{aligned}$$

If $\zeta_0 = \bar{\zeta}_0$ or $-1/\zeta_0$, then $\theta_0^2 = 2 \operatorname{tr} A_0^2$ and $\theta_1^2 = 2 \operatorname{tr} A_1^2$ must be real numbers. If $\zeta_0 = \bar{\zeta}_1$ or $-1/\zeta_1$, then $\theta_0^2 = 2 \operatorname{tr} A_0^2$ and $\theta_1^2 = 2 \operatorname{tr} A_1^2$ must coincide.

(b) If B_1 has two double poles, then

$$\begin{aligned} B_1 &= \frac{A_2}{(z-\zeta)^2} + \frac{\sqrt{-1}\,C}{z-\zeta} + \frac{-\sqrt{-1}\,C}{z+1/\overline{\zeta}} + \frac{-{}^t\overline{A}_2/\overline{\zeta}^2}{(z+1/\overline{\zeta})^2} \\ &= \frac{\overline{A}_2}{(z-\overline{\zeta})^2} + \frac{\sqrt{-1}\,\overline{C}}{z-\overline{\zeta}} + \frac{-\sqrt{-1}\,\overline{C}}{z+1/\zeta} + \frac{-{}^tA_2/\zeta^2}{(z+1/\zeta)^2}, \end{aligned}$$

where $C = -{}^t\overline{C}$. If $\zeta = \overline{\zeta}$, then $\theta^2 = 2(\mathrm{tr}(A_2C))^2/\mathrm{tr}\,A_2^2$ must be a real number. If $\zeta = -1/\zeta$, then $\theta^2 = 2(\mathrm{tr}(A_2C))^2/\mathrm{tr}\,A_2^2 = 0$.

Theorem 3.7. *If $\xi_1\xi_2\xi_3 = 0$, then the anti-self-dual equations reduce to the following Painlevé equations*:
(a) *A family of Painlevé VI with two real parameters,*

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta_0 - 1)^2, \frac{1}{2}\theta_0^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}(1 + \theta_1^2)\right),$$

or one complex parameter,

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{1}{2}(\theta - 1)^2, \frac{1}{2}\overline{\theta}^2, -\frac{1}{2}\theta^2, \frac{1}{2}(1 + \overline{\theta}^2)\right).$$

(b) *A family of Painlevé III with one real parameter,*

$$(\alpha, \beta, \gamma, \delta) = (4\theta, 4(1 + \theta), 4, -4).$$

4. Hermitian structure

In this section, we study the geometric meaning of the $SU(2)$-invariant anti-self-dual metric specified by the solutions of Painlevé III. Painlevé III is the deformation equation of

$$\left(\frac{d}{dz} - B_1\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0,$$

where B_1 has two double poles. By a direct calculation, we obtain the following lemma.

Lemma 4.1. *The poles of B_1 are determined by the following equation*:

$$\begin{aligned} (17)\quad & z^4\left((\alpha_1 - \alpha_2) - \sqrt{-1}\,X_3\right) - 2z^3\left(X_1 - \sqrt{-1}\,X_2\right) \\ &\quad + 2z^2(\alpha_1 + \alpha_2 - 2\alpha_3) + 2z\left(X_1 + \sqrt{-1}\,X_2\right) \\ &\quad + \left((\alpha_1 - \alpha_2) + \sqrt{-1}\,X_3\right) = 0, \end{aligned}$$

where

$$X_1 = \frac{w_2^2 - w_3^2}{w_2 w_3}\xi_1, \qquad X_2 = \frac{w_3^2 - w_1^2}{w_3 w_1}\xi_2, \qquad X_3 = \frac{w_1^2 - w_2^2}{w_1 w_2}\xi_3.$$

Since the equation (17) is preserved by $z \mapsto -1/\bar{z}$ and the conjugate action, if the equation (17) has a solution $z = \zeta$ of order two, then $z = -1/\bar{\zeta}$ is also a solution of order two.

Lemma 4.2. *Let g be a non-diagonal $SU(2)$-invariant metric. Then B_1 has two double poles, if and only if there exists a function $f(t)$ satisfying*

$$\begin{aligned} X_1^2 &= 4(f - \alpha_2)(f - \alpha_3), \\ X_2^2 &= 4(f - \alpha_3)(f - \alpha_1), \\ X_3^2 &= 4(f - \alpha_1)(f - \alpha_2). \end{aligned} \tag{18}$$

And then the anti-self-dual equation reduce to (5), (6) *and* $\dot{f} = f^2$.

Proof. If $X_1 = X_2 = X_3 = 0$, then the discriminant of (17) is

$$16\,(\alpha_1 - \alpha_2)^2\,(\alpha_2 - \alpha_3)^2\,(\alpha_3 - \alpha_1)^2.$$

Therefore, if

$$(\alpha_1 - \alpha_2)\,(\alpha_2 - \alpha_3)\,(\alpha_3 - \alpha_1) = 0,$$

then B_1 has two double poles. This case is in the form of (18) by $f = \alpha_1 = \alpha_2$ or $f = \alpha_2 = \alpha_3$ or $f = \alpha_3 = \alpha_1$. By the equation (5), (6), (7), we obtain $\dot{f} = f^2$. If $f = 0$, then we obtain the diagonal scalar-flat-Kähler metric given by Pedersen and Poon [10].

If $X_1 X_2 X_3 = 0$, then, from Remark 2.3, at least two of X_1, X_2, X_3 must be zero. Assume that $X_1 \neq 0$ and $X_2 = X_3 = 0$. Then the discriminant of (17) is $\left(X_1^2 + (\alpha_2 - \alpha_3)^2\right)\left(X_1^2 - 4\,(\alpha_1 - \alpha_2)\,(\alpha_1 - \alpha_3)\right)^2$. Therefore, the equation

$$X_1^2 = 4\,(\alpha_1 - \alpha_2)\,(\alpha_1 - \alpha_3)$$

is the condition that B_1 has two double poles. This is (18) where $f = \alpha_1$. In this case, we obtain the double poles on

$$\zeta = \frac{\sqrt{\alpha_3 - \alpha_1} \pm \sqrt{-1}\,\sqrt{\alpha_2 - \alpha_1}}{\sqrt{\alpha_2 + \alpha_3 + 2\alpha_1}}. \tag{19}$$

X_1, X_2, X_3 satisfy the equation (5), (6), (7), if and only if $\dot{\alpha_1} = \alpha_1^2$.

If $X_1X_2X_3 \neq 0$, the discriminant of (17) is too complicated to calculate. Therefore, we attack by an other way. We obtain (17) in the following form:

$$\bar{a}z^4 - \bar{b}z^3 + cz^2 + bz + a = 0, \tag{20}$$

where a, b are complex coefficients and c is a real coefficient. By a linear fractional transformation

$$z \mapsto \frac{\left(b - |b|\right)\zeta - b + |b|}{\left(-\bar{b} + |b|\right)\zeta - \bar{b} + |b|}, \tag{21}$$

the equation (17) turns into the following form:

$$\zeta^4 - \bar{b}_0\zeta^3 + c_0\zeta^2 + b_0\zeta + 1 = 0, \tag{22}$$

where b_0 is a complex coefficient and c_0 is a real coefficients. Since (21) preserves the antipodal pairs on $\mathbb{CP}^1$, if $\zeta = \zeta_0$ is a solution of (22) of order two, then $\zeta = -1/\bar{\zeta}_0$ is also a solution of order two. Therefore

$$\zeta^4 - \bar{b}_0\zeta^3 + c_0\zeta^2 + b_0\zeta + 1 = (\zeta - \zeta_0)^2\left(\zeta + \frac{1}{\zeta_0}\right)^2, \tag{23}$$

and then $\zeta_0 = \pm\bar{\zeta}_0$, which implies ζ_0 is real or pure-imaginary. Therefore $b_0 = 2\zeta\left(-1 + \zeta\bar{\zeta}\right)/\bar{\zeta}^2$ must be real or pure-imaginary. By a direct calculation, we obtain the following. The real part of b_0 vanishes, if and only if

$$X_2^4\left(X_1^2 + X_2^2\right)^2 = 0,$$

which never occurs. The imaginary part of b_0 vanishes, if and only if

$$X_2^4\left(\left(X_1^2 - X_2^2\right)X_3 - 2X_1X_2\left(\alpha_1 - \alpha_2\right)\right) = 0.$$

Therefore,

$$\left(X_1^2 - X_2^2\right)X_3 = 2X_1X_2\left(\alpha_1 - \alpha_2\right), \tag{24}$$

if and only if B_1 has two double poles. By the linear transformation $z \mapsto (z + \sqrt{-1})/(z - \sqrt{-1})$, the suffixes of X_i and α_i on (17) are replaced cyclically. Therefore, if B_1 has two double poles, then the following must be also satisfied:

$$\left(X_2^2 - X_3^2\right)X_1 = 2X_2X_3\left(\alpha_2 - \alpha_3\right), \tag{25}$$

$$\left(X_3^2 - X_1^2\right)X_2 = 2X_3X_1\left(\alpha_3 - \alpha_1\right). \tag{26}$$

By (24), (25) and (26), X_1, X_2, X_3 must satisfy (18) with an auxiliary function f. Actually, if X_1, X_2, X_3 satisfy (18), then (17) has two solutions of order two:

$$\zeta = \frac{X_1X_2 \pm \sqrt{X_2^2X_3^2 + X_3^2X_1^2 + X_1^2X_2^2}}{X_3(X_1 - \sqrt{-1}\,X_2)}. \tag{27}$$

In this case, X_1, X_2, X_3 satisfy the equation (5), (6), (7), if and only if $\dot{f} = f^2$. □

Therefore, we obtain the following theorem.

Theorem 4.3. *The $SU(2)$-invariant anti-self-dual metric is specified by the solution of Painlevé III, if and only if X_1, X_2, X_3 satisfy* (18) *and $\dot{f} = f^2$.*

If we restrict $\Theta_1|_{z=\zeta(t)}$ and $\Theta_2|_{z=\zeta(t)}$ for some $z = \zeta(t)$, then we obtain (1, 0)-forms on M, which determine an $SU(2)$-invariant almost complex structure on M.

Theorem 4.4. *Let g be an $SU(2)$-invariant anti-self-dual scalar-flat metric. There exists an $SU(2)$-invariant hermitian structure (g, I) if and only if B_1 has double poles.*

Proof. Let $G(z)$ be the left hand side of (17). Then $G(z)$ is the denominator of B_1. We obtain

$$\Theta_3 \equiv dz + H_0\,dt + H_1\,\hat{\sigma}_1 \quad (\text{mod } \Theta_1, \Theta_2),$$

where $H_1 = 0$ is equivalent with $G(z) = 0$, and $dz + H_0\,dt = 0$ is equivalent with $dG = 0$. Therefore, the almost complex structure determined by $\{\Theta_1|_{z=\zeta(t)}, \Theta_2|_{z=\zeta(t)}\}$ is integrable, if and only if $G(z)$ admits a multiple zero on $z = \zeta(t)$. □

Theorem 4.5. *The hermitian structure (g, I) determined on* Theorem 4.4 *is Kähler, if and only if*

$$X_1^2 = 4\alpha_2\alpha_3, \qquad X_2^2 = 4\alpha_3\alpha_1, \qquad X_3^2 = 4\alpha_1\alpha_2. \tag{28}$$

Proof. If $X_1X_2X_3 \neq 0$, the Kähler form is determined by (27) as

$$\Omega = \frac{X_2X_3}{\sqrt{X_2^2X_3^2 + X_3^2X_1^2 + X_1^2X_2^2}}\Omega_1^+ + \frac{X_3X_1}{\sqrt{X_2^2X_3^2 + X_3^2X_1^2 + X_1^2X_2^2}}\Omega_2^+$$

$$+\frac{X_1X_2}{\sqrt{X_2^2X_3^2+X_3^2X_1^2+X_1^2X_2^2}}\Omega_3^+.$$

By the equations (5), (6), (7) and $\dot{f}=f^2$, we obtain

$$\begin{aligned} d\Omega = &\frac{2f\,w_1X_2X_3}{\sqrt{X_2^2X_3^2+X_3^2X_1^2+X_1^2X_2^2}}\,dt\wedge\hat{\sigma}_2\wedge\hat{\sigma}_3 \\ &+\frac{2f\,w_2X_3X_1}{\sqrt{X_2^2X_3^2+X_3^2X_1^2+X_1^2X_2^2}}\,dt\wedge\hat{\sigma}_3\wedge\hat{\sigma}_1 \\ &+\frac{2f\,w_3X_1X_2}{\sqrt{X_2^2X_3^2+X_3^2X_1^2+X_1^2X_2^2}}\,dt\wedge\hat{\sigma}_1\wedge\hat{\sigma}_2. \end{aligned}$$

Since $w_1w_2w_3\neq 0$ and $X_1X_2X_3\neq 0$, we obtain $d\Omega=0$, if and only if $f=0$.

If $X_1X_2X_3=0$, then f must be α_1, α_2 or α_3. Suppose that $f=\alpha_1$, then we obtain $X_1^2=4(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)$, $X_2=0$, $X_3=0$. The Kähler form is determined by (19) as

$$\Omega=\frac{\sqrt{\alpha_2-\alpha_1}}{\sqrt{\alpha_2+\alpha_3-2\alpha_1}}\Omega_2^+ + \frac{\sqrt{\alpha_3-\alpha_1}}{\sqrt{\alpha_2+\alpha_3-2\alpha_1}}\Omega_3^+. \tag{29}$$

Then we obtain

$$d\Omega=\frac{2w_2\alpha_1\sqrt{\alpha_2-\alpha_1}}{\sqrt{\alpha_2+\alpha_3-2\alpha_1}}\,dt\wedge\hat{\sigma}_3\wedge\hat{\sigma}_1+\frac{2w_3\alpha_1\sqrt{\alpha_3-\alpha_1}}{\sqrt{\alpha_2+\alpha_3-2\alpha_1}}\,dt\wedge\hat{\sigma}_1\wedge\hat{\sigma}_2. \tag{30}$$

If the metric is non-diagonal, then $X_1^2=4(\alpha_2-\alpha_1)(\alpha_3-\alpha_1)\neq 0$. Therefore, we obtain $d\Omega=0$, if and only if $\alpha_1=0$. □

By a conformal rescaling $g\mapsto Fg$ where F satisfies $(1/2)(dF/dt)=f$, we can eliminate f of lemma 4.2 (see Remark 2.2).

Theorem 4.6. *An $SU(2)$-invariant anti-self-dual metric is specified by a solution of Painlevé III, if and only if the metric is conformally equivalent with a scalar-flat Kähler metric.*

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References

[1] M.F. Atiyah and N.J. Hitchin: *Low energy scattering of non-Abelian monopoles*, Phys. Lett. A **107** (1985), 21–25.

[2] M.F. Atiyah, N.J. Hitchin and I.M. Singer: *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc, London Ser. A **362** (1978) 425–461.

[3] A.L. Besse: Einstein Manifolds, Springer-Verlag, Berlin-Heidelberg, 1987.

[4] V.A. Belinski, G.W. Gibbons, D.W. Page and C.N. Pope: *Asymptotically Euclidean Bianchi IX metrics in quantum gravity*, Phys. Lett. B **76** (1978), 433–435.

[5] A.S. Dancer: *Scalar-flat Kähler metrics with* $SU(2)$ *symmetry*, J. reine. angew. Math. **479** (1996), 99–120.

[6] N.J. Hitchin: *Twistor spaces, Einstein metrics and isomonodromic deformations*, J. Differential Geom. **42** (1995), 30–112.

[7] M. Jimbo, T. Miwa and K. Ueno: *Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients*. I, Physica **2D** (1981), 306–352.

[8] M. Jimbo and T. Miwa: *Monodromy Preserving Deformation of Linear Ordinary Differential Equations with Rational Coefficients*. II, Physica **2D** (1981), 407–448.

[9] R. Maszczyk, L.J. Mason and N.M.J. Woodhouse: *Self-dual Bianchi metrics and the Painlevé transcendents*, Classical Quantum Gravity **11** (1994), 65–71.

[10] H. Pedersen and Y.S. Poon: *Kähler surfaces with zero scalar curvature*, Classical Quantum Gravity **7** (1990), 1707–1719.

[11] R. Penrose: *Nonlinear gravitons and curved twistor theory*, Gen. Rel. Grav. **7** (1976), 31–52.

[12] K.P. Tod: *Self-dual Einstein metrics from the Painlevé VI equation*, Phys. Lett. A **190** (1994), 221–224.

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