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SOME HOMOTOPY OF THE UNITARY GROUPS DETECTED BY THE K-THEORY OF 2-CELL COMPLEXES

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1. Introduction

Let $k \geqslant 1$ and $m \geqslant 2k+1$. Consider the real Hopf-Whitehead J-homomorphism $J\colon \pi_{4k-1}(\mathrm{SO}(2m)) \longrightarrow \pi_{2m+4k-1}(S^{2m})$. Since the quotient $\mathrm{SO}/\mathrm{SO}(2m)$ is $2m$-connected, by real Bott periodicity, we have $\pi_{4k-1}(\mathrm{SO}(2m)) \cong \pi_{4k-1}(\mathrm{SO}) \cong \mathbb{Z}$. For $s \geqslant m$, $B\mathrm{U}$ and $B\mathrm{U}(s)$ admit CW-complex structures with the same $(2m+1)$-skeleton, so, we have isomorphisms $[S^{2m}, B\mathrm{U}(s)] \cong [S^{2m}, B\mathrm{U}] \cong \widetilde{K}^0(S^{2m}) \cong \mathbb{Z}$, using complex Bott periodicity. By the Freudenthal Suspension Theorem, there is an isomorphism $\pi_{2m+4k-1}(S^{2m}) \cong \pi^S_{4k-1}$. (We refer to p. 480 in [12], p. 216 in [8], Theorem I in [2], and Theorem VI.2.10 in [4] for the details.) We prove the following result:

Theorem 1.1. *For $k \geqslant 1$, $m \geqslant 2k+1$ and $m \leqslant s < m+2k$, let $j_{4k-1} \in \pi^S_{4k-1}$ denote the image of a generator of $\pi_{4k-1}(\mathrm{SO})$ under the J-homomorphism, and let x_{2m} be the Bott generator of $\widetilde{K}^0(S^{2m})$. Then, the composition $x_{2m} \circ j_{4k-1}$ represents a non-zero element in $\pi_{2m+4k-1}(B\mathrm{U}(s))$, whose order is given by*

$$\begin{cases} \operatorname{denom}\left(\dfrac{B_k}{4k}\right), & \text{if } k \text{ is even} \\ \operatorname{denom}\left(\dfrac{B_k}{4k}\right) \ \text{or}\ \dfrac{1}{2}\operatorname{denom}\left(\dfrac{B_k}{4k}\right), & \text{if } k \text{ is odd,} \end{cases}$$

where B_k is the k-th Bernoulli number. When k is odd and s is equal to $m+2k-1$, the order of $x_{2m} \circ j_{4k-1}$ is $(1/2)\operatorname{denom}\left(B_k/(4k)\right)$.

Unfortunately, we were unable to determine in full generality the precise order when k is odd. Notice that for given k and m, the order might depend on s (neither could we settle this question.)

We single out that the element $x_{2m} \circ j_{4k-1}$ of $\pi_{2m+4k-1}(B\mathrm{U}(s))$ can be written down explicitly by means of the J-homomorphism and of the real and the complex Bott periodicity isomorphisms. Let us now give some numerical examples, where the indicated homotopy groups of the Grassmannians $B\mathrm{U}(n)$ can for instance be found either

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in Mimura's survey article [9] or in Lundell's tables [6].

EXAMPLES 1.2. i) For $k = 1$ and $m = 3$, denom$\big(B_1/4\big) = 24$ holds and we can take $s = 3$ or 4; the corresponding groups are $\pi_9(BU(3)) \cong \mathbb{Z}/12$ and $\pi_9(BU(4)) \cong \mathbb{Z}/24$. We see that $x_6 \circ j_3$ is a generator of the former, but only generates a subgroup of index 2 in the latter. Changing m yields in each case an element of order 12 or 24 in the first indicated group and of order 12 in the second one:

$$\underline{m = 4}:$$
$$\pi_{11}(BU(4)) \cong \frac{\mathbb{Z}}{2} \oplus \frac{\mathbb{Z}}{24} \oplus \frac{\mathbb{Z}}{5} \qquad \pi_{11}(BU(5)) \cong \frac{\mathbb{Z}}{24} \oplus \frac{\mathbb{Z}}{5};$$
$$\underline{m = 5}:$$
$$\pi_{13}(BU(5)) \cong \frac{\mathbb{Z}}{72} \oplus \frac{\mathbb{Z}}{5} \qquad \pi_{13}(BU(6)) \cong \frac{\mathbb{Z}}{144} \oplus \frac{\mathbb{Z}}{5}.$$

ii) Since denom$\big(B_2/8\big) = 24$, for $k = 2$ and $m = 5$, we get an element of order 24 in the groups

$$\pi_{17}(BU(5)) \cong \frac{\mathbb{Z}}{2} \oplus \frac{\mathbb{Z}}{48} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}$$
$$\pi_{17}(BU(6)) \cong \frac{\mathbb{Z}}{144} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}$$
$$\pi_{17}(BU(7)) \cong \frac{\mathbb{Z}}{576} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}$$
$$\pi_{17}(BU(8)) \cong \frac{\mathbb{Z}}{1152} \oplus \frac{\mathbb{Z}}{5} \oplus \frac{\mathbb{Z}}{7}.$$

We observe that even for k even, the element $x_{4k+2} \circ j_{4k-1}$ does generally *not* generate a direct summand in $\pi_{8k+1}(BU(2k + 1))$.
iii) In Theorem 1.1, the case of most interest for $k \geqslant 1$ fixed is when m and s are as small as possible, namely $m = s = 2k + 1$: it predicts that $x_{4k+2} \circ j_{4k-1}$ is of order denom$\big(B_k/4k\big)$ (or possibly half of it for k odd) in the group $\pi_{8k+1}(BU(2k + 1))$. As an illustration, for $k = 6$, we get the element $x_{26} \circ j_{23}$ of order 65520 in $\pi_{49}(BU(13))$.

Here is a brief outline of the content of the paper. In Section 2, we study the K-theory of 2-cell complexes with even dimensional cells, say $X = S^{2m} \cup_f e^{2m+2l}$. In particular, we determine the Chern classes of the elements of $K^0(X)$ in terms of the Adams e-invariant of the attaching map f. The connection with the homotopy of $BU(n)$ is obtained by studying the set of bundles over X that restrict to a given multiple of the Bott generator x_{2m} over the sphere S^{2m}. Section 3 contains the proof of Theorem 1.1.

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2. On the K-theory of 2-cell complexes

In this section, we recall some basic and well-known properties of the K-theory of 2-cell complexes, in order to establish a key ingredient (Proposition 2.2 below) for the proof of Theorem 1.1.

Let $f\colon S^{2m+2l-1} \longrightarrow S^{2m}$ be a pointed map with $m,\ l \geqslant 1$, and let X be the mapping cone of f, i.e. the 2-cell complex $S^{2m} \cup_f e^{2m+2l}$. Denote by ι the inclusion of S^{2m} in X, and let $p\colon X \twoheadrightarrow X/S^{2m} \simeq S^{2m+2l}$ be the collapsing map; they fit in the cofibre sequence $S^{2m} \hookrightarrow X \twoheadrightarrow S^{2m+2l}$. For a sphere S^{2q}, we designate the Bott generator of $\widetilde{K}(S^{2q})$ by x_{2q}. Taking $\xi \in (\iota^*)^{-1}(x_{2m})$ and $\eta := p^*(x_{2m+2l})$, we get

$$K^0(X) \cong \mathbb{Z} \oplus \mathbb{Z} \cdot \xi \oplus \mathbb{Z} \cdot \eta \cong \mathbb{Z}^3.$$

Notice that ξ is uniquely determined up to addition of an integral multiple of η. Similarly, the integral cohomology of X is given by

$$H^*(X;\, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \cdot y \oplus \mathbb{Z} \cdot z \cong \mathbb{Z}^3,$$

with y corresponding via ι^* to a generator of $H^{2m}(S^{2m};\, \mathbb{Z})$, and z corresponding via p^* to a generator of $H^{2m+2l}(S^{2m+2l};\, \mathbb{Z})$; we use the same notation for the rational cohomology of X. The ring structure is given by $xy = 0$, $z^2 = 0$ and $y^2 = H(f)\cdot z$, where $H(f)$ denotes the Hopf invariant of $[f] \in \pi_{4m-1}(S^{2m})$ when $m = l$, and $H(f) := 0$ when $m \neq l$. The Chern character is given by $ch(\xi) = y + \lambda \cdot z$ and $ch(\eta) = z$, for some rational number λ. Because of the different possible choices for ξ, the rational number λ is only determined modulo 1, i.e. it represents a unique element $e(f)$ in the group $\mathbb{Q}/\mathbb{Z}$, called the Adams e-invariant of f (also denoted by $e_{\mathbb{C}}(f)$). It only depends on the homotopy class of f. Without loss of generality, we can consider $e(f)$ as a uniquely determined element of $\mathbb{Q} \cap]-1/2, 1/2]$. (See [1], pp. 321–323 for some more details on the e-invariant.) Since ch is an injective ring homomorphism (X being torsion-free), the product in $\widetilde{K}(X)$ is given by $\xi^2 = H(f) \cdot \eta$, $\xi\eta = 0$ and $\eta^2 = 0$. We would like to compute the Chern classes of ξ and η. They are closely related to the Chern character, as we now recall. For a connected finite CW-complex Y, we denote by ch_{2k} the component of ch in $H^{2k}(Y;\, \mathbb{Q})$. One has $ch_{2k} = (1/k!)s_k(c_1, \ldots, c_k)$ (for $k \geqslant 1$), where the s_k's are the Newton polynomials. They are defined by the relation $s_k(\sigma_1, \ldots, \sigma_k) = t_1^k + \cdots + t_k^k$, with σ_j the j-th elementary symmetric polynomial in $t_1, \ldots, t_k$ (see for example [5], p. 92). Newton's formula reads

$$s_k - c_1 s_{k-1} + c_2 s_{k-2} - \cdots + (-1)^{k-1} c_{k-1} s_1 + (-1)^k k \cdot c_k = 0$$

(see *loc. cit.*). Coming back to X, it is straightforward to check that

$$c_m(\xi) = (-1)^{m-1}(m-1)! \cdot y \quad \text{and} \quad c(\eta) = 1 + (-1)^{m+l-1}(m+l-1)! \cdot z.$$

Clearly, for $j \notin \{m,\, m+l\}$ and $1 \leqslant k \leqslant m-1$, one has the equalities $s_m(\xi) = m!\cdot y$ and $c_j(\xi) = s_k(\xi) = 0$. In Newton's formula for s_{m+l}, the only possible nonzero contributions are $(-1)^{m+l}(m+l)c_{m+l}$ and, if $m = l$, the product $(-1)^m c_m s_m$. After a short computation, we get

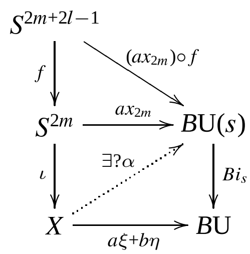
$$c(\xi) = 1 + (-1)^{m-1}(m-1)! \cdot y + \left(\frac{((m-1)!)^2}{2} \cdot H(f) + (-1)^{m+l-1}(m+l-1)! \cdot e(f)\right) \cdot z.$$

Now, for $a,\, b \in \mathbb{Z}$, we find

$$\begin{aligned} c(a\xi + b\eta) &= c(\xi)^a \cdot c(\eta)^b \\ &= 1 + (-1)^{m-1}(m-1)! a \cdot y \\ &\quad + \left(\frac{((m-1)!)^2}{2} a^2 \cdot H(f) + (-1)^{m+l-1}(m+l-1)!(a \cdot e(f) + b)\right) \cdot z. \end{aligned}$$

Recall that for a connected finite CW-complex Y, the *geometric dimension* of a stable bundle $\vartheta \in \widetilde{K}(Y) = [Y,\, B\mathrm{U}]$ is the smallest integer $n \geqslant 0$ such that ϑ lifts, up to homotopy, to a map $Y \longrightarrow B\mathrm{U}(n)$, in other words, such that $n + y \in \mathbb{Z} \oplus \widetilde{K}(Y)$ can be represented by a complex n-bundle over Y; we denote it by $n = \text{g-dim}(\vartheta)$. We also define $\text{c-dim}(\vartheta)$ as the smallest positive integer i such that $c_j(\vartheta) = 0$ in $H^{2j}(Y;\, \mathbb{Z})$ for all $j > i$. Clearly, $\text{c-dim}(\vartheta) \leqslant \text{g-dim}(\vartheta)$. (The reader may refer to [7] for details on the functions g-dim and c-dim.)

Now, suppose that $l < m$ (as a consequence of which $H(f) = 0$ holds). Fix an integer a and let $a\xi + b\eta \in \widetilde{K}(X)$, where b is considered as an unknown integral parameter; let s satisfy $m \leqslant s \leqslant m + l - 1$. Denote by i_s the inclusion of $\mathrm{U}(s)$ in U, and consider the following diagram representing a lifting and extension problem:



Clearly, there exists, up to homotopy, an extension of ax_{2m} to X if and only if the composition $(ax_{2m}) \circ f$ is zero in $\pi_{2m+2l-1}(B\mathrm{U}(s))$. In this case, the composition $Bi_s \circ \alpha \in \widetilde{K}(X)$ is a stable vector bundle ζ over X such that $\iota^*(\zeta) = ax_{2m}$ and with $\text{g-dim}(\zeta) \leqslant s$. It follows that there exists an integer b (our parameter!) such that $\zeta = a\xi + b\eta$ and $\text{c-dim}(\zeta) \leqslant s \leqslant m + l - 1$, and therefore $c_{m+l}(\zeta) = 0$. We have thus proved that

$$(ax_{2m}) \circ f = 0 \in \pi_{2m+2l-1}(B\mathrm{U}(s)) \Longrightarrow \exists b \in \mathbb{Z} \ \text{ s.t. } c_{m+l}(a\xi + b\eta) = 0.$$

We call this condition (♠). Since $H(f) = 0$, the above computation of the Chern classes for X shows that

$$c_{m+l}(a\xi + b\eta) = 0 \Longleftrightarrow a \cdot e(f) + b = 0.$$

This means that the denominator of $e(f) \in \mathbb{Q} \cap]-1/2, 1/2]$, expressed in lowest terms, must divide a. By Theorem 41.5 in Steenrod [11], we have

$$\text{g-}\dim(a\xi + b\eta) < m + l \Longleftrightarrow c_{m+l}(a\xi + b\eta) = 0.$$

So, for $s = m + l - 1$, condition (♠) is an equivalence. Now, the following lemma provides the necessary control, with respect to a, of the element $(ax_{2m}) \circ f$.

Lemma 2.1. *For $l < m$ and for $a \in \mathbb{Z}$, we have*

$$(ax_{2m}) \circ f = a \cdot (x_{2m} \circ f) \in \pi_{2m+2l-1}(B\mathrm{U}(s)).$$

Proof. For $l < m$, the Freudenthal Suspension Theorem (see [4], Theorem VI.2.10) implies that f is a suspension and the lemma follows directly from Theorem VI.2.3 in [4]. □

The group $\pi_{2m+2l-1}(B\mathrm{U}(s))$ is finite for $1 \leqslant s \leqslant m + l - 1$, as is well-known (see for example Lemma 4.2 in [7] for a proof). We now collect the results obtained so far in a proposition.

Proposition 2.2. *For $1 \leqslant l \leqslant m - 1$, let $f\colon S^{2m+2l-1} \longrightarrow S^{2m}$ be a pointed map; let x_{2m} be the Bott generator of $\widetilde{K}(S^{2m}) \cong [S^{2m}, BU(s)]$, $m \leqslant s \leqslant m + l - 1$. Then, the composition $x_{2m} \circ f$ represents a class in $\pi_{2m+2l-1}(B\mathrm{U}(s))$, whose order is a multiple of* $\mathrm{denom}(e(f))$, *the denominator of the Adams e-invariant $e(f)$ expressed in lowest terms. For $s = m + l - 1$, the order of $x_{2m} \circ f$ is precisely* $\mathrm{denom}(e(f))$.

3. The proof of Theorem 1.1

We apply Proposition 2.2 with $f = j_{4k-1}\colon S^{2m+4k-1} \longrightarrow S^{2m}$ and with $l = 2k$. By Adams [1] and Quillen [10], the image of J is a direct summand in $\pi_{2m+4k-1}(S^{2m})$ and is of order exactly $M_k := \mathrm{denom}\left(B_k/4k\right)$ (see also Switzer [12], p. 488). This means that j_{4k-1} is of order M_k and generates a direct summand. On the other hand, by Theorem 1 of Dyer [3], the Adams e-invariant $e(j_{4k-1})$ (expressed in lowest terms) has denominator M_k/b_k, where b_k is equal to 1 (resp. 2) for k even (resp. odd). (This result is also a consequence of Adams [1], Proposition 7.14 and Theorem 7.16.) The proof is complete. □

References

[1] J.F. Adams: *On the groups $J(X)$ IV*, Topology **5** (1966), 21–71.
[2] R. Bott: *The stable homotopy of the classical groups*, Ann. of Math. **70** (1959), 313–337.
[3] E. Dyer: *Chern characters of certain complexes*, Math. Z. **80** (1962) 363–373.
[4] P.J. Hilton: An introduction to homotopy theory, Cambridge, at the University Press, 1953.
[5] F. Hirzebruch: Topological methods in algebraic geometry, Springer-Verlag, Berlin, 1995.
[6] A.T. Lundell: Concise tables of James numbers and some homotopy of classical Lie groups and associated homogeneous spaces, In *Algebraic topology* (*San Feliu de Guíxols*, 1990), 250–272. Springer, Berlin, 1992.
[7] M. Matthey and U. Suter: *The positive cone of spheres and some products of spheres*, Enseign. Math. **47** (2001), 131–160.
[8] M. Mimura and H. Toda: Topology of Lie groups I, II, American Mathematical Society, Providence, RI, 1991.
[9] M. Mimura: Homotopy theory of Lie groups, In *Handbook of algebraic topology*, 951–991. North-Holland, Amsterdam, 1995.
[10] D. Quillen: *The Adams conjecture*, Topology **10** (1971), 67–80.
[11] N. Steenrod: The topology of fibre bundles, Princeton University Press, Princeton, NJ, 1999.
[12] R.M. Switzer: Algebraic topology—homotopy and homology, Springer-Verlag, New York, 1975. Die Grundlehren der mathematischen Wissenschaften, Band 212.

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