Kamei, S.
Osaka J. Math.
41 (2004), 357–370

CONES OVER THE BOUNDARIES OF NONSHELLABLE BUT CONSTRUCTIBLE 3-BALLS

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(Received August 21, 2002)

1. Introduction

Shellability is a fundamental and important concept for the study of combinatorics of simplicial complexes. After the proof of the Upper Bound Conjecture for convex polytopes, due to McMullen ([11]), many researchers study this concept in many fields of combinatorics. It is known that every shellable pseudomanifold is either a ball or a sphere. Furthermore in dimension 2, if a pseudomanifold is a ball or a sphere, it is always shellable. On the other hand, many examples of nonshellable balls and spheres are known in dimension more than 2. Many related examples appear in [16].

Constructibility can be viewed as a relaxation of shellability. This notion appears in different combinatorial contexts in [1], [4], and [14]. The same as shellability, it can be shown that every constructible pseudomanifold is either a ball or a sphere, and for the converse, examples of nonconstructible balls and spheres are studied in dimension more than 2 in [5], [6], [8] and [9]. As is mentioned in [1], constructibility is strictly weaker than shellability. In fact, it is known that the examples of nonshellable 3-balls which are presented by Rudin, Grünbaum, and Ziegler in [13], [4], and [17] respectively are all constructible ([6]). On the other hand, there are still no examples of nonshellable but constructible 3-spheres. Then it rouse our interest whether there exists a nonshellable but constructible 3-sphere or not.

To obtain a 3-sphere, it is a natural way to take a cone over the boundary of some 3-ball. So it is a natural approach for exploring the difference between shellability and constructibility of 3-spheres to study cones over the boundaries of nonshellable but constructible 3-balls. Recently Hachimori constructed shellings of cones over the boundaries of above nonshellable but constructible 3-balls by using the computer program which he developed ([7]). In this paper we will consider a theoretical explanation for the shellings of the spheres, and study more complicated cases. Concretely we will prove the following theorem.

Theorem 3.3. *Let $B_1, B_2, \dots, B_n$ be constructible* 3*-balls which satisfy the following condition*; *each B_i can be decomposed into two* 3*-balls C_i and C_i' such that each C_i and C_i' has a shelling starting with an arbitrary facet and that $C_i \cap C_i'$ is a* 2*-ball. Consider a boundary connected sum of $B_1, B_2, \dots, B_n$ which is homeomorphic* *to a* 3*-ball such that each* C_i (C_i') *is glued at most one other ball* B_j *together. Then a cone over the boundary of the boundary connected sum is shellable.*

The condition of this theorem seems very strict. However, the examples of nonshellable but constructible 3-balls mentioned above all satisfy the condition. Furthermore we will prove the following theorem.

Theorem 4.1. *Let* B_1, $B_2, \dots, B_n$ *be constructible* 3*-balls which satisfy the following condition*; *each* B_i *can be decomposed into two* 3*-balls* C_i *and* C_i' *such that each* C_i *and* C_i' *has a shelling starting with an arbitrary facet and that* $C_i \cap C_i'$ *is a* 2*-ball and that there are no inner edges of* $\partial C_i \cap \partial B_i$ *and* $\partial C_i' \cap \partial B_i$ *of which vertices are both contained in* $\partial C_i \cap \partial C_i'$*. Consider any boundary connected sum of* B_1, $B_2, \dots, B_n$ *which is homeomorphic to a* 3*-ball. Then a cone over the boundary of the boundary connected sum is shellable.*

It seems that the examples stated in Section 2 do not satisfy the condition of this theorem. But later we will see another example which satisfies the condition.

In Section 2, we define notations, and see some examples. In Section 3, we consider shellings of some easy cases and prove Theorem 3.3. In Section 4, we prove Theorem 4.1.

REMARK. There exists an easy example of a 3-sphere as a pseudosimplicial complex which is nonshellable but constructible. Consider a Ziegler's ball, that will be stated in the next section precisely, and its mirror symmetry. Glue them together along the corresponding 2-faces. Then the obtained pseudosimplicial complex is nonshellable but constructible. See [10] for the definition of the pseudosimplicial complex. Also see [15].

2. Definitions and examples

A simplicial complex C is a finite set of simplices in some Euclidean space such that (1) if $\sigma \in C$, all the faces of σ (including the empty set) are contained in C, and (2) if σ, $\sigma' \in C$, then $\sigma \cap \sigma'$ is a face of both σ and σ'. The 0-dimensional simplices in C are the *vertices* and the 1-dimensional simplices are the *edges* of C. The inclusion-maximum faces are called *facets*. The dimension of C is the largest dimension of facets. A *d-complex* is short for a d-dimensional simplicial complex. If all the facets of C have the same dimension, then C is called *pure*. In particular, the simplicial complex which has only the empty set as a face is a pure complex of dimension -1, with a single facet. For a set of simplices $C' \subseteq C$, the simplicial complex $\overline{C'}$ consists of the simplices in C' together with all their faces. The union $|C|$ of the simplices of C is called the *underlying space* of C. If $|C|$ is homeomorphic to a manifold M, then C is a *triangulation* of M. If C is a triangulation of a d-ball or of a d-sphere, then C will be simply called a *d-ball* or a *d-sphere*. For any triangulation C of a manifold, the *boundary complex* ∂C is the collection of all simplices of C which lie on the boundary of the manifold. The interior $\operatorname{int} C$ is the set $C \setminus \partial C$. A d-dimensional pure simplicial complex is *strongly connected* if for any two of its facets F and F', there is a sequence of facets $F = F_1, F_2, \ldots, F_k = F'$ such that $F_i \cap F_{i+1}$ is a face of dimension $d - 1$, for $1 \leq i \leq k - 1$. If a d-dimension pure simplicial complex is strongly connected and each $(d-1)$-dimensional face belongs to at most two facets, then it is called a *pseudomanifold*. Every triangulation of a connected manifold is a pseudomanifold. For a simplicial complex C and a face σ, the *star neighborhood* $\operatorname{star}_C \sigma$ is the subcomplex of C which contains all faces of facets of C containing σ. For a simplex σ and a vertex $v \notin \sigma$, the *join* $v * \sigma$ is a simplex whose vertices are those of σ plus the extra vertex v. The join $v * C$ of a simplicial complex C with a new vertex v is defined as $v * C = \{v * \tau : \tau \in C\} \cup C$. For some simplicial complex C, we consider a *cone over the boundary*, that is, by forming $C \cup (v * \partial C)$. $v * \partial C$ is called the *cone part* of $C \cup (v * \partial C)$.

DEFINITION. A pure d-dimensional simplicial complex is *shellable* if its facets can be ordered $F_1, F_2, \ldots, F_t$ so that $\left(\bigcup_{i=1}^{j-1} \overline{F_i}\right) \cap \overline{F_j}$ is a pure $(d-1)$-complex for $2 \leq j \leq t$. This ordering of the facets is called a *shelling*.

In the followings, we also use another definition of shellability, that is, a pure d-dimensional simplicial complex C is shellable if (1) C is a simplex, or (2) there exist a d-dimensional simplex Δ and d-dimensional shellable subcomplex C' such that $C = \Delta \cup C'$ and that $\Delta \cap C'$ is a $(d-1)$-dimensional shellable complex. We can see this definition is equivalent to the definition above. We call the shelling order of the first definition the *regular order* and of the second definition the *reverse order*. There will be the cases where the orders are not mentioned. In the cases we will consider the regular order.

DEFINITION. A pure d-dimensional simplicial complex C is *constructible* if
(1) C is a simplex, or
(2) there exist d-dimensional constructible subcomplexes C_1 and C_2 such that $C = C_1 \cup C_2$ and that $C_1 \cap C_2$ is a $(d-1)$-dimensional constructible simplicial complex.

Now we will see some examples. The following examples are all nonshellable 3-balls. Furthermore the first three examples are showed constructible in [6]. In fact, we can decompose each 3-ball into two shellable 3-balls C_1 and C_2, where C_i is a simplicial complex specified at the lists below, and also we can check $C_1 \cap C_2$ is a 2-ball.

EXAMPLE 1. The first example which is presented by Ziegler has 10 vertices and 21 facets ([17]). The following list is all facets of the ball.

C_1 :	$\{1, 2, 3, 4\}$	$\{1, 2, 5, 6\}$	$\{2, 3, 6, 7\}$	$\{4, 1, 8, 5\}$	$\{1, 5, 6, 9\}$	$\{1, 6, 2, 9\}$
	$\{1, 2, 4, 9\}$	$\{1, 4, 8, 9\}$	$\{1, 8, 5, 9\}$	$\{2, 5, 6, 0\}$	$\{2, 6, 7, 0\}$	$\{2, 7, 3, 0\}$
	$\{2, 3, 1, 0\}$	$\{2, 1, 5, 0\}$				
C_2 :	$\{3, 4, 7, 8\}$	$\{3, 6, 7, 8\}$	$\{3, 2, 4, 8\}$	$\{3, 2, 6, 8\}$	$\{4, 5, 7, 8\}$	$\{4, 1, 3, 7\}$
	$\{4, 1, 5, 7\}$					

EXAMPLE 2. The second example which is presented by Rudin has 14 vertices and 41 facets ([13]). The following list is all facets of the ball.

C_1 :	$\{3, 4, 7, 11\}$	$\{3, 4, 7, 12\}$	$\{4, 7, 11, 12\}$	$\{4, 8, 11, 12\}$	$\{5, 6, 9, 13\}$
	$\{5, 6, 9, 14\}$	$\{6, 9, 13, 14\}$	$\{6, 10, 13, 14\}$	$\{7, 11, 12, 13\}$	$\{3, 7, 12, 13\}$
	$\{3, 9, 12, 13\}$	$\{9, 13, 14, 11\}$	$\{5, 9, 14, 11\}$	$\{5, 7, 14, 11\}$	$\{1, 3, 9, 13\}$
	$\{1, 3, 7, 13\}$	$\{1, 7, 11, 13\}$	$\{1, 5, 7, 11\}$	$\{1, 5, 9, 11\}$	$\{1, 9, 11, 13\}$
C_2 :	$\{4, 5, 8, 12\}$	$\{4, 5, 8, 13\}$	$\{5, 8, 12, 13\}$	$\{5, 9, 12, 13\}$	$\{6, 3, 10, 14\}$
	$\{6, 3, 10, 11\}$	$\{3, 10, 14, 11\}$	$\{3, 7, 14, 11\}$	$\{8, 12, 13, 14\}$	$\{4, 8, 13, 14\}$
	$\{4, 10, 13, 14\}$	$\{10, 14, 11, 12\}$	$\{6, 10, 11, 12\}$	$\{6, 8, 11, 12\}$	$\{2, 4, 10, 14\}$
	$\{2, 4, 8, 14\}$	$\{2, 8, 12, 14\}$	$\{2, 6, 8, 12\}$	$\{2, 6, 10, 12\}$	$\{2, 10, 12, 14\}$
	$\{11, 12, 13, 14\}$				

EXAMPLE 3. The third example which is presented by Grünbaum has 14 vertices and 29 facets. This example appears in [3] first, but the facet list in it has a typo. The correct list appears in [6]. The following list is all facets of the ball.

C_1 :	$\{1, 2, 3, 7\}$	$\{1, 2, 4, 8\}$	$\{1, 2, 7, 8\}$	$\{1, 3, 5, 7\}$	$\{1, 4, 8, 10\}$
	$\{1, 5, 6, 13\}$	$\{1, 5, 7, 13\}$	$\{1, 6, 11, 13\}$	$\{1, 7, 8, 10\}$	$\{1, 7, 11, 13\}$
	$\{2, 3, 7, 9\}$	$\{2, 7, 8, 9\}$	$\{3, 5, 7, 9\}$	$\{5, 7, 9, 13\}$	$\{7, 8, 9, 13\}$
C_2 :	$\{2, 4, 6, 8\}$	$\{2, 5, 6, 14\}$	$\{2, 5, 12, 14\}$	$\{2, 6, 8, 14\}$	$\{2, 8, 12, 14\}$
	$\{4, 6, 8, 10\}$	$\{5, 6, 13, 14\}$	$\{5, 12, 13, 14\}$	$\{6, 8, 10, 14\}$	$\{6, 11, 13, 14\}$
	$\{7, 8, 10, 14\}$	$\{7, 8, 13, 14\}$	$\{7, 11, 13, 14\}$	$\{8, 12, 13, 14\}$	

REMARK. (1) Example 1, 2 and 3 can be realized in $\mathbf{R}^3$. See [17], [13], and [6] respectively. (2) Each C_i in Example 1, 2 and 3 has a shelling starting with an arbitrary facet. We will use this property in the followings.

The next example is classically known as a nonshellable 3-ball. It is also proved nonconstructible in [6].

EXAMPLE 4. Consider a pile of cubes with a plugged knotted hole, and triangulate each cube so that the edges of the cubes are also the edges of the triangulation. This example is called “Furch’s knotted hole ball”. For more details, see [6] and [17].

A cone over the boundary of the 3-ball of Example 4 is showed nonconstructible in [9]. In the next section, we will see that cones over the boundary of the 3-balls which are stated in Example 1, 2 and 3 are all shellable.

3. Unions of shellable 3-balls and cones over their boundaries

The following terminology was defined by Danaraj and Klee. See [4] and [16].

DEFINITION. A simplicial complex is *extendably shellable* if for every shellable subcomplex of the same dimension there is a shelling of the whole complex that shells the subcomplex first.

For 2-balls and 2-spheres, the following property is classically known ([12], [4]).

Lemma 3.1. *Every* 2*-sphere and* 2*-ball is extendably shellable.*

From this lemma, we can see that for every 2-ball there is a shelling starting with an arbitrary facet, and for every 2-sphere there is a shelling starting with an arbitrary facet and ending with another arbitrary facet.

Theorem 3.2. *Let B be a constructible* 3*-ball which can be decomposed into two shellable* 3*-balls C and C' such that $C \cap C'$ is a* 2*-ball. Then a cone over the boundary of B is shellable.*

Proof. Let v be a cone point. We will remove facets in turn and construct the reverse order of a shelling of $B \cup (v * \partial B)$ concretely.

First we remove the facets of C along the regular order of a shelling of C. At each step, the union of the removed facets is a 3-ball so that the complement is also a 3-ball and the intersection of them is a 2-sphere. The simplicial complex ∂B is a 2-sphere, then there is a shelling of ∂B which shells $\partial B \cap \partial C$ first from Lemma 3.1. So remove the facets of $v * \partial B$ along the regular order of the shelling. The remained subcomplex is C'. Then remove the facets of C' along the reverse order of a shelling of C'. At last we obtain the reverse order of a shelling of $C \cup (v * \partial B) \cup C'$. □

From this theorem, we can see cones over the boundaries of the 3-balls which are stated in Example 1, 2 and 3 are all shellable. On the other hand, we can construct constructible 3-balls which do not satisfy the condition of Theorem 3.2. To see this, we define an operation as the following.

DEFINITION. Let C_1, C_2 be 3-dimensional simplicial complexes with boundaries. Let δ_i be a 2-face of ∂C_i. Consider an isomorphic map from δ_1 to δ_2 and glue C_1 and C_2 together along the map. The simplicial complex thus obtained is called a *boundary* *connected sum* of C_1 and C_2.

A boundary connected sum of some two 3-balls which are stated in Example 1, 2 and 3 is also a constructible 3-ball. It is obvious that the 3-ball cannot be decomposed into two shellable 3-balls such that the intersection of the decomposed 3-balls is shellable. But for some simple cases, we can prove the following theorem.

Theorem 3.3. *Let $B_1, B_2, \dots, B_n$ be constructible 3-balls which satisfy the following condition; each B_i can be decomposed into two 3-balls C_i and C_i' such that each C_i and C_i' has a shelling starting with an arbitrary facet and that $C_i \cap C_i'$ is a 2-ball. Consider a boundary connected sum of $B_1, B_2, \dots, B_n$ which is homeomorphic to a 3-ball such that each C_i (C_i') is glued at most one other ball B_j together. Then a cone over the boundary of the boundary connected sum is shellable.*

Proof. Let v be a cone point. We may reorder the index so that B_i and B_{i+1} are glued together at 2-faces of C_i' and C_{i+1} for $1 \leq i \leq n-1$.

First we remove facets of C_1 along the regular order of a shelling of C_1. Let δ_1 be $C_1' \cap C_2$. Consider a shelling of ∂C_1 which shells $\partial C_1 \cap \partial B_1$ first and ends with δ_1. Remove the facets of $v * (\overline{\partial B_1 \setminus \delta_1})$ along the regular order of the shelling. Furthermore remove the facets of C_1' along the reverse order of a shelling starting with the facet containing δ_1.

Continuously we remove the facets the same as above. Then we can remove all facets and construct a shelling the same as Theorem 3.2. □

4. More complicated cases

In this section, we will study more complicated cases. For a surface S, a 1-face of S is called an *inner edge* if it is not contained in ∂S.

Theorem 4.1. *Let $B_1, B_2, \dots, B_n$ be constructible 3-balls which satisfy the following condition; each B_i can be decomposed into two 3-balls C_i and C_i' such that each C_i and C_i' has a shelling starting with an arbitrary facet and that $C_i \cap C_i'$ is a 2-ball and that there are no inner edges of $\partial C_i \cap \partial B_i$ and $\partial C_i' \cap \partial B_i$ of which vertices are both contained in $\partial C_i \cap \partial C_i'$. Consider any boundary connected sum of $B_1, B_2, \dots, B_n$ which is homeomorphic to a 3-ball. Then a cone over the boundary of the boundary connected sum is shellable.*

REMARK. Take two 3-balls which satisfy the condition of Theorem 3.3, and consider a boundary connected sum of those 3-balls. We can see a cone over the boundary of the boundary connected sum as a connected sum of two cones over the boundaries of those 3-balls. (We will see this in the proof of Lemma 4.7 again.) So if the statement "any shellable 3-sphere has a shelling starting with an arbitrary facet" should

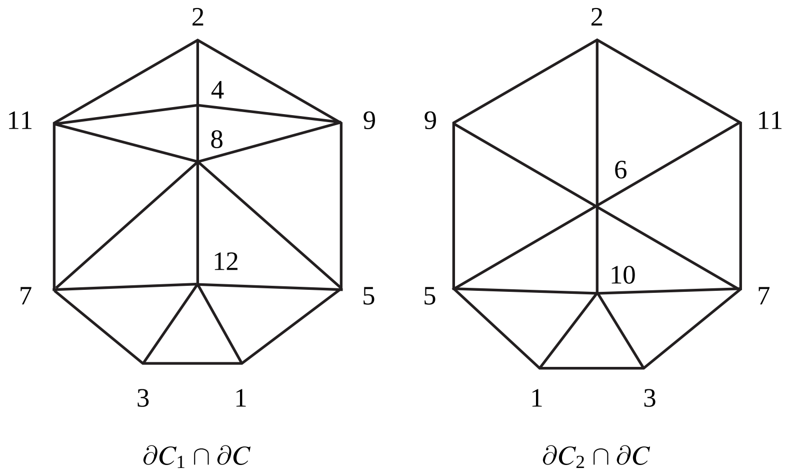

Fig. 1. The decomposition of the boundary of the 3-ball which is stated in Example 5.

be available, any 3-sphere which is a connected sum of two shellable 3-spheres is always shellable and thus Theorem 4.1 follows immediately. A similar statement for constructible 3-spheres is proved in [9, Theorem 4]. Also see the added comments of the theorem.

There is an example which satisfies the condition of Theorem 4.1. This example is also presented by Ziegler. In [17], Example 1 is constructed by modifying this example.

EXAMPLE 5. This 3-ball is also nonshellable but constructible. It has 12 vertices and 25 facets. The following list is all facets of the 3-ball.

C_1 : $\{1,2,3,4\}$ $\{1,4,8,5\}$ $\{3,4,7,8\}$ $\{1,2,4,9\}$ $\{1,4,8,9\}$ $\{1,5,8,9\}$
$\{3,7,8,11\}$ $\{3,4,8,11\}$ $\{2,3,4,11\}$ $\{1,3,4,12\}$ $\{1,4,5,12\}$ $\{4,5,8,12\}$
$\{4,7,8,12\}$ $\{3,4,7,12\}$
C_2 : $\{1,2,5,6\}$ $\{2,3,6,7\}$ $\{1,5,6,9\}$ $\{1,6,2,9\}$ $\{2,5,6,10\}$ $\{2,6,7,10\}$
$\{2,3,7,10\}$ $\{1,2,3,10\}$ $\{1,2,5,10\}$ $\{2,3,6,11\}$ $\{3,6,7,11\}$

This 3-ball can be decomposed into two shellable 3-balls C_1 and C_2 such that $C_1 \cap C_2$ is a 2-ball the same as Example 1, 2 and 3. Let C be $C_1 \cup C_2$. Fig. 1 specifies $\partial C_1 \cap \partial C$ and $\partial C_2 \cap \partial C$. We can check this example satisfies the condition of Theorem 4.1.

To prove the main theorem, we prepare some lemmas.

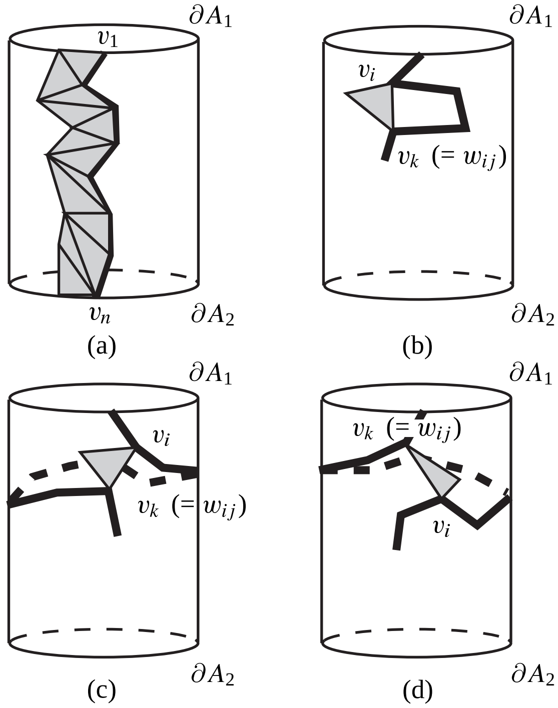

Fig. 2. Constructions of Σ_i.

Lemma 4.2. *Let A be a simplicial complex which is homeomorphic to an annulus. Let ∂A_1 and ∂A_2 be the boundary components of A. Suppose that there are no inner edges of which vertices are both contained in ∂A_1 or ∂A_2. Then there is a subcomplex Σ of A which is homeomorphic to a* 2*-ball such that each $\Sigma \cap \partial A_i$ is a* 1-*face* (Fig. 2a).

Proof. In the followings, we set ∂A_1 above and ∂A_2 below as Fig. 2, and determine the right and the left directions. Let P be a simple path which connects ∂A_1 and ∂A_2. We will construct Σ along P. Let $v_1, v_2, \dots, v_n$ be vertices on P ordered from $v_1 = \partial A_1 \cap P$ to $v_n = \partial A_2 \cap P$. For $i > 3$, if some v_i is connected to ∂A_1 by an edge, we take the largest number of such i and exchange v_2 for v_i. Also choose the leftmost point which is connected to v_2 by an edge, and exchange v_1 for the point. Similarly we improve v_{n-1} and v_n.

The 2-ball $\mathrm{star}_A\, v_i$ is divided by the 1-ball $\mathrm{star}_P\, v_i$. Take a subcomplex of $\mathrm{star}_A\, v_i$ which contains all faces which belong to the left side of $\mathrm{star}_P\, v_i$, and denote it by Σ_i. We construct the union $\bigcup_{i=1}^{k} \Sigma_i$ in turn, and denote the vertices contained in $(\Sigma_i \setminus \Sigma_{i-1}) \setminus \mathrm{star}_P\, v_i$ by w_{ij} which are ordered from the point close to ∂A_1. If each $\Sigma_i \cap P$ coincides with $\mathrm{star}_P\, v_i$, the union $\bigcup_{i=1}^{n} \Sigma_i$ satisfies the assertion. So we see w_{ij} in lexicographically order and assume that some w_{ij} coincides with some v_k first. There are the following three cases.

CASE 1. w_{ij} coincides with v_k such that $k > i$ and that the simple closed curve $\{v_i, v_{i+1}, \dots, v_k, v_i\}$ is null-homotopic in A (Fig. 2b). In this case, we take a new path as P with exchanging the subcomplex $\{v_i, v_{i+1}, \dots, v_k\}$ for the subcomplex $\{v_i, v_k\}$.

CASE 2. w_{ij} coincides with v_k such that $k > i$ and that the simple closed curve $\{v_i, v_{i+1}, \dots, v_k, v_i\}$ is not null-homotopic in A (Fig. 2c). In this case, we take a new path P with exchanging the subcomplex $\{v_i, v_{i+1}, \dots, v_k\}$ for the subcomplex $\{v_i, v_k\}$.

CASE 3. w_{ij} coincides with v_k such that $k < i$ and that the simple closed curve $\{v_k, v_{k+1}, \dots, v_i, v_k\}$ is not null-homotopic in A (Fig. 2d). In this case, we take a new path P with exchanging the subcomplex $\{v_k, v_{k+1}, \dots, v_i\}$ for the subcomplex $\{v_k, v_i\}$. Notice that Σ_k which we take newly never contains a vertex of the subcomplex $\{v_1, \dots, v_{k-2}\}$ if $k > 2$. Then after we take the new Σ_k, the above two cases may occur but this case does not occur again.

We can proceed the index i in the first two cases and j in the third case, then the construction is terminated after finite steps. If $\bigcup_{i=1}^n \Sigma_i$ is a 2-ball, we adopt $\bigcup_{i=1}^n \Sigma_i$ as Σ. If $\bigcup_{i=1}^n \Sigma_i$ is not a 2-ball, we fill up the holes which are bounded by edges of $\partial\big(\bigcup_{i=1}^n \Sigma_i\big) \setminus (P \cup \partial A_1 \cup \partial A_2)$. Then we obtain a 2-ball Σ. From the improvement of the path P and the condition that there are no inner edges of which vertices are both contained in ∂A_1 or ∂A_2, only one facet of Σ contains $\Sigma \cap \partial A_i$ $(i = 1, 2)$. Thus $\Sigma \cap \partial A_i$ $(i = 1, 2)$ is a 1-face. At last we obtain a 2-ball Σ which satisfies the assertion. □

Lemma 4.3. *Let D be a 2-ball such that there are no inner edges of which vertices are both contained in ∂D. Let δ be a facet of D such that $\delta \cap \partial D$ is empty, and e be a 1-face of δ. Then there is a subcomplex Σ of D which is homeomorphic to a 2-ball such that $\Sigma \cap \partial D$ is a 1-face and that $\Sigma \cap \delta$ is a 1-ball containing e.*

Proof. In the followings, we set δ above and ∂D below so that we can determine the right and the left directions the same as Lemma 4.2.

Let v_1 be a vertex of e. First we assume that there is a path which connect v_1 and vertices of ∂D. Furthermore we assume that e belongs to the left side of v_1. Consider the leftmost edge connecting v_1 and ∂D and denote it by f. Let v_2 be the vertex of ∂D incident to f. Consider the union of the faces of $\mathrm{star}_{\overline{D\setminus\delta}}\, v_1 \cup \mathrm{star}_{\overline{D\setminus\delta}}\, v_2$ which belongs to the left side of f and denote it by Σ'. If $\Sigma' \cup \delta$ is a 2-ball, Σ' satisfies the condition of Σ. If $\Sigma' \cup \delta$ is not a 2-ball, the union of the subcomplex of $\partial\Sigma'$ and of $\partial\delta$ bound a 2-ball in $\overline{D \setminus (\delta \cup \Sigma')}$ (Fig. 3). Then Σ' and the bounded disk form a 2-ball which satisfies the condition of Σ. In the case where f belongs to the right

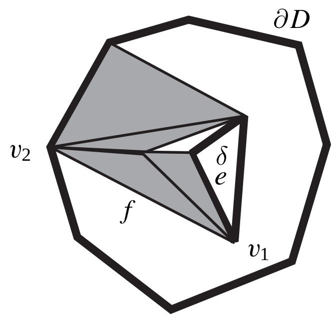

Fig. 3. An example of the case where $\Sigma' \cup \delta$ is not a 2-ball.

side of v_1, we can discuss the same as above.

Let w be the vertex opposite to e on $\partial\delta$. Consider the case where there are edges connecting w and a vertex of ∂D, and no edges connecting the vertices of e and of ∂D. Let e' be the 1-face of δ which belongs to the left side of w. We can construct a 2-ball Σ' such that $\Sigma' \cap \delta = e'$ and that $\Sigma' \cap \partial D$ and $\Sigma' \cap \partial\delta$ are 1-faces the same as above. Let v_1 be $\Sigma' \cap e$. If $\Sigma' \cup \mathrm{star}_{\overline{D\setminus\delta}} v_1$ is a 2-ball, we adopt $\Sigma' \cup \mathrm{star}_{\overline{D\setminus\delta}} v_1$ as Σ. If $\Sigma' \cup \mathrm{star}_{\overline{D\setminus\delta}} v_1$ is not a 2-ball, we fill up the holes the same as Lemma 4.2. Then we obtain a 2-ball which satisfies the condition of Σ.

Consider the case where there are no edges connecting $\partial\delta$ and ∂D. From Lemma 4.2, we can construct a 2-ball Σ' such that $\Sigma' \cap \partial D$ and $\Sigma' \cap \partial\delta$ are 1-faces. Assume that $\Sigma' \cap \partial\delta$ is not e. If $\mathrm{star}_{\overline{D\setminus\delta}} v_1 \cup \Sigma'$ is a 2-ball, we adopt $\Sigma' \cup \mathrm{star}_{\overline{D\setminus\delta}} v_1$ as Σ. Consider the case where $\Sigma' \cup \mathrm{star}_{\overline{D\setminus\delta}} v_1$ is not a 2-ball. Let v_1 be $e \cap \Sigma'$, P_1 be a component of $\overline{\partial\Sigma' \setminus (\delta \cup \partial D)}$ containing v_1 and P_2 be another component. If no vertices of $\mathrm{star}_{\overline{D\setminus(\delta\cup\Sigma')}} v_1$ coincide with vertices of P_2, we fill up the holes. If some vertices of $\mathrm{star}_{\overline{D\setminus(\delta\cup\Sigma')}} v_1$ coincide with vertices of P_2, we adopt P_2 as P and construct Σ' in the right side of P again. After all we obtain a 2-ball Σ. □

Lemma 4.4. *Let D be a 2-ball such that there are no inner edges of which vertices are both contained in ∂D. Let δ be a facet of D such that $\delta \cap \partial D$ is not empty, and e be a 1-face of δ. Then there is a subcomplex Σ of D which is homeomorphic to a 2-ball or a 1-ball such that $\Sigma \cap \partial D$ is a 1-face and that $\Sigma \cap \delta$ is a 1-ball containing e.*

Proof. Consider the case where $\partial D \cap \delta$ is a vertex which is not contained in e (Fig. 4a). In this case, we can construct a 2-ball the same as Lemma 4.3. Notice that the constructed 2-ball Σ satisfies $\Sigma \cap \delta = e$ since there are no inner edges of which vertices are both contained in ∂D.

Consider the case where $\partial D \cap \delta$ is a vertex which is contained in e (Fig. 4b). Let v be the vertex. The simplicial complex $\mathrm{star}_{\overline{D\setminus\delta}} v$ can be seen as the union of two

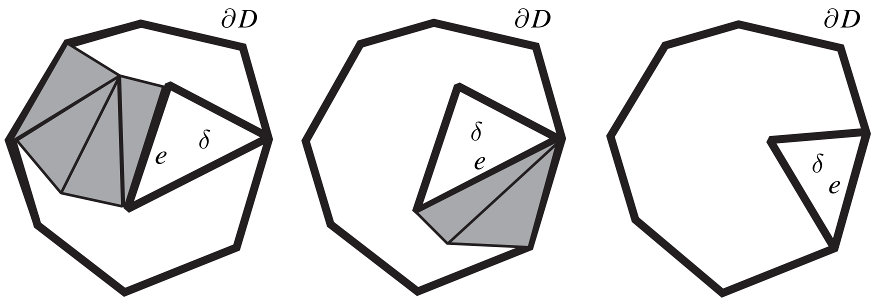

Fig. 4. Examples of balls which satisfies the condition of Lemma 4.5.

2-balls such that the intersection of them is only v. Then one of the 2-balls containing e satisfies the condition of Σ.

Finally we consider the case where $\partial D \cap \delta$ is e (Fig. 4c). In this case, e satisfies the condition of Σ. □

Lemma 4.5. *Let D be a 2-ball such that there are no inner edges of which vertices are both contained in ∂D. Let δ be a facet of D. Let D' be a subcomplex of D which is homeomorphic to a 2-ball such that $D' \cap \delta$ is a 1-face and that $D' \cap \partial D$ is a 1-ball. Then there is a subcomplex of D' which is homeomorphic to a 2-ball such that $\Sigma \cap \partial D$ and $\Sigma \cap \delta$ are 1-faces.*

Proof. Consider a component of $\overline{\partial D' \setminus (\partial\delta \cup \partial D)}$ as a path and construct a 2-ball the same as Lemma 4.2. Then we obtain a 2-ball which satisfies the condition of Σ. □

We will prove Theorem 4.1 by induction. The following lemma is the initial state of the induction and we will proceed the induction by Lemma 4.7.

Lemma 4.6. *Let B be a constructible 3-ball which can be decomposed into two shellable 3-balls B_1 and B_2 such that each B_i has a shelling starting with an arbitrary facet and that $B_1 \cap B_2$ is a 2-ball. Then a cone over the boundary of B has a shelling ending with an arbitrary facet of the cone part.*

Proof. Let v be a cone point. We will remove facets of $(v * \partial B) \cup B$ in turn and construct the reverse order of a shelling concretely. Let Δ be some facet of $v * \partial B$. There is a facet Δ' of B such that $\Delta \cap \Delta'$ is a 2-face of ∂B. We may assume that Δ' belongs to B_1. There is a shelling of B_1 starting with Δ'. Then we remove Δ at first and continuously remove facets of B_1 along the regular order of the shelling. At each step, the union of the removed facets is a 3-ball because the intersection of the removed subcomplex contained in B_1 and Δ is always $\Delta \cap \Delta'$. So we can remove the facets of $B_1 \cup \Delta$. Next consider a shelling of ∂B which starts with the facet $\Delta \cap \Delta'$ and shells $\partial B \cap \partial B_1$ first. Remove the facets of $\overline{(v * \partial B) \setminus \Delta}$ along the regular order of the shelling. Then the remainder is only B_2. We remove facets of B_2 along the reverse order of a shelling of B_2. At last we obtain the reverse order of a shelling of $(v * \partial B) \cup B$ which satisfies the assertion. □

Lemma 4.7. *Let B be a 3-ball such that a cone over the boundary of B has a shelling ending with some facet of the cone part. Let B' be a constructible 3-ball which satisfies the following condition*: (1) *B' can be decomposed into two 3-balls B'_1 and B'_2 such that each B'_i has a shelling starting with an arbitrary facet and that $B_1 \cap B_2$ is a 2-ball,* (2) *there are no inner edges on $\partial B'_1 \cap \partial B'$ and $\partial B'_2 \cap \partial B'$ of which vertices are both contained in $\partial B_i \cap \partial B'_i$. Consider any boundary connected sum of B and B'. Then a cone over the boundary of the boundary connected sum has a shelling ending with some facet of the cone part.*

Proof. Consider some boundary connected sum of B and B', and denote it by $B \natural B'$. We consider cones over the boundaries of B, B', and $B \natural B'$ with cone points v, v', and w, respectively. Let δ and δ' be the 2-faces of ∂B and $\partial B'$ such that B and B' are glued together at δ and δ', and let Δ and Δ' be the facets of $v * \partial B$ and $v' * \partial B'$ which satisfy $\Delta \cap \partial B = \delta$ and $\Delta' \cap \partial B' = \delta'$. Remove Δ and Δ' from $(v * \partial B) \cup B$ and $(v' * \partial B') \cup B'$, and glue them together along an orientation reversing isomorphic map from $\partial\Delta$ to $\partial\Delta'$ such that v coincides with v' and that δ coincides with δ' the same as the connected sum. Then the obtained simplicial complex is isomorphic to $(w * \partial(B \natural B')) \cup (B \natural B')$. We use this correspondence. In the followings, we assume that δ' belongs to $\partial B'_1$.

First we assume that there is a shelling of $(v * \partial B) \cup B$ ending with some facet of $v * \partial B$ except Δ. We remove facets of $(w * (\overline{\partial B \setminus \delta})) \cup B$ along the reverse order of the shelling, and at the step that the facet corresponding to Δ will be removed next, we remove the facets of $(w * (\overline{\partial B' \setminus \delta'})) \cup B'$ as the followings.

Let F be the facet of B which satisfies $F \cap \partial B = \delta$. Assume that F was already removed. In this case, we remove facets of B'_1 along the regular order of a shelling starting with the facet F' which satisfies $F' \cap F = \delta$. For removing more facets, we consider three facets of $w * (\overline{\partial B \setminus \delta})$ each of which contains a 1-face of δ. If all of the three facets were removed, the shelling ends with Δ and contradicts the assumption. Then some of the three facets were not removed. Also notice that at least one facet had to be removed because we removed the facet of $w * (\overline{\partial B \setminus \delta})$ at first and the removed subcomplex contained in $w * (\overline{\partial B \setminus \delta})$ must be a 3-ball. Let D be $\partial B'_1 \cap \partial B'$. Let e be a 1-face of $\partial\delta$ which is contained in the remained facet of $w * (\overline{\partial B \setminus \delta})$. Then we can take a 1-ball or a 2-ball Σ' which satisfies the condition of Lemma 4.3

or 4.4. If $\Sigma' \cap \delta'$ contains two 1-faces and the facet of $w * (\overline{\partial B \setminus \delta})$ which contains $\overline{\partial\delta \setminus \Sigma'}$ remains, we denote $\overline{\partial\delta \setminus \Sigma'}$ by e anew. From Lemma 4.5, we can take Σ' such that $\Sigma' \cap \delta$ is a 1-face again. Let σ be the facet of $\partial B'_2 \cap \partial B'$ such that $\Sigma' \cap \sigma$ is a 1-face. We denote $\Sigma' \cup \sigma$ by Σ. Let γ be a facet of $\overline{\partial B' \setminus (\delta' \cup \Sigma)}$ such that the facet of $w * (\overline{\partial B \setminus \delta'})$ which contains $\gamma \cap \delta$ was already removed. We remove facets of $w * (\overline{\partial B' \setminus (\delta' \cup \Sigma)})$ along the regular order of a shelling of $\overline{\partial B' \setminus (\delta' \cup \Sigma)}$ which starts with γ and shells $\partial B'_1 \cap \overline{\partial B' \setminus (\delta' \cup \Sigma)}$ first. Continuously remove B'_2 along the reverse order of a shelling starting with the facet containing σ. Finally we remove facets of $w * \Sigma$ along the regular order of a shelling of Σ starting with σ.

Assume that F was not removed. There are the following two cases: (1) some of the three facets were not removed, (2) all of them were removed. Consider the case (1). Let D be $\partial B'_1 \cap \partial B'$. Let e be a 1-face of $\partial\delta$ which is contained in the removed facet of $w * (\overline{\partial B' \setminus \delta'})$. Then we can take a subcomplex Σ' of $\partial B'_1$ the same as above. Similarly take the facet σ and denote $\Sigma' \cup \sigma$ by Σ. Let γ be a facet of $\overline{\partial B' \setminus \Sigma}$ such that $\gamma \cap \delta$ is contained in the remained facet of $w * (\overline{\partial B' \setminus \delta'})$. In the case (2), we denote a 1-face of $\partial\delta$ by e and construct Σ similarly. Let γ be a facet of $\overline{\partial B' \setminus \Sigma}$ such that $\gamma \cap \delta$ is a 1-face. We remove facets of $w * \Sigma$ along the regular order of a shelling of Σ starting with the facet containing e. Continuously we remove facets of B'_2 along the regular order of a shelling starting with the facet containing σ. Finally we remove facets of $w * (\overline{\partial B' \setminus (\Sigma \cup \delta')})$ along the reverse order of a shelling of $\overline{\partial B' \setminus (\Sigma \cup \delta')}$ which starts with γ and shells $\partial B'_1 \cap \overline{\partial B' \setminus (\Sigma \cup \delta')})$ first.

In the above cases, we continue removing the remained facets of $(w * \partial(B \natural B')) \cup (B \natural B')$ along the reverse order of the shelling of $(v * \partial B) \cup B$. Then we can remove all facets and the order satisfies the condition of the reverse order of a shelling.

Next we assume that there is a shelling of $(v * \partial B) \cup B$ ending with Δ. In this case, we remove the facets of $(w * (\overline{\partial B' \setminus \delta'})) \cup B'$ and continuously remove the facets of $(w * (\overline{\partial B \setminus \delta})) \cup B$ the same as Lemma 4.6.

At last we can construct a shelling of $(w * \partial(B \natural B')) \cup (B \natural B')$ starting with some facet of the cone part. □

Proof of Theorem 4.1. The assertion follows from Lemma 4.6 and Lemma 4.7. □

ACKNOWLEDGEMENT. The author would like to thank Dr. Masahiro Hachimori for his helpful suggestions and many discussions.

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