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ON THE STABILITY OF CONTACT DISCONTINUITY FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH FREE BOUNDARY

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1. Introduction

The one-dimensional compressible Navier-Stokes equations read in the *Eulerian* coordinates

$$\begin{cases} \tilde\rho_t + (\tilde\rho\tilde u)_{\tilde x} = 0, \\ (\tilde\rho\tilde u)_t + (\tilde\rho\tilde u^2 + \tilde p)_{\tilde x} = \mu\tilde u_{\tilde x\tilde x}, \\ \left(\tilde\rho\left(\tilde e + \dfrac{\tilde u^2}{2}\right)\right)_t + \left(\tilde\rho\tilde u\left(\tilde e + \dfrac{\tilde u^2}{2}\right) + \tilde p\tilde u\right)_{\tilde x} = \kappa\tilde\theta_{\tilde x\tilde x} + (\mu\tilde u\tilde u_{\tilde x})_{\tilde x}, \end{cases} \tag{1.1}$$

where $\tilde u(\tilde x, t)$ is the velocity, $\tilde\rho(\tilde x, t) > 0$ the density, $\tilde\theta(\tilde x, t)$ the absolute temperature, $\mu > 0$ the viscosity constant and $\kappa > 0$ the coefficient of heat conduction. The pressure $p = \tilde p(\tilde\rho, \tilde\theta)$ and the internal energy $\tilde e = \tilde e(\tilde\rho, \tilde\theta)$ are related by the second law of thermodynamics.

There have been a lot of works on the asymptotic behaviors of the solutions for the system (1.1). Most of these results are concerned with the rarefaction wave and viscous shock wave. We refer to [10–15] for 2×2 case and [4–5, 7–8] for 3×3 case and references therein. However there is no result on the contact discontinuity for the system (1.1) until now due to various difficulties. Although some progress on the contact discontinuity were obtained by Liu and Xin [9] and Xin [17] in which the asymptotic toward the contact discontinuity was investigated for the initial value problem (IVP) of viscous conservation laws with uniformly artificial viscosity, no result is known for the physical system, especially for the compressible N-S equations (1.1). Therefore we really want to give a positive result on the contact discontinuity for the physical system (1.1). To simplify our problem, we focus our attention on the perfect gas. In this situation,

$$\tilde p(\tilde\rho, \tilde\theta) = R\tilde\theta\tilde\rho, \tag{1.2}$$

$$\tilde e(\tilde\rho, \tilde\theta) = \frac{R}{\gamma - 1}\tilde\theta + const. \tag{1.3}$$

where $R > 0$ is the gas constant and $\gamma > 1$ is the adiabatic exponent. As observed by [9, 17], the contact discontinuity can not be the asymptotic state, and a diffusive wave, which approximates the contact discontinuity on any finite time interval, instead dominates the large time behavior of the solutions for their viscous system. For the system (1.1), we also expect that the asymptotic behavior of the solutions is governed by a nonlinear diffusive wave. We call it viscous contact discontinuity. We observe that the sign of the first derivative of the velocity in the nonlinear diffusive wave is important to the a priori estimate and is not good in the whole space. Thus it is difficult to obtain the asymptotic stability of the viscous contact discontinuity for IVP of the system (1.1). We further observe that the effect of the sign could exactly be neglected for initial boundary value problem (IBVP) of the system (1.1) because there has the inequality of Poincar*é* type in the half space. It might be possible to obtain the a priori estimate for IBVP of the system (1.1) although IBVP is usually much more difficult than IVP in many situations. Motivated by this observation, we consider the system (1.1) in the part $\tilde{x} > \tilde{x}(t)$, where $\tilde{x} = \tilde{x}(t)$ is a free boundary, with the following boundary conditions

$$\frac{d\tilde{x}(t)}{dt} = \tilde{u}(\tilde{x}(t), t), \quad \tilde{x}(0) = 0, \quad \tilde{\theta}(\tilde{x}(t), t) = \theta_- > 0, \tag{1.4}$$

and

$$(\tilde{p} - \mu\tilde{u}_{\tilde{x}})|_{\tilde{x}=\tilde{x}(t)} = p_-, \tag{1.5}$$

which means the gas is attached at the free boundary $\tilde{x} = \tilde{x}(t)$ to the atmosphere with pressure p_- (see [15]) and the initial data

$$(\tilde{\rho}, \tilde{u}, \tilde{\theta})(\tilde{x}, 0) = (\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)(\tilde{x}), \quad \lim_{\tilde{x}\to+\infty}(\tilde{\rho}_0, \tilde{u}_0, \tilde{\theta}_0)(\tilde{x}) = (\rho_+, 0, \theta_+), \tag{1.6}$$

where ρ_+, θ_+ are positive constants and $\theta_0(0) = \theta_-$. Because here we only consider the case of a single contact discontinuity, we require

$$p_- = p_+ = R\theta_+\rho_+. \tag{1.7}$$

Since the boundary condition (1.4) means the particles always stay on the free boundary $\tilde{x} = \tilde{x}(t)$, if we use the *Lagrangian* coordinates, then the free boundary becomes a fixed boundary. Thus we transform the *Eulerian* coordinates $(\tilde{x}, t)$ to the *Lagrangian* coordinates (x, t) by

$$x = \int_{\tilde{x}(t)}^{\tilde{x}} \tilde{\rho}(y, t)\, dy, \quad t = t, \tag{1.8}$$

and our free boundary value problem (1.1–1.7) is then changed into

$$
\left\{\begin{array}{ll}
v_t - u_x = 0, & x > 0,\ t > 0,\\
u_t + p(v,\theta)_x = \mu\left(\dfrac{u_x}{v}\right)_x, & x > 0,\ t > 0,\\
\left(e(v,\theta) + \dfrac{u^2}{2}\right)_t + \big(p(v,\theta)\,u\big)_x = \left(\kappa\dfrac{\theta_x}{v} + \mu\dfrac{uu_x}{v}\right)_x, & x > 0,\ t > 0,\\
\theta|_{x=0} = \theta_-, & \\
\left(p(v,\theta) - \mu\dfrac{u_x}{v}\right)(0,t) = p_+,\ \ t > 0, & \\
(v,u,\theta)(x,0) = (v_0,u_0,\theta_0)(x) \to (v_+,0,\theta_+) \text{ as } x \to +\infty, &
\end{array}\right. \tag{1.9}
$$

where $u(x,t) = \tilde{u}(\tilde{x},t)$ etc. $v = 1/\rho$, $e(v,\theta) = \{R/(\gamma-1)\}\theta + const.$, and $p(v,\theta) = R\theta/v$.

We now construct the viscous contact discontinuity $(V,U,\Theta)(x,t)$ which is expected to describe the large time behavior of the solution to the system (1.9). To achieve our goal, we consider the following Riemann problem

$$
\left\{\begin{array}{l}
v_t - u_x = 0,\\
u_t + p(v,\theta)_x = 0,\\
\left(e(v,\theta) + \dfrac{u^2}{2}\right)_t + \big(p(v,\theta)\,u\big)_x = 0,\\
(v,u,\theta)(x,0) = (v_-,0,\theta_-),\ \text{ if } x < 0,\\
(v,u,\theta)(x,0) = (v_+,0,\theta_+),\ \ \text{ if } x > 0,
\end{array}\right. \tag{1.10}
$$

where $v_\pm$ and $\theta_\pm$ are given positive constants. It is known from [16] that the Riemann problem (1.10) admits a contact discontinuity

$$
(\bar{V},\bar{U},\bar{\Theta})(x,t) = \left\{\begin{array}{ll}(v_-,0,\theta_-), & x < 0,\\ (v_+,0,\theta_+), & x > 0,\end{array}\right. \tag{1.11}
$$

provided that

$$
p_- = \frac{R\theta_-}{v_-} = p_+ = \frac{R\theta_+}{v_+}. \tag{1.12}
$$

From the structure of the Riemann solution, we conjecture

$$
P(V,\Theta) = \frac{R\Theta}{V} = p_+. \tag{1.13}
$$

On the other hand, it is observed that the third equation of (1.9) can be reduced to

$$
\frac{R}{\gamma-1}\theta_t + p(v,\theta)u_x = \left(\kappa\frac{\theta_x}{v}\right)_x + \mu\frac{u_x^2}{v}, \tag{1.14}
$$

by $(1.9)_2$. Motivated by (1.14), we further conjecture that (V, U, Θ) is governed by the following equations

$$
\begin{cases} V_t - U_x = 0, \\ \dfrac{R}{\gamma - 1}\Theta_t + p_+ U_x = \kappa\left(\dfrac{\Theta_x}{V}\right)_x, \\ \Theta(0,t) = \theta_-, \Theta(+\infty, t) = \theta_+. \end{cases} \tag{1.15}
$$

Direct computations from (1.13) and (1.15) yields

$$
\Theta_t = a\left(\frac{\Theta_x}{\Theta}\right)_x, \Theta(0,t) = \theta_-, \Theta(+\infty, t) = \theta_+, \quad a = \frac{\kappa p_+(\gamma - 1)}{\gamma R^2} > 0. \tag{1.16}
$$

We seek the self similarity solution $\Theta(\xi)$, $\xi = x/\sqrt{1+t}$ of (1.16). Namely $\Theta(\xi)$ is the solution of the following equation

$$
-\frac{1}{2}\xi\Theta' = a\left(\frac{\Theta'}{\Theta}\right)', \quad \Theta(0) = \theta_-, \Theta(+\infty) = \theta_+, \quad ' = \frac{d}{d\xi}. \tag{1.17}
$$

It is known from [1] that there exists a unique solution $\Theta(\xi)$ of (1.17) which is a monotone function, increasing if $\theta_+ > \theta_-$ and decreasing if $\theta_+ < \theta_-$. Furthermore, $\Theta'(0)$ has the following property due to [2],

$$
C_1|\theta_+ - \theta_-| \le |\Theta'(0)| \le C_2|\theta_+ - \theta_-|, \tag{1.18}
$$

where C_1 and C_2 are positive constants depending on $\theta_\pm$. From (1.17), we have

$$
\Theta'' = \frac{\Theta}{a}\left(-\frac{1}{2}\xi + \frac{a\Theta'}{\Theta^2}\right)\Theta' =: A(\xi)\Theta', \tag{1.19}
$$

which yields

$$
\Theta'(\xi) = \Theta'(0)e^{\int_0^\xi A(s)\,ds}. \tag{1.20}
$$

Due to [3], the asymptotic behavior of $e^{\int_0^\xi A(s)\,ds}$ is

$$
e^{\int_0^\xi A(s)\,ds} = O\left(e^{-C_3\xi^2}\right), \quad \text{as} \quad \xi \to \infty, \tag{1.21}
$$

for some positive constant C_3 depending on $\theta_\pm$. Thus by (1.18)–(1.21), it is easy to verify the following estimates of Θ.

Lemma 1.1. *Let $|\theta_+ - \theta_-| = \delta$, the following estimates hold*

$$
\int_0^\infty \Theta_x^4\,dx \le C\delta^4(1+t)^{-3/2}, \quad \int_0^\infty \Theta_{xx}^2\,dx \le C\delta^2(1+t)^{-3/2}, \tag{1.22}
$$

$$\int_0^\infty \Theta_{xxx}^2\,dx \le C\delta^2(1+t)^{-5/2}, \quad \int_0^\infty x(\Theta_x^2 + |\Theta_{xx}|)\,dx \le C\delta. \tag{1.23}$$

After $\Theta(x,t) = \Theta(\xi)$ is obtained, we define

$$V(x,t) = \frac{R}{p_+}\Theta(x,t), \quad U(x,t) = \frac{\kappa(\gamma-1)}{\gamma R}\frac{\Theta(x,t)_x}{\Theta(x,t)}. \tag{1.24}$$

It is easy to see (V, U, Θ) has the following property

$$\|(V - \bar{V}, U - \bar{U}, \Theta - \bar{\Theta})\|_{L^p(0,+\infty)} \le C[\kappa(1+t)]^{1/(2p)}, \text{ all } p \ge 1, \tag{1.25}$$

which means the nonlinear wave (V, U, Θ) approximates the contact discontinuity $(\bar{V}, \bar{U}, \bar{\Theta})$ in L^p norm, $p \ge 1$ on any finite time interval as κ tends to zero. We call $(V, U, \Theta)(x,t)$ viscous contact discontinuity.

Therefore, we conjecture that the solution $(v, u, \theta)(x,t)$ of the system (1.9) asymptotically tends to the viscous contact discontinuity $(V, U, \Theta)(x,t)$. The aim of this paper is to justify the above conjecture. Our result is, roughly speaking, as follows.

If $|\theta_+ - \theta_-|$ is small, then the viscous contact discontinuity is asymptotic stable.

Our plan of this paper is as follows. in Sec. 2, we reformulate the original problem and state our main result; in Sec. 3, we establish the a priori estimate and prove our main result; in Sec. 4, we show the local existence of the solution.

NOTATIONS. Throughout this paper, several positive generic constants are denoted by c, C without confusions. For function spaces, $H^l(\Omega)$ denotes the l-th order Sobolev space with its norm

$$\|f\|_l = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2\right)^{1/2}, \quad \text{when } \|\cdot\| := \|\cdot\|_{L^2(\Omega)}. \tag{1.26}$$

The domain Ω will be often abbreviated without confusions.

2. Reformulated problem and main result

To state our main result, we put the perturbation $(\phi, \psi, \zeta)(x,t)$ by

$$\begin{aligned} v(x,t) &= V(x,t) + \phi(x,t), \\ u(x,t) &= U(x,t) + \psi(x,t), \\ \theta(x,t) &= \Theta(x,t) + \zeta(x,t), \end{aligned} \tag{2.1}$$

where $(V, U, \Theta)(x, t)$ is the viscous contact discontinuity defined in (1.16) and (1.19). By the definition of the viscous discontinuity, we have

$$
\begin{cases}
V_t - U_x = 0, \\
U_t + \left(\dfrac{R\Theta}{V}\right)_x = \mu\left(\dfrac{U_x}{V}\right)_x + F, \\
\dfrac{R}{\gamma - 1}\Theta_t + p_+ U_x = \kappa\left(\dfrac{\Theta_x}{V}\right)_x + \mu\dfrac{U_x^2}{V} + G,
\end{cases}
\tag{2.2}
$$

where

$$
F = \frac{\kappa(\gamma-1)}{R\gamma}\left[(\ln\Theta)_{xt} - \mu\left(\frac{p_+}{R\Theta}(\ln\Theta)_{xx}\right)_x\right],
\tag{2.3}
$$

$$
G = -\frac{\mu p_+}{R\Theta}\left(\frac{\kappa(\gamma-1)}{R\gamma}(\ln\Theta)_{xx}\right)^2 = O\left(\Theta_{xx}^2 + \Theta_x^4\right).
\tag{2.4}
$$

Substituting (2.2) into $(1.9)_1$, $(1.9)_2$ and (1.14) yields

$$
\begin{cases}
\phi_t - \psi_x = 0, \\
\psi_t + \left(\dfrac{R(\Theta+\zeta)}{V+\phi} - \dfrac{R\Theta}{V}\right)_x = \mu\left(\dfrac{U_x+\psi_x}{V+\phi} - \dfrac{U_x}{V}\right)_x - F, \\
\dfrac{R}{\gamma-1}\zeta_t + \dfrac{R(\Theta+\zeta)}{V+\phi}(U_x+\psi_x) - p_+ U_x \\
\quad = \kappa\left(\dfrac{\Theta_x+\zeta_x}{V+\phi}\right)_x - \kappa\left(\dfrac{\Theta_x}{V}\right)_x + \mu\dfrac{(U_x+\psi_x)^2}{V+\phi} - \mu\dfrac{U_x^2}{V} - G, \\
\left(\dfrac{R\theta_-}{V+\phi} - \mu\dfrac{U_x+\psi_x}{V+\phi}\right)\Big|_{x=0} = p_+, \\
\zeta(0,t) = 0, \\
(\phi, \psi, \zeta)(x, 0) = (v_0 - V, u_0 - U, \theta - \Theta)(x, 0).
\end{cases}
\tag{2.5}
$$

We assume

$$
(\phi_0, \psi_0)(x) = (\phi, \psi)(x, 0) \in H^1(0, +\infty), \quad \zeta_0(x) = \zeta(x, 0) \in H_0^1(0, +\infty).
\tag{2.6}
$$

Our main result is

Theorem 2.1 (Stability theorem). *There exist positive constants δ_0 and ε_0 such that for any $|\theta_+ - \theta_-| = \delta \le \delta_0$, if $\|(\phi_0, \psi_0, \zeta_0)\|_1 \le \varepsilon_0$, then the problem* (2.5) *has a unique global solution $(\phi, \psi, \zeta)(x, t)$ satisfying*

$$
\begin{gathered}
(\phi, \psi)(x, t) \in C\left(0, +\infty; H^1(0, +\infty)\right), \quad \zeta(x, t) \in C\left(0, +\infty; H_0^1(0, +\infty)\right) \\
\phi_x(x, t) \in L^2\left(0, +\infty; L^2(0, +\infty)\right), \\
(\psi, \zeta)_x(x, t) \in L^2\left(0, +\infty; H^1(0, +\infty)\right),
\end{gathered}
$$

and

$$\sup_{x\geq 0}|(\phi,\psi,\zeta)(x,t)|\to 0,\quad \text{as}\quad t\to+\infty.$$

REMARK 1. Although Theorem 2.1 is only concerned with the perfect gas case, it is not difficult to obtain the similar result for general case by the same argument.

REMARK 2. One would expect similar result for Dirichlet problem $u(0,t)=0$, $\theta(0,t)=\theta_-$. However, if we still use the same viscous contact discontinuity constructed in (1.16) and (1.24), then the decay rate of the boundary value $\psi(0,t)=O((1+t)^{-1/2})$ is not good. It is difficult to control the terms from the boundary.

We shall prove the stability Theorem 2.1 by the local existence and the a priori estimate. We look for the solution (ϕ,ψ,ζ) in the solution space $X_{m,M}(0,+\infty)=\bigcup_{T>0}X_{m,M}(0,T)$, where

$$\begin{aligned} X_{m,M}(0,T) = \{&(\phi,\psi,\zeta):(\phi,\psi)\in C([0,T];H^1),\zeta\in C([0,T];H_0^1),\\ &\|(\phi,\psi,\zeta)\|_1\leq M,\inf_{x,t}(V+\phi)\geq m>0,\phi_x\in L^2(0,T;L^2),\\ &(\psi,\zeta)_x\in L^2(0,T;H^1)\} \end{aligned}\tag{2.7}$$

for $T,\ m,\ M>0$, and $H^1=H^1(0,+\infty)$. Let

$$N(T)=\sup_{t\in[0,T]}\|(\phi,\psi,\zeta)(t)\|_1.\tag{2.8}$$

In the present paper, the a priori estimate will be investigated in the next section and the local existence will be left to the last section. Nevertheless we first state our local existence.

Proposition 2.2 (Local Existence). *There exists a positive constant b such that if $\|(\phi_0,\psi_0,\zeta_0)\|_1\leq M$ and $\inf_{\mathbb{R}_+}(V+\phi_0)\geq m$, then there exists a positive constant $T_0=T_0(m,M)$ such that there exists a unique solution $(\phi,\psi,\zeta)\in X_{(1/2)m,bM}(0,T_0)$ to* (2.5).

3. A priori estimate

Before establishing the a priori estimate, we first estimate the value of $\phi(0,t)$ on the boundary $x=0$ by the boundary condition (2.5). Let $\phi(t)=\phi(0,t)$. Since $U_x(0,t)=0$, the boundary condition of (2.5) yields

$$\frac{R\theta_-}{v_-+\phi(t)}-\mu\frac{\phi(t)_t}{v_-+\phi(t)}=p_+,\ \ t>0.\tag{3.1}$$

Direct computation gives

$$\phi_t = -\frac{p_+}{\mu}\phi(t), \quad \phi(0) = \phi_0(0). \tag{3.2}$$

It follows then, that

$$\phi(t) = \phi_0(0)e^{-(p_+/\mu)t}. \tag{3.3}$$

Now we turn to the a priori estimate. We have

Proposition 3.1 (A Priori Estimates). *There exist positive constants* δ_0 *and* ε_1 *such that, for any given* $|\theta_+ - \theta_-| = \delta \leq \delta_0$, *if* $(\phi, \psi, \zeta) \in X_{(1/2)m_0, b\varepsilon_1}(0, T)$ *is a solution of* (2.5) *for some positive* T, *then* (ϕ, ψ, ζ) *satisfies the a priori estimate*

$$\|(\phi, \psi, \zeta)\|_1^2 + \int_0^t \{\|\phi_x\|^2 + \|(\psi_x, \zeta_x)\|_1^2\}\, d\tau \leq C\big(\delta^{4/3} + \|(\phi_0, \psi_0, \zeta_0)\|_1^{4/3}\big), \tag{3.4}$$

where $m_0 = (1/2)\min\{v_-, v_+\}$, $v_- = R\theta_-/p_+$.

Proposition 3.1 is proved by a series of Lemmas. We first give the following key Lemma.

Lemma 3.2. *It follows that*

$$\begin{aligned} &\|(\phi, \psi, \zeta)(t)\|^2 + \int_0^t \{\|(\psi_x, \zeta_x)(\tau)\|^2\}\, d\tau \\ \leq& C\Big(\delta^{4/3} + \|(\phi_0, \psi_0, \zeta_0)\|_1^{4/3} + \delta\int_0^t \|\phi_x(\tau)\|^2\, dt\Big). \end{aligned} \tag{3.5}$$

Proof. Multiplying $(2.5)_2$ by ψ, we have

$$\begin{aligned} &\Big(\frac{1}{2}\psi^2\Big)_t + R\Big[\Big(\frac{\Theta+\zeta}{v} - \frac{\Theta}{V}\Big)\psi\Big]_x - \mu\Big[\Big(\frac{U_x+\psi_x}{v} - \frac{U_x}{V}\Big)\psi\Big]_x \\ &\quad - R\Big(\frac{\Theta+\zeta}{v} - \frac{\Theta}{V}\Big)\psi_x + \mu\Big(\frac{U_x+\psi_x}{v} - \frac{U_x}{V}\Big)\psi_x = -F\psi. \end{aligned} \tag{3.6}$$

Since $\phi_t = \psi_x$ and

$$R\Big(\frac{\Theta+\zeta}{v} - \frac{\Theta}{V}\Big) = R\Theta\Big(\frac{1}{v} - \frac{1}{V}\Big) + \frac{R\zeta}{v}, \tag{3.7}$$

we get

$$\begin{aligned} &\Big(\frac{1}{2}\psi^2\Big)_t - R\Theta\Big(\frac{1}{v} - \frac{1}{V}\Big)\phi_t - \frac{R}{v}\zeta\psi_x + H_x \\ &\quad + \mu\frac{\psi_x^2}{v} + \mu\psi_x U_x\Big(\frac{1}{v} - \frac{1}{V}\Big) = -F\psi, \end{aligned} \tag{3.8}$$

where

$$H = R\Big[\Big(\frac{\Theta+\zeta}{v} - \frac{\Theta}{V}\Big)\psi\Big] - \mu\Big[\Big(\frac{U_x+\psi_x}{v} - \frac{U_x}{V}\Big)\psi\Big].$$

Let

$$\Phi(s) = s - 1 - \ln s. \tag{3.9}$$

It is easy to check that $\Phi'(1) = 0$ and $\Phi(s)$ is strictly convex around $s = 1$. We compute

$$\begin{aligned}\Big\{R\Theta\Phi\Big(\frac{v}{V}\Big)\Big\}_t &= R\Theta_t\Phi\Big(\frac{v}{V}\Big) + R\Theta\Big(-\frac{1}{v}+\frac{1}{V}\Big)\phi_t \\ &\quad + R\Theta\Big(-\frac{v}{V^2}+\frac{1}{V}\Big)V_t + R\Theta\Big(-\frac{1}{v}+\frac{1}{V}\Big)V_t \\ &= R\Theta\Big(-\frac{1}{v}+\frac{1}{V}\Big)\phi_t - p_+\Psi\Big(\frac{v}{V}\Big)V_t,\end{aligned} \tag{3.10}$$

where

$$\Psi(s) = s^{-1} - 1 + \ln s. \tag{3.11}$$

Substituting (3.10) into (3.8) yields

$$\begin{aligned}&\Big(\frac{1}{2}\psi^2 + R\Theta\Phi\Big(\frac{v}{V}\Big)\Big)_t + p_+\Psi\Big(\frac{v}{V}\Big)U_x \\ &- \frac{R}{v}\zeta\psi_x + H_x + \mu\frac{\psi_x^2}{v} + \mu\psi_x U_x\Big(\frac{1}{v}-\frac{1}{V}\Big) = -F\psi.\end{aligned} \tag{3.12}$$

On the other hand, we calculate

$$\Big[\Theta\Phi\Big(\frac{\theta}{\Theta}\Big)\Big]_t = \Big(1-\frac{\Theta}{\theta}\Big)\theta_t - \ln\frac{\theta}{\Theta}\Theta_t = \Big(1-\frac{\Theta}{\theta}\Big)\zeta_t - \Psi\Big(\frac{\theta}{\Theta}\Big)\Theta_t, \tag{3.13}$$

and

$$\begin{aligned}&\frac{R}{\gamma-1}\Big(1-\frac{\Theta}{\theta}\Big)\zeta_t \\ =&\Big(1-\frac{\Theta}{\theta}\Big)\Big(-pu_x + p_+U_x + \kappa\Big(\frac{\theta_x}{v}-\frac{\Theta_x}{V}\Big)_x + \mu\Big(\frac{u_x^2}{v}-\frac{U_x^2}{V}\Big) - G\Big) \\ =&-\frac{R}{v}\zeta\psi_x + \frac{\zeta}{\theta}(p_+-p)U_x + \kappa\Big(\frac{\zeta}{\theta}\Big(\frac{\theta_x}{v}-\frac{\Theta_x}{V}\Big)\Big)_x \\ &+\kappa\Big(-\frac{\zeta_x}{\theta}+\frac{\theta_x\zeta}{\theta^2}\Big)\Big(\frac{\zeta_x}{v}+\Big(\frac{1}{v}-\frac{1}{V}\Big)\Theta_x\Big) + \frac{\mu\zeta}{\theta}\Big(\frac{u_x^2}{v}-\frac{U_x^2}{V}\Big) - \frac{\zeta G}{\theta} \\ =&-\frac{R}{v}\zeta\psi_x + \frac{\zeta}{\theta}(p_+-p)U_x + \kappa\Big(\frac{\zeta}{\theta}\Big(\frac{\theta_x}{v}-\frac{\Theta_x}{V}\Big)\Big)_x - \frac{\kappa}{v\theta}\zeta_x^2 \\ &+\kappa\frac{\theta_x}{\theta^2 v}\zeta\zeta_x + \kappa\frac{\zeta_x\phi}{\theta vV}\Theta_x - \kappa\frac{\theta_x\zeta\phi}{\theta^2 vV}\Theta_x + \frac{\mu\zeta}{\theta}\Big(\frac{u_x^2}{v}-\frac{U_x^2}{V}\Big) - \frac{\zeta G}{\theta}.\end{aligned} \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) gives

$$\begin{aligned}(3.15)\qquad &\Big(\frac{1}{2}\psi^2 + R\Theta\Phi\Big(\frac{v}{V}\Big) + \frac{R}{\gamma-1}\Theta\Phi\Big(\frac{\theta}{\Theta}\Big)\Big)_t + \frac{\kappa}{v\theta}\zeta_x^2 + \mu\frac{\psi_x^2}{v}\\ &+ \tilde{H}_x + Q = -F\psi - \frac{\zeta G}{\theta},\end{aligned}$$

where

$$(3.16)\qquad \tilde{H} = H - \kappa\frac{\zeta}{\theta}\Big(\frac{\theta_x}{v} - \frac{\Theta_x}{V}\Big),$$

and

$$\begin{aligned}(3.17)\qquad Q =& p_+\Psi\Big(\frac{v}{V}\Big)U_x + \frac{p_+}{\gamma-1}\Psi\Big(\frac{\theta}{\Theta}\Big)U_x + \mu\psi_x U_x\Big(\frac{1}{v} - \frac{1}{V}\Big)\\ &- \frac{\zeta}{\theta}(p_+ - p)U_x - \kappa\frac{\theta_x}{\theta^2 v}\zeta\zeta_x - \kappa\frac{\zeta_x\phi}{\theta v V}\Theta_x + \kappa\frac{\theta_x\zeta\phi}{\theta^2 v V}\Theta_x - \frac{\mu\zeta}{\theta}\Big(\frac{u_x^2}{v} - \frac{U_x^2}{V}\Big),\end{aligned}$$

satisfies

$$(3.18)\qquad |Q| \le \big(\delta_1 + C_{\delta_1}N(T)\big)\big(\psi_x^2 + \zeta_x^2\big) + C_{\delta_1}\big(\zeta^2 + \phi^2\big)\big(|\Theta_{xx}| + \Theta_x^2\big),$$

if ε_1 is suitably small.

Note that $\Phi(0) = \Phi'(0) = 0$ and $\Phi(s)$ is a strictly convex function around $s = 1$ and

$$\int_0^\infty |F\psi|\,dx \le \delta_1\|\psi_x\|^2 + C_{\delta_1}\|F\|_{L^1}^{4/3}.$$

Choosing δ_1 suitably small and integrating (3.15) over $R_+ \times (0,t)$ give

$$\begin{aligned}(3.19)\qquad &\|(\phi,\psi,\zeta)\|^2 + \int_0^t \|(\psi_x,\zeta_x)\|^2\,dt - \int_0^t \tilde{H}(0,t)\,dt\\ &\le \int_0^t\int_0^\infty C\big(U_x^2 + (\zeta^2+\phi^2)(|\Theta_{xx}| + \Theta_x^2) + |G|\big)\,dx\,dt\\ &\quad + C\int_0^t \|F\|_{L^1}^{4/3}\,dt + C\|(\phi_0,\psi_0,\zeta_0)\|^2.\end{aligned}$$

From Lemma 1.1, it is easy to see

$$(3.20)\qquad \int_0^t\int_0^\infty U_x^2 + |G|\,dx\,dt + \int_0^t \|F\|_{L^1}^{4/3}\,dt \le C\delta^{4/3}.$$

Due to the good property of (3.20), the estimates of the terms containing F and G are omitted in what follows. Since

$$(3.21)\qquad |\zeta(x,t)| \le x^{1/2}\|\zeta_x\|, \quad |\phi(x,t)| \le |\phi(0,t)| + x^{1/2}\|\phi_x\|,$$

applying Lemma 1.1 and the boundary condition (3.3), we have

$$\int_0^t\int_0^\infty (\zeta^2+\phi^2)(|\Theta_{xx}|+\Theta_x^2)\,dx\,dt \le C\delta\|\phi_0\|_1^2 + C\delta\int_0^t \|(\phi_x,\zeta_x)\|^2\,dt. \tag{3.22}$$

We now estimate the term from the boundary. We compute from (2.5) and (3.3),

$$\begin{aligned}\int_0^t |\tilde{H}(0,t)|\,dt &\le C\int_0^t |\phi(0,t)\psi(0,t)| + |\phi_t(0,t)\psi(0,t)|\,dt\\ \le C|\phi_0(0)|\int_0^t e^{-(p_+/\mu)t}|\psi(0,t)|\,dt &\le \frac{1}{2}\int_0^t \|\psi_x\|^2\,dt + C\|\phi_0\|_1^{4/3}.\end{aligned} \tag{3.23}$$

Thus there exists a positive constant δ_0. For any $\delta \le \delta_0$, Lemma 3.2 is obtained by (3.19–3.23). □

Lemma 3.3. *It follows that*

$$\|\phi_x(t)\|^2 + \int_0^t \|\phi_x(\tau)\|^2\,d\tau \le C\big(\delta^{4/3} + \|(\phi_0,\psi_0,\zeta_0)\|_1^{4/3}\big). \tag{3.24}$$

Proof. Following [13], we introduce a new variable $\tilde{v} = v/V$. Then $(2.5)_2$ can be rewritten by the new variable as

$$\Big(\mu\Big(\frac{\tilde{v}_x}{\tilde{v}}\Big) - \psi\Big)_t - p_x = F. \tag{3.25}$$

Multiplying (3.25) by $\tilde{v}_x/\tilde{v}$, we have

$$\begin{aligned}&\Big(\frac{\mu}{2}\Big(\frac{\tilde{v}_x}{\tilde{v}}\Big)^2 - \psi\frac{\tilde{v}_x}{\tilde{v}}\Big)_t + \Big(\psi\frac{\tilde{v}_t}{\tilde{v}}\Big)_x + p\Big(\frac{\tilde{v}_x}{\tilde{v}}\Big)^2 - \frac{R}{v}\zeta_x\frac{\tilde{v}_x}{\tilde{v}}\\ &+ p\Big(\frac{1}{\Theta} - \frac{1}{\theta}\Big)\Theta_x\frac{\tilde{v}_x}{\tilde{v}} = \frac{\psi_x^2}{v} + \psi_x U_x\Big(\frac{1}{v} - \frac{1}{V}\Big) + F\frac{\tilde{v}_x}{\tilde{v}}.\end{aligned} \tag{3.26}$$

The Cauchy inequality yields that

$$\Big|\frac{R}{v}\zeta_x\frac{\tilde{v}_x}{\tilde{v}}\Big| + \Big|\psi_x U_x\Big(\frac{1}{v} - \frac{1}{V}\Big)\Big| \le \delta_1\Big(\frac{\tilde{v}_x}{\tilde{v}}\Big)^2 + C_{\delta_1}\zeta_x^2 + C_{\delta_1}\big(\psi_x^2 + \phi^2 U_x^2\big), \tag{3.27}$$

$$\Big|p\Big(\frac{1}{\Theta} - \frac{1}{\theta}\Big)\Theta_x\frac{\tilde{v}_x}{\tilde{v}}\Big| \le \delta_1\Big(\frac{\tilde{v}_x}{\tilde{v}}\Big)^2 + C_{\delta_1}\zeta^2\Theta_x^2, \tag{3.28}$$

and

$$c_1\phi_x^2 - c_2\phi^2 V_x^2 \le \Big(\frac{\tilde{v}_x}{\tilde{v}}\Big)^2 \le c_3\big(\phi_x^2 + \phi^2 V_x^2\big). \tag{3.29}$$

On the boundary $x = 0$, we have

$$\psi\frac{\tilde{v}_t}{\tilde{v}} = \psi\frac{\phi_t}{v}. \tag{3.30}$$

Note that the right hand sides of (3.27)–(3.30) have already been investigated in Lemma 3.2. Integrating (3.26) over $R_+ \times (0, t)$, using Lemma 3.2, (3.27)–(3.30), the boundary condition (3.3), the fact that

$$\left|\psi \frac{\tilde{v}_x}{\tilde{v}}\right| \le \frac{\mu}{4}\left(\frac{\tilde{v}_x}{\tilde{v}}\right)^2 + C\psi^2,$$

and choosing δ_1 is suitably small, we get the desired estimate (3.24). □

Lemma 3.4. *It follows that*

$$\|(\psi_x, \zeta_x)(t)\|^2 + \int_0^t \{\|(\psi_{xx}, \zeta_{xx})(\tau)\|^2\}\, d\tau \le C\big(\delta^{4/3} + \|(\phi_0, \psi_0, \zeta_0)\|_1^{4/3}\big). \tag{3.31}$$

Proof. Multiplying $(2.5)_2$ by $-\psi_{xx}$ and $(2.5)_3$ by $-\zeta_{xx}$ and combining the resulting equalities, we have

$$\begin{aligned}&\left(\frac{1}{2}\psi_x^2 + \frac{R}{2(\gamma-1)}\zeta_x^2\right)_t - \left(\psi_t\psi_x + \frac{R}{\gamma-1}\zeta_t\zeta_x\right)_x + \mu\frac{\psi_{xx}^2}{v}\\ &+ \kappa\frac{\zeta_{xx}^2}{v} - p_x\psi_{xx} + \mu\left(\frac{u_x^2}{v} - \frac{U_x^2}{V}\right)\zeta_{xx} + \tilde{Q} = F\psi_{xx} + G\zeta_{xx},\end{aligned} \tag{3.32}$$

where

$$\begin{aligned}\tilde{Q} =& \mu\left(\frac{U_x}{v} - \frac{U_x}{V}\right)_x\psi_{xx} - (p - p_+)U_x\zeta_{xx} + \mu\psi_x\left(\frac{1}{v}\right)_x\psi_{xx}\\ &+ \kappa\zeta_x\left(\frac{1}{v}\right)_x\zeta_{xx} - p\psi_x\zeta_{xx} + \kappa\left(\frac{\Theta_x}{v} - \frac{\Theta_x}{V}\right)_x\zeta_{xx}.\end{aligned} \tag{3.33}$$

Since p_+ is constant, we have

$$|p_x\psi_{xx}| \le \delta_1\psi_{xx}^2 + C_{\delta_1}\big(\phi^2 + \zeta^2\big)\Theta_x^2 + C_{\delta_1}\big(\phi_x^2 + \zeta_x^2\big). \tag{3.34}$$

On the other hand, the Cauchy inequality yields

$$\left|\mu\left(\frac{u_x^2}{v} - \frac{U_x^2}{V}\right)\zeta_{xx}\right| \le \delta_1\zeta_{xx}^2 + C_{\delta_1}\big(\psi_x^2 + U_x^4 + \psi_x^2|\zeta_{xx}|\big) \tag{3.35}$$

and

$$\begin{aligned}|\tilde{Q}| \le\ & \delta_1\big(\psi_{xx}^2 + \zeta_{xx}^2\big) + C_{\delta_1}\big(\Theta_x^4 + \Theta_{xx}^2 + \Theta_{xxx}^2 + \phi_x^2 + \psi_x^2 + \zeta_x^2\big)\\ &+ C|\phi_x|\big(|\psi_x||\psi_{xx}| + |\zeta_x||\zeta_{xx}|\big).\end{aligned} \tag{3.36}$$

The integral of the term $\psi_x^2|\zeta_{xx}|$ can be estimated as follows.

$$\begin{aligned}\int_0^\infty \psi_x^2|\zeta_{xx}|\,dx &\le \sup\{|\psi_x|\}\|\psi_x\|\|\zeta_{xx}\| \\ &\le \sqrt{2}\,\|\psi_x\|^{3/2}\|\psi_{xx}\|^{1/2}\|\zeta_{xx}\| \\ &\le \delta_1\big(\|\psi_{xx}\|^2+\|\zeta_{xx}\|^2\big)+C'_{\delta_1}N(T)\|\psi_x\|^2.\end{aligned} \tag{3.37}$$

The last term of (3.36) can be also treated as in (3.37). It is noted that $\psi_x(0,t) = \phi_t(0,t)$ exponentially decays and $\zeta_t = 0$. On the boundary $x = 0$, we have

$$\psi_t\phi_t = (\psi\phi_t)_t - \psi\phi_{tt} = (\psi\phi_t)_t - \phi_0(0)\frac{p_+^2}{\mu^2}\psi e^{-(p_+/\mu)t}. \tag{3.38}$$

The right hand side of (3.38) can be estimated by the same way as (3.23). Thus integrating (3.32) over $R_+ \times (0,t)$, choosing δ_1 is suitably small, and using (3.33)–(3.38) and Lemma 3.2, Lemma 3.3 imply (3.31). Lemma 3.4 is proved. □

Proof of Proposition 3.1. Proposition 3.1 is obtained at once from Lemma 3.2–Lemma 3.4. □

REMARK. We note that Proposition 3.1 is only formally obtained in the previous argument because our solution space is not enough regular. To give a rigorous justification, we use the standard mollifier to smooth the solution. Since our solution is regular with respect to the space variable, we only need to smooth the solution with respect to the time variable. Let $j(t) \ge 0 \in C_0^\infty(0,1)$ satisfying $\int_0^1 j(t)\,dt = 1$ and $j_\varepsilon(t) = (1/\varepsilon)j(t/\varepsilon)$. Let $f(x,t)$ be any measurable function, we define $f^\varepsilon = f * j_\varepsilon$. For any $0 < \tau < t < \infty$, there exists a positive constant $\varepsilon_0(\tau)$ such that for any $0 < \varepsilon \le \varepsilon_0$, $f^\varepsilon \in C^\infty([\tau,t];H^1)$ holds if $f \in C([0,t+1];H^1)$. Then we mollify the equations (2.5) as following:

$$\begin{cases} \phi_t^\varepsilon - \psi_x^\varepsilon = 0, \\ \psi_t^\varepsilon + \Big(\dfrac{R(\Theta^\varepsilon+\zeta^\varepsilon)}{V^\varepsilon+\phi^\varepsilon} - \dfrac{R\Theta^\varepsilon}{V^\varepsilon}\Big)_x = \mu\Big(\dfrac{U_x^\varepsilon+\psi_x^\varepsilon}{V^\varepsilon+\phi^\varepsilon} - \dfrac{U_x^\varepsilon}{V^\varepsilon}\Big)_x - F^\varepsilon + R_1^\varepsilon, \\ \dfrac{R}{\gamma-1}\zeta_t^\varepsilon + \dfrac{R(\Theta^\varepsilon+\zeta^\varepsilon)}{V^\varepsilon+\phi^\varepsilon}(U_x^\varepsilon+\psi_x^\varepsilon) - p_+U_x^\varepsilon \\ \quad = \kappa\Big(\dfrac{\Theta_x^\varepsilon+\zeta_x^\varepsilon}{V^\varepsilon+\phi^\varepsilon}\Big)_x - \kappa\Big(\dfrac{\Theta_x^\varepsilon}{V^\varepsilon}\Big)_x + \mu\dfrac{(U_x^\varepsilon+\psi_x^\varepsilon)^2}{V^\varepsilon+\phi^\varepsilon} - \mu\dfrac{(U_x^\varepsilon)^2}{V^\varepsilon} - G^\varepsilon + R_2^\varepsilon, \\ \psi_x^\varepsilon(0,t) = -\dfrac{p_+}{\mu}\phi^\varepsilon(0,t), \\ \zeta^\varepsilon(0,t) = 0, \end{cases} \tag{3.39}$$

where R_i^ε, $i = 1, 2$ tends to zero in $L^2([0,T];L^2)$ norm as $\varepsilon \to 0$. Following the same method in Proposition 3.1, we get

$$\begin{aligned}&\|(\phi^\varepsilon,\psi^\varepsilon,\zeta^\varepsilon)\|_1^2+\int_\tau^t\{\|\phi_x^\varepsilon\|^2+\|(\psi_x^\varepsilon,\zeta_x^\varepsilon)\|_1^2\}\,d\tau\\ \le&C\big(\delta^{4/3}+\|(\phi^\varepsilon,\psi^\varepsilon,\zeta^\varepsilon)(\tau)\|_1^{4/3}\big)+C\int_\tau^t\|R_1^\varepsilon\|\|\psi^\varepsilon\|+\|R_2^\varepsilon\|\|\zeta^\varepsilon\|\,dt\\ &+C\int_\tau^t\|R_1^\varepsilon\|^2+\|R_2^\varepsilon\|^2\,dt.\end{aligned}\tag{3.40}$$

Since $(\phi^\varepsilon,\psi^\varepsilon,\zeta^\varepsilon)\to(\phi,\psi,\zeta)$ in norm H^1, $(\psi^\varepsilon,\zeta^\varepsilon)\to(\psi,\zeta)$ in norm $L^2([0,T];H^2)$ and $(\phi^\varepsilon,\psi^\varepsilon,\zeta^\varepsilon)(x,\tau)\to(\phi,\psi,\zeta)(x,\tau)$ in norm H^1, letting $\varepsilon\to0$ implies

$$\begin{aligned}&\|(\phi,\psi,\zeta)\|_1^2+\int_\tau^t\{\|\phi_x\|^2+\|(\psi_x,\zeta_x)\|_1^2\}\,d\tau\\ \le&C\big(\delta^{4/3}+\|(\phi,\psi,\zeta)(\tau)\|_1^{4/3}\big).\end{aligned}\tag{3.41}$$

Letting $\tau\to0$ yields Proposition 3.1.

Proof of Theorem 2.1. Theorem 2.1 is easy from the Propositions 2.2 and 3.1. □

4. Local existence

Proof of Proposition 2.2. From $(2.5)_1$, ϕ has an explicit form

$$\phi(x,t)=\phi_0(x)+\int_0^t\psi_x(x,\tau)\,d\tau.\tag{4.1}$$

The equation $(2.5)_2$ and $(2.5)_3$ are regarded as the initial-boundary value problem for the parabolic equation of ψ and ζ respectively:

$$\begin{cases}\psi_t-\mu\Big(\dfrac{\psi_x}{V+\phi}\Big)_x=g_1:=g_1(\phi,\zeta,\phi_x,\zeta_x)\\ \mu\psi_x(0,t)=-p_+\phi(0,t),\\ \psi|_{t=0}=\psi_0,\end{cases}\tag{4.2}$$

$$\begin{cases}\zeta_t-\kappa\Big(\dfrac{\zeta_x}{V+\phi}\Big)_x=g_2:=g_2(\phi,\zeta,\phi_x,\psi_x)\\ \zeta(0,t)=0,\\ \zeta|_{t=0}=\zeta_0,\end{cases}\tag{4.3}$$

where

$$g_1=-\Big(\frac{R(\Theta+\zeta)}{V+\phi}-\frac{R\Theta}{V}\Big)_x+\mu\left(\frac{U_x}{V+\phi}-\frac{U_x}{V}\right)_x-F,\tag{4.4}$$

$$g_2 = -\frac{R(\Theta+\zeta)}{V+\phi}(U_x+\phi_x)+\kappa\left(\frac{\Theta_x}{V+\phi}-\frac{\Theta_x}{V}\right)_x + p_+U_x+\mu\left(\frac{(U_x+\psi_x)^2}{V+\phi}-\frac{U_x^2}{V}\right)-G. \tag{4.5}$$

To use the iteration method, we approximate $(\phi_0,\psi_0) \in H^1$, $\zeta_0 \in H_0^1$ by $(\phi_{0k},\psi_{0k},\zeta_{0k}) \in H^3$ which will be determined later. We define the sequence $\{(\phi_k^{(n)},\psi_k^{(n)},\zeta_k^{(n)})(x,t)\}$ for each k so that

$$\left(\phi_k^{(0)},\psi_k^{(0)},\zeta_k^{(0)}\right)(x,t) = (\phi_{0k},\psi_{0k},\zeta_{0k})(x), \tag{4.6}$$

and for a given $(\phi_k^{(n-1)},\psi_k^{(n-1)},\zeta_k^{(n-1)})(x,t)$, $\psi_k^{(n)}$ and $\zeta_k^{(n)}$ are respectively the solutions to

$$\begin{cases} \psi_{kt}^{(n)} - \mu\left(\dfrac{\psi_{kx}^{(n)}}{V+\phi_k^{(n-1)}}\right)_x = g_1^{(n-1)} = g_1\left(\phi_k^{(n-1)},\zeta_k^{(n-1)},\phi_{kx}^{(n-1)},\zeta_{kx}^{(n-1)}\right) \\ \psi_{kx}^{(n)}(0,t) = -\dfrac{p_+}{\mu}\phi_{0k}(0)e^{(p_+/\mu)t}, \\ \psi_k^{(n)}|_{t=0} = \psi_{0k}, \end{cases} \tag{4.7}$$

$$\begin{cases} \zeta_{kt}^{(n)} - \kappa\left(\dfrac{\zeta_{kx}^{(n)}}{V+\phi_k^{(n-1)}}\right)_x = g_2^{(n-1)} = g_2\left(\phi_k^{(n-1)},\zeta_k^{(n-1)},\phi_{kx}^{(n-1)},\psi_{kx}^{(n)}\right) \\ \zeta_k^{(n)}(0,t) = 0, \\ \zeta_k^{(n)}|_{t=0} = \zeta_{0k}, \end{cases} \tag{4.8}$$

and

$$\phi_k^{(n)}(x,t) = \phi_{0k}(x) + \int_0^t \psi_{kx}^{(n)}(x,\tau)\,d\tau. \tag{4.9}$$

We now construct $(\phi_{0k},\psi_{0k},\zeta_{0k})$. First we choose $\phi_{0k} \in H^3$ such that $\phi_{0k} \to \phi_0$ strongly in H^1 as $k \to \infty$. Let

$$A_1(x) = \psi_0(0)e^{-x^2}, \quad A_2(x) = -\frac{p_+}{\mu}\phi_{0k}(0)xe^{-x^2}. \tag{4.10}$$

It is easy to check that $A_i(x) \in H^3$, $i = 1, 2$ satisfying

$$A_1(0) = \psi_0(0), \quad A_{1x}(0) = A_2(0) = 0, \quad A_{2x}(0) = -\frac{p_+}{\mu}\phi_{0k}(0). \tag{4.11}$$

Let $\bar{\psi}_0(x) = \psi_0(x) - A_1(x) - A_2(x)$. Then we have $\bar{\psi}_0(x) \in H_0^1$. We choose $\bar{\psi}_{0k} \in H_0^3$ such that $\bar{\psi}_{0k} \to \bar{\psi}_0$ strongly in H^1. Let $\psi_{0k}(x) = \bar{\psi}_0(x) + A_1(x) + A_2(x)$, then we have $\psi_{0k} \to \psi_0$ strongly in H^1. Furthermore we have $\psi_{0kx}(0) = -(p_+/\mu)\phi_{0k}(0)$ which guarantees the compatibility condition for the equation (4.7). In the same way, we can construct the approximation $\zeta_{0k} \in H_0^2 \cap H^3$ satisfying $\zeta_{0k} \to \zeta_0$ strongly in H^1 and

$$\kappa \frac{\zeta_{0kxx}(0)}{v_- + \phi_{0k}(0)} = -g_2\big(\phi_{0k}(0), 0, \phi_{0kx}(0), \psi_{0kx}(0)\big) \tag{4.12}$$

which guarantees the compatibility condition for the equation (4.8). It is obvious that we can choose the above approximation $(\phi_{0k}, \psi_{0k}, \zeta_{0k})$ satisfying $\|(\phi_{0k}, \psi_{0k}, \zeta_{0k})\|_1 \leq (3/2)M$, $\inf_{R_+}(V + \phi_{0k}) \geq (2/3)m$ for any k.

By the linear theory, if $g_i^{(n-1)} \in C([0, T]; H^2)$, $i = 1, 2$, and $(\psi_{0k}, \zeta_{0k}) \in H^3$, there exists a unique-local solution $(\psi_k^{(n)}, \zeta_k^{(n)})$ to (4.7) and (4.8) satisfying

$$(\psi_k^{(n)}, \zeta_k^{(n)}) \in C([0, T]; H^3) \cap C^1([0, T]; H^1) \cap L^2(0, T; H^4). \tag{4.13}$$

Thus, if $(\phi_k^{(n-1)}, \psi_k^{(n-1)}, \zeta_k^{(n-1)}) \in X_{(1/2)m, bM}$, then the elementary energy estimate gives

$$\|\psi_k^{(n)}(t)\|_1^2 \leq \left(\left(\frac{3}{2}M\right)^2 + C(m, M)t_0\right) \exp\big(C(m, M)t_0\big) + \frac{C\mu}{p_+}M^2 \leq (bM)^2, \tag{4.14}$$

if $t_0 = t_0(m, M)$ is suitably small. Also we have

$$\int_0^{t_0} \|\psi_x\|_1^2 \, d\tau \leq C(m, M)(bM)^2. \tag{4.15}$$

Similarly we obtain

$$\|\zeta_k^{(n)}(t)\|_1^2 \leq (bM)^2, \tag{4.16}$$

where we have used the fact that

$$\begin{aligned} \int_0^\infty \big(\psi_{kx}^{(n)}\big)^2 |\zeta_{kxx}^{(n)}| \, dx &\leq \|\psi_{kx}^{(n)}\|_{L^\infty} \|\psi_{kx}^{(n)}\| \|\zeta_{kxx}^{(n)}\| \\ &\leq \delta_2 \|\zeta_{kxx}^{(n)}\|^2 + \delta_2 \|\psi_{kxx}^{(n)}\|^2 + C_{\delta_2} M^2, \end{aligned} \tag{4.17}$$

for suitably small constant $\delta_2 > 0$. On the other hand, direct computation on (4.9) and (4.16) yields

$$\|\phi_k^{(n)}(t)\|_1^2 \leq (2M)^2, \tag{4.18}$$

and $\inf_{R_+ \times [0, t_0]}(V + \phi_k^{(n)})(x, t) \geq (1/2)m$. Therefore we have $(\phi_k^{(n)}, \psi_k^{(n)}, \zeta_k^{(n)}) \in X_{(1/2)m, bM}(0, t_0)$. Since $\|\phi_{0k}, \psi_{0k}, \zeta_{0k}\|_3 \leq C_k$, $(\phi_k^{(n)}, \psi_k^{(n)}, \zeta_k^{(n)})$ can be shown a Cauchy sequence in $C(0, t_0; H^2)$. Thus we have a solution $(\phi_k, \psi_k, \zeta_k)(x, t)$ by letting n tends to infinity. In particular, on the boundary $x = 0$, we have

$$\phi_{kt} = \psi_{kx} = -\frac{p_+}{\mu} \phi_k. \tag{4.19}$$

In the same way, we can show $(\phi_k, \psi_k, \zeta_k)$ is also a Cauchy sequence in $C(0, T_0; H^1)$ (taking T_0 smaller than t_0 if necessary). Letting $k \to \infty$, we obtain the desired unique-local solution $(\phi, \psi, \zeta)(x, t) \in X_{(1/2)m,bM}(0, T_0)$ to (2.5). $\square$

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