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FORMAL GEVREY THEORY FOR SINGULAR FIRST ORDER SEMI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction and main result

In this paper we are concerned with formal power series solutions of the following first order semi-linear partial differential equation:

$$P(x, D)u(x) \equiv \sum_{i=1}^{d} a_i(x)D_i u(x) = f(x, u(x)), \quad u(0) = 0, \qquad x = (x_1, \ldots, x_d) \in \mathbf{C}^d, \quad D_i = \frac{\partial}{\partial x_i}, \tag{1.1}$$

where coefficients $a_i(x)$ $(i = 1, \ldots, d)$ and $f(x, u)$ are holomorphic in a neighborhood of $x = 0$ and $(x, u) = (0, 0)$, respectively.

If $a_i(0) \neq 0$ for some i, the solvability is well known by Cauchy-Kowalevsky's theorem. Therefore we shall study the case where

$$a_i(0) = 0 \quad \text{for all} \quad i = 1, \ldots, d, \tag{1.2}$$

which is called a singular or degenerate case. In the following we always assume (1.2).

The first purpose of this paper is to prove the existence and the uniqueness of the formal power series solution $u(x) = \sum_{|\alpha|\geq 1} u_\alpha x^\alpha$ $(\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d$, $\mathbf{N} = \{0, 1, 2, \ldots\}$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d})$ centered at the origin for the singular equation (1.1). As we will see later, we can prove it under some condition on the principal part $P(x, D)$. However, this formal power series solution $u(x)$ does not necessarily converge. So we would like to obtain the rate of divergence, which is called the Gevrey order, of the formal solution (cf. Definition 1.1). This is the second purpose of this paper.

1.1. Motivation. In the paper Hibino [2], we considered the following singular first order linear partial differential equation:

$$\widetilde{P}(x, D)u(x) \equiv \sum_{i=1}^{d} a_i(x)D_i u(x) + b(x)u(x) = f(x), \tag{1.3}$$

where $a_i(x)$ are the same as the above and we assume (1.2); $b(x)$ and $f(x)$ are holomorphic at $x = 0$. We remark that we do not demand $u(0) = 0$ here.

In Hibino [2], we obtained the condition under which the formal power series solution $u(x) = \sum_{\alpha\in\mathbf{N}^d} u_\alpha x^\alpha$ of the equation (1.3) exists uniquely, and obtained the Gevrey order of $u(x)$. Firstly, let us introduce this result.

Let $D_x a(0) := (D_i a_j(0))_{i,j=1,\dots,d}$ be the Jacobi matrix at the origin of the mapping $a = (a_1, \dots, a_d)$ and let its Jordan canonical form be

$$\begin{pmatrix} A & & & & \\ & B_1 & & & \\ & & \ddots & & \\ & & & B_k & \\ & & & & O_p \end{pmatrix},$$

where

$$A = \begin{pmatrix} \lambda_1 & \delta_1 & & \\ & \lambda_2 & \ddots & \\ & & \ddots & \delta_{m-1} \\ & & & \lambda_m \end{pmatrix}, \quad B_h = \underbrace{\begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}}_{n_h}, \quad \begin{aligned} &\lambda_i \neq 0 \ (i = 1, \dots, m), \\ &\delta_i = 0 \text{ or } 1 \ (i = 1, \dots, m-1), \\ &h = 1, \dots, k, \end{aligned}$$

and O_p is a zero-matrix of order p ($m, k, p \geq 0$; $n_h \geq 2$; $m + n_1 + \cdots + n_k + p = d$).

Let us assume the following condition (Po) according to the value of m ("Po" derives from Poincaré):

$$(\mathbf{Po}) \quad \begin{cases} \left| \displaystyle\sum_{i=1}^m \lambda_i \alpha_i + b(0) \right| > \delta|\alpha| \quad \text{for all} \quad \alpha \in \mathbf{N}^m & (\text{if } m \geq 1), \\ b(0) \neq 0 & (\text{if } m = 0), \end{cases}$$

where δ is a positive constant independent of $\alpha \in \mathbf{N}^m$.

Before stating the main result in Hibino [2], let us give the definition of the Gevrey order, which gives the rate of divergence of formal power series.

Definition 1.1. Let $u(x) = \sum_{\alpha\in\mathbf{N}^d} u_\alpha x^\alpha$ be a formal power series centered at the origin. We say that $u(x)$ *belongs to* $G^{\{s\}}$ ($s = (s_1, \dots, s_d) \in \mathbf{R}^d$), if the power series

$$v(\xi) = \sum_{\alpha\in\mathbf{N}^d} u_\alpha \frac{\xi^\alpha}{(\alpha!)^{s-1^{(d)}}}$$

converges in a neighborhood of $\xi = 0$, where $1^{(d)} = (\overbrace{1, \dots, 1}^{d})$, $s - 1^{(d)} = (s_1 - 1, \dots, s_d -$

1) and $(\alpha!)^{s-1^{(d)}} = (\alpha_1!)^{s_1-1}\cdots(\alpha_d!)^{s_d-1}$. Especially, $u(x) \in G^{\{1^{(d)}\}}$ if and only if $u(x)$ is a convergent power series near $x = 0$.

Now the main result in Hibino [2] is stated as follows:

Theorem 1.1 (Hibino [2]). *Under the condition* (Po), *the equation* (1.3) *has a unique formal power series solution* $u(x) = \sum_{\alpha\in\mathbf{N}^d} u_\alpha x^\alpha$. *Furthermore the formal solution* $u(x)$ *belongs to* $G^{\{2N,\dots,2N\}}$, *where*

$$N = \begin{cases} \max\{n_1,\dots,n_k\} & (\textit{if } k \geq 1),\\ 1 & (\textit{if } k = 0 \textit{ and } p \geq 1),\\ \dfrac{1}{2} & (\textit{if } k = p = 0). \end{cases}$$

Therefore in the case $k = p = 0$ *the formal solution converges, but in other cases it diverges in general.*

The purpose of this paper is to generalize this result up to semi-linear equations.

Now let us consider the equation (1.3) again and let us try to calculate $u(0)$. Since the condition (Po) implies that $b(0) \neq 0$, it is easy to prove that $u(0) = f(0)/b(0)$. Therefore it follows from a change of unknown functions $v(x) = u(x) - u(0)$ that under the condition $b(0) \neq 0$ (especially the condition (Po)) the equation (1.3) is equivalent to the following one:

$$\sum_{i=1}^{d} a_i(x) D_i v(x) + b(x)v(x) = g(x), \quad v(0) = 0, \tag{1.4}$$

where $g(x)$ is holomorphic in a neighborhood of the origin with $g(0) = 0$.

Therefore corresponding to the condition $g(0) = 0$, it is natural to assume the following condition for our equation (1.1):

$$f(0,0) = 0. \tag{1.5}$$

In the following we always assume (1.5).

1.2. Main result. Let us state the main result in this paper. First, we state the condition. Instead of the condition (Po), we assume the following condition (Po2):

$$\textbf{(Po2)} \quad \begin{cases} \left|\displaystyle\sum_{i=1}^{m} \lambda_i\alpha_i - f_u(0,0)\right| > \delta|\alpha| \quad \text{for all} \quad \alpha \in \mathbf{N}^m & (\text{if } m \geq 1),\\ f_u(0,0) \neq 0 & (\text{if } m = 0), \end{cases}$$

where $f_u(0,0) = (\partial f/\partial u)(0,0)$.

Now our main result is stated as follows:

Theorem 1.2. *Under the condition* (Po2), *the equation* (1.1) *has a unique formal power series solution* $u(x) = \sum_{|\alpha|\geq 1} u_\alpha x^\alpha$. *Furthermore the formal solution* $u(x)$ *belongs to* $G^{\{2N,\dots,2N\}}$, *where* N *is same as in* Theorem 1.1.

In order to prove Theorem 1.2, we shall transform the equation (1.1) in the next section. For that transformed equation we can obtain the precise Gevrey order in individual variables of the formal solution (Theorem 2.1). We shall prove the unique existence of the formal solution and its Gevrey order separately. Admitting the unique existence of the formal solution, we will prove its Gevrey order in §4 (in the case $m = 0$) and §5 (in the case $m \geq 1$) by using the contraction mapping principle in Banach spaces which consist of formal power series. The Banach spaces employed in the proof will be introduced in §3. The unique existence of the formal solution will be proved in §6.

REMARK 1.1. The studies in this paper and Hibino [2] are inspired by the study in Ōshima [8]. He studied a characterization of the kernel and the cokernel of the linear mapping

$$\widetilde{P}(x,D)\colon \mathcal{O} \to \mathcal{O},$$

where $\mathcal{O}$ is the set of holomorphic functions at the origin. He studied the case $m \geq 1$ and $k = 0$ in our notation, and obtained the condition under which the formal solution converges. As mentioned in our theorem, when $m \geq 1$, $k = 0$ and $p \geq 1$, the formal solution diverges in general and it belongs to $G^{\{2,\dots,2\}}$. In this sence, our theorem gives one of the generalizations of Ōshima [8].

Many mathematicians have generalized Ōshima's result. The cases of higher order equations are studied by Miyake [4] and Miyake-Hashimoto [5]. Nonlinear equations are studied in Gérard-Tahara [1] and Miyake-Shirai [6]. Moreover for linear equations, Kashiwara-Kawai-Sjöstrand [3] and Miyake-Yoshino [7] give different characterizations of convergence of formal solutions.

2. Reduction of equation and Newton polyhedron

In order to prove Theorem 1.2 we shall transform the equation (1.1) by a linear transform of independent variables which reduces $D_x a(0)$ to its Jordan canonical form. A reduced equation is written as follows according to the values of m, k and p:

CASE (i). $m \geq 1,\ k \geq 1,\ p \geq 1$:

$$(2.1)\qquad \begin{aligned} &P_1 u = g_0(x, y^1, \dots, y^k, z) + g\bigl(x, y^1, \dots, y^k, z, u(x, y^1, \dots, y^k, z)\bigr),\\ &u(0, 0, \dots, 0, 0) = 0, \end{aligned}$$

where $x = (x_1, \dots, x_m) \in \mathbf{C}^m$, $y^h = (y^h_1, \dots, y^h_{n_h}) \in \mathbf{C}^{n_h}$ ($h = 1, \dots, k$) and $z = (z_1, \dots, z_p) \in \mathbf{C}^p$. g_0 and g are holomorphic at the origin which satisfy $g_0(0, 0, \dots, 0, 0) = 0$ and $g(x, y^1, \dots, y^k, z, 0) \equiv g_u(x, y^1, \dots, y^k, z, 0) \equiv 0$, respectively. Furthermore P_1 is a linear partial differential operator which has the following form:

$$P_1 = \sum_{i=1}^{m} \lambda_i x_i \frac{\partial}{\partial x_i} - f_u(0, 0) + P_1' + P_1'' + P_1''' + P_1'''' + h, \tag{2.2}$$

where

$$\begin{aligned}
P_1' &= \sum_{i=1}^{m-1} \delta_i x_{i+1} \frac{\partial}{\partial x_i} \\
&\quad + \sum_{i=1}^{m} \Bigg(\sum_{\substack{|\alpha|+|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2 \\ |\alpha|\geq 1}}^{\text{finite}} c_{i\alpha\beta^1\dots\beta^k\gamma}(x, y^1, \dots, y^k, z) x^\alpha (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} z^\gamma \Bigg) \frac{\partial}{\partial x_i}, \\
P_1'' &= \sum_{h=1}^{k} \sum_{j_h=1}^{n_h} \Bigg(\sum_{\substack{|\alpha|+|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2 \\ |\alpha|\geq 1}}^{\text{finite}} d^h_{j_h\alpha\beta^1\dots\beta^k\gamma}(x, y^1, \dots, y^k, z) \\
&\qquad\qquad \times x^\alpha (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} z^\gamma \Bigg) \frac{\partial}{\partial y^h_{j_h}} \\
&\quad + \sum_{q=1}^{p} \Bigg(\sum_{\substack{|\alpha|+|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2 \\ |\alpha|\geq 1}}^{\text{finite}} e_{q\alpha\beta^1\dots\beta^k\gamma}(x, y^1, \dots, y^k, z) x^\alpha (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} z^\gamma \Bigg) \frac{\partial}{\partial z_q}, \\
P_1''' &= \sum_{h=1}^{k} \sum_{j_h=1}^{n_h-1} y^h_{j_h+1} \frac{\partial}{\partial y^h_{j_h}} \\
&\quad + \sum_{h=1}^{k} \sum_{j_h=1}^{n_h} \Bigg(\sum_{|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2}^{\text{finite}} d^h_{j_h\beta^1\dots\beta^k\gamma}(x, y^1, \dots, y^k, z) (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} z^\gamma \Bigg) \frac{\partial}{\partial y^h_{j_h}} \\
&\quad + \sum_{q=1}^{p} \Bigg(\sum_{|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2}^{\text{finite}} e_{q\beta^1\dots\beta^k\gamma}(x, y^1, \dots, y^k, z) (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} z^\gamma \Bigg) \frac{\partial}{\partial z_q}, \\
P_1'''' &= \sum_{i=1}^{m} \Bigg(\sum_{|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2}^{\text{finite}} c_{i\beta^1\dots\beta^k\gamma}(x, y^1, \dots, y^k, z) (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} z^\gamma \Bigg) \frac{\partial}{\partial x_i}, \\
h &= h(x, y^1, \dots, y^k, z) \\
&= \sum_{|\alpha|+|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 1}^{\text{finite}} h_{\alpha\beta^1\dots\beta^k\gamma}(x, y^1, \dots, y^k, z) x^\alpha (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} z^\gamma.
\end{aligned}$$

In the above expressions, all coefficients $c_{i\alpha\beta^1\dots\beta^k\gamma}$, etc., are holomorphic at the origin, and none of them vanish at the origin unless they vanish identically. In the following expressions, we assume the same conditions for those functions appearing in the coefficients.

CASE (ii). $m \geq 1,\ k \geq 1,\ p = 0$:

$$
(2.3)\qquad \begin{aligned}
&P_1 u = g_0(x, y^1, \dots, y^k) + g\big(x, y^1, \dots, y^k, u(x, y^1, \dots, y^k)\big),\\
&u(0, 0, \dots, 0) = 0,
\end{aligned}
$$

where g_0 and g are holomorphic at the origin which satisfy $g_0(0, 0, \dots, 0) = 0$ and $g(x, y^1, \dots, y^k, 0) \equiv g_u(x, y^1, \dots, y^k, 0) \equiv 0$, respectively. The linear partial differential operator P_1 is same as (2.2), where

$$
\begin{aligned}
P_1' &= \sum_{i=1}^{m-1} \delta_i x_{i+1} \frac{\partial}{\partial x_i}\\
&\quad + \sum_{i=1}^{m} \Bigg(\sum_{\substack{|\alpha|+|\beta^1|+\dots+|\beta^k|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} c_{i\alpha\beta^1\dots\beta^k}(x, y^1, \dots, y^k) x^\alpha (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} \Bigg) \frac{\partial}{\partial x_i},\\
P_1'' &= \sum_{h=1}^{k} \sum_{j_h=1}^{n_h} \Bigg(\sum_{\substack{|\alpha|+|\beta^1|+\dots+|\beta^k|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} d^h_{j_h\alpha\beta^1\dots\beta^k}(x, y^1, \dots, y^k) x^\alpha (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} \Bigg) \frac{\partial}{\partial y^h_{j_h}},\\
P_1''' &= \sum_{h=1}^{k} \sum_{j_h=1}^{n_h-1} y^h_{j_h+1} \frac{\partial}{\partial y^h_{j_h}}\\
&\quad + \sum_{h=1}^{k} \sum_{j_h=1}^{n_h} \Bigg(\sum_{|\beta^1|+\dots+|\beta^k|\geq 2}^{\text{finite}} d^h_{j_h\beta^1\dots\beta^k}(x, y^1, \dots, y^k) (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} \Bigg) \frac{\partial}{\partial y^h_{j_h}},\\
P_1'''' &= \sum_{i=1}^{m} \Bigg(\sum_{|\beta^1|+\dots+|\beta^k|\geq 2}^{\text{finite}} c_{i\beta^1\dots\beta^k}(x, y^1, \dots, y^k) (y^1)^{\beta^1} \cdots (y^k)^{\beta^k} \Bigg) \frac{\partial}{\partial x_i},\\
h &= h(x, y^1, \dots, y^k)\\
&= \sum_{|\alpha|+|\beta^1|+\dots+|\beta^k|\geq 1}^{\text{finite}} h_{\alpha\beta^1\dots\beta^k}(x, y^1, \dots, y^k) x^\alpha (y^1)^{\beta^1} \cdots (y^k)^{\beta^k}.
\end{aligned}
$$

CASE (iii). $m \geq 1,\ k = 0,\ p \geq 1$:

$$
(2.4)\qquad P_1 u = g_0(x, z) + g\big(x, z, u(x, z)\big), \quad u(0, 0) = 0,
$$

where g_0 and g are holomorphic at the origin with $g_0(0, 0) = 0$ and $g(x, z, 0) \equiv g_u(x, z, 0) \equiv 0$, respectively. The linear partial differential operator P_1 is same as (2.2), where

$$
\begin{aligned}
P_1' &= \sum_{i=1}^{m-1} \delta_i x_{i+1} \frac{\partial}{\partial x_i} + \sum_{i=1}^{m} \Bigg(\sum_{\substack{|\alpha|+|\gamma|\geq 2 \\ |\alpha|\geq 1}}^{\textit{finite}} c_{i\alpha\gamma}(x,z) x^\alpha z^\gamma \Bigg) \frac{\partial}{\partial x_i}, \\
P_1'' &= \sum_{q=1}^{p} \Bigg(\sum_{\substack{|\alpha|+|\gamma|\geq 2 \\ |\alpha|\geq 1}}^{\textit{finite}} e_{q\alpha\gamma}(x,z) x^\alpha z^\gamma \Bigg) \frac{\partial}{\partial z_q}, \\
P_1''' &= \sum_{q=1}^{p} \Bigg(\sum_{|\gamma|\geq 2}^{\textit{finite}} e_{q\gamma}(x,z) z^\gamma \Bigg) \frac{\partial}{\partial z_q}, \\
P_1'''' &= \sum_{i=1}^{m} \Bigg(\sum_{|\gamma|\geq 2}^{\textit{finite}} c_{i\gamma}(x,z) z^\gamma \Bigg) \frac{\partial}{\partial x_i}, \\
h &= h(x,z) \\
&= \sum_{|\alpha|+|\gamma|\geq 1}^{\textit{finite}} h_{\alpha\gamma}(x,z) x^\alpha z^\gamma .
\end{aligned}
$$

CASE (iv). $m \geq 1,\ k = p = 0$:

$$
P_1 u = g_0(x) + g\big(x, u(x)\big), \quad u(0) = 0, \tag{2.5}
$$

where g_0 and g are holomorphic at the origin with $g_0(0) = 0$ and $g(x,0) \equiv g_u(x,0) \equiv 0$, respectively. The operator P_1 is given by

$$
\begin{aligned}
P_1 = &\sum_{i=1}^{m} \lambda_i x_i \frac{\partial}{\partial x_i} - f_u(0,0) + \sum_{i=1}^{m-1} \delta_i x_{i+1} \frac{\partial}{\partial x_i} \\
&+ \sum_{i=1}^{m} \Bigg(\sum_{|\alpha|\geq 2}^{\textit{finite}} c_{i\alpha}(x) x^\alpha \Bigg) \frac{\partial}{\partial x_i} + \sum_{|\alpha|\geq 1}^{\textit{finite}} h_\alpha(x) x^\alpha .
\end{aligned} \tag{2.6}
$$

CASE (v). $m = 0,\ k \geq 1,\ p \geq 1$:

$$
\begin{aligned}
&P_1 u = g_0(y^1, \dots, y^k, z) + g\big(y^1, \dots, y^k, z, u(y^1, \dots, y^k, z)\big), \\
&u(0, \dots, 0, 0) = 0,
\end{aligned} \tag{2.7}
$$

where g_0 and g are holomorphic at the origin which satisfy $g_0(0, \dots, 0, 0) = 0$ and $g(y^1, \dots, y^k, z, 0) \equiv g_u(y^1, \dots, y^k, z, 0) \equiv 0$, respectively. Furthermore P_1 is a linear partial differential operator which has the following form:

$$
P_1 = -f_u(0,0) + P_1''' + h, \tag{2.8}
$$

where

$$
\begin{aligned}
P_1''' &= \sum_{h=1}^{k} \sum_{j_h=1}^{n_h-1} y^h_{j_h+1} \frac{\partial}{\partial y^h_{j_h}} \\
&\quad + \sum_{h=1}^{k} \sum_{j_h=1}^{n_h} \left(\sum_{|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2}^{\text{finite}} d^h_{j_h\beta^1\ldots\beta^k\gamma}(y^1,\ldots,y^k,z)(y^1)^{\beta^1}\cdots(y^k)^{\beta^k} z^{\gamma} \right) \frac{\partial}{\partial y^h_{j_h}}, \\
&\quad + \sum_{q=1}^{p} \left(\sum_{|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 2}^{\text{finite}} e_{q\beta^1\ldots\beta^k\gamma}(y^1,\ldots,y^k,z)(y^1)^{\beta^1}\cdots(y^k)^{\beta^k} z^{\gamma} \right) \frac{\partial}{\partial z_q}, \\
h &= h(y^1,\ldots,y^k,z) \\
&= \sum_{|\beta^1|+\cdots+|\beta^k|+|\gamma|\geq 1}^{\text{finite}} h_{\beta^1\ldots\beta^k\gamma}(y^1,\ldots,y^k,z)(y^1)^{\beta^1}\cdots(y^k)^{\beta^k} z^{\gamma}.
\end{aligned}
$$

CASE (vi). $m = 0,\ k \geq 1,\ p = 0$:

$$
P_1 u = g_0(y^1,\ldots,y^k) + g\big(y^1,\ldots,y^k,u(y^1,\ldots,y^k)\big), \quad u(0,\ldots,0) = 0, \tag{2.9}
$$

where g_0 and g are holomorphic at the origin with $g_0(0,\ldots,0) = 0$ and $g(y^1,\ldots,y^k, 0) \equiv g_u(y^1,\ldots,y^k,0) \equiv 0$, respectively. The linear partial differential operator P_1 is same as (2.8), where

$$
\begin{aligned}
P_1''' &= \sum_{h=1}^{k} \sum_{j_h=1}^{n_h-1} y^h_{j_h+1} \frac{\partial}{\partial y^h_{j_h}} \\
&\quad + \sum_{h=1}^{k} \sum_{j_h=1}^{n_h} \left(\sum_{|\beta^1|+\cdots+|\beta^k|\geq 2}^{\text{finite}} d^h_{j_h\beta^1\ldots\beta^k}(y^1,\ldots,y^k)(y^1)^{\beta^1}\cdots(y^k)^{\beta^k} \right) \frac{\partial}{\partial y^h_{j_h}}, \\
h &= h(y^1,\ldots,y^k) \\
&= \sum_{|\beta^1|+\cdots+|\beta^k|\geq 1}^{\text{finite}} h_{\beta^1\ldots\beta^k}(y^1,\ldots,y^k)(y^1)^{\beta^1}\cdots(y^k)^{\beta^k}.
\end{aligned}
$$

CASE (vii). $m = k = 0,\ p \geq 1$:

$$
P_1 u = g_0(z) + g\big(z, u(z)\big), \quad u(0) = 0, \tag{2.10}
$$

where g_0 and g are holomorphic at the origin satisfying $g_0(0) = 0$ and $g(z,0) \equiv g_u(z,0) \equiv 0$, respectively. P_1 is same as (2.8), where

$$
P_1''' = \sum_{q=1}^{p} \left(\sum_{|\gamma|\geq 2}^{\text{finite}} e_{q\gamma}(z) z^{\gamma} \right) \frac{\partial}{\partial z_q},
$$

$$\begin{aligned} h &= h(z) \\ &= \sum_{|\gamma|\ge 1}^{\mathit{finite}} h_\gamma(z) z^\gamma. \end{aligned}$$

Now we shall study the equations (2.1), (2.3), (2.4), (2.5), (2.7), (2.9) and (2.10).

In order to give the Gevrey orders in an individual variable for formal solutions of the above equations, we study the Newton polyhedron of linear partial differential operators (see also Hibino [2] and Yamazawa [9]).

Newton polyhedron. Let

$$P(\xi, D_\xi) = \sum_{|\alpha|,|\beta|\ge 0}^{\mathit{finite}} a_{\alpha\beta}(\xi)\xi^\alpha D_\xi^\beta$$

($\xi = (\xi_1, \dots, \xi_d)$, $D_\xi^\beta = (\partial/\partial\xi_1)^{\beta_1} \cdots (\partial/\partial\xi_d)^{\beta_d}$) be a linear partial differential operator, where all coefficients are holomorphic at the origin and do not vanish at the origin unless they vanish identically.

Let us define $Q(\alpha, \beta) \subset \mathbf{R}^{d+1}$ by

$$Q(\alpha, \beta) = \{(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}_1, \dots, \mathcal{X}_d, \mathcal{Y}) \in \mathbf{R}^{d+1};\ \mathcal{X}_i \ge \alpha_i - \beta_i\ (i = 1, \dots, d),\ \mathcal{Y} \le |\beta|\}$$

and let us define the Newton polyhedron $N(P)$ of the operator P by

$$N(P) = \begin{cases} \mathrm{Ch}\left\{ \displaystyle\bigcup_{(\alpha,\beta)\ \text{with}\ a_{\alpha\beta}\not\equiv 0} Q(\alpha, \beta) \right\} & (\text{if } P \ne 0), \\ Q(0, 0) & (\text{if } P = 0), \end{cases}$$

where $\mathrm{Ch}\, A$ denotes the convex hull of a set $A \subset \mathbf{R}^{d+1}$.

Now we shall apply the above general definition to our operator P_1. We remark that the correspondence of variables between $(x, y^1, \dots, y^k, z)$ and ξ is given by

	ξ
Case (i)	$(x, y^1, \dots, y^k, z)$
Case (ii)	$(x, y^1, \dots, y^k)$
Case (iii)	(x, z)
Case (iv)	—
Case (v)	$(y^1, \dots, y^k, z)$
Case (vi)	$(y^1, \dots, y^k)$
Case (vii)	z

In order to state the main theorem in this section, we shall define the sets S_i ($i = 1$, 2, 3, 5, 6, 7), $\widetilde{S}_j$, $\widetilde{S}'_j$, $\widetilde{S}''_j$, S'_j, S''_j ($j = 1$, 2, 3) whose elements give the Gevrey orders of formal solutions.

CASE (i). We define $\widetilde{\Pi}_1(\rho, \sigma^1, \dots, \sigma^k, \tau)$ and $\Pi_1(\rho, \sigma^1, \dots, \sigma^k, \tau)$ $((\rho, \sigma^1, \dots, \sigma^k, \tau) \in [1, +\infty)^d,\ \rho = (\rho_1, \dots, \rho_m),\ \sigma^h = (\sigma^h_1, \dots, \sigma^h_{n_h})\ (h = 1, \dots, k),\ \tau = (\tau_1, \dots, \tau_p))$ by

$$\begin{aligned}\widetilde{\Pi}_1(\rho, \sigma^1, \dots, \sigma^k, \tau) = \Big\{&(\mathcal{X}, \mathcal{Y}^1, \dots, \mathcal{Y}^k, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} \\ &+ \sum_{h=1}^{k} (\sigma^h - 1^{(n_h)}) \cdot \mathcal{Y}^h + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq -1\Big\}\end{aligned}$$

and

$$\begin{aligned}\Pi_1(\rho, \sigma^1, \dots, \sigma^k, \tau) = \Big\{&(\mathcal{X}, \mathcal{Y}^1, \dots, \mathcal{Y}^k, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} \\ &+ \sum_{h=1}^{k} (\sigma^h - 1^{(n_h)}) \cdot \mathcal{Y}^h + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq 0\Big\},\end{aligned}$$

respectively, and define $\widetilde{S}_1$, $\widetilde{S}'_1$, $\widetilde{S}''_1$, S_1, S'_1 and S''_1 as follows:

$$\begin{aligned}\widetilde{S}_1 &= \{(\rho, \sigma^1, \dots, \sigma^k, \tau) \in [1, +\infty)^d;\ N(P'_1) \subset \widetilde{\Pi}_1(\rho, \sigma^1, \dots, \sigma^k, \tau)\},\\ \widetilde{S}'_1 &= \{(\rho, \sigma^1, \dots, \sigma^k, \tau) \in [1, +\infty)^d;\ N(P''_1) \subset \widetilde{\Pi}_1(\rho, \sigma^1, \dots, \sigma^k, \tau)\},\\ \widetilde{S}''_1 &= \{(\rho, \sigma^1, \dots, \sigma^k, \tau) \in [1, +\infty)^d;\ N(P''''_1) \subset \widetilde{\Pi}_1(\rho, \sigma^1, \dots, \sigma^k, \tau)\},\\ S_1 &= \{(\rho, \sigma^1, \dots, \sigma^k, \tau) \in [1, +\infty)^d;\ N(P'''_1) \subset \Pi_1(\rho, \sigma^1, \dots, \sigma^k, \tau)\},\\ S'_1 &= \{(\rho, \sigma^1, \dots, \sigma^k, \tau) \in [1, +\infty)^d;\ N(P''_1) \subset \Pi_1(\rho, \sigma^1, \dots, \sigma^k, \tau)\},\\ S''_1 &= \{(\rho, \sigma^1, \dots, \sigma^k, \tau) \in [1, +\infty)^d;\ N(P''''_1) \subset \Pi_1(\rho, \sigma^1, \dots, \sigma^k, \tau)\}.\end{aligned}$$

CASE (ii). We set $\widetilde{\Pi}_2(\rho, \sigma^1, \dots, \sigma^k)$ and $\Pi_2(\rho, \sigma^1, \dots, \sigma^k)$ $\big((\rho, \sigma^1, \dots, \sigma^k) \in [1, +\infty)^d\big)$ by

$$\begin{aligned}&\widetilde{\Pi}_2(\rho, \sigma^1, \dots, \sigma^k)\\ &= \Big\{(\mathcal{X}, \mathcal{Y}^1, \dots, \mathcal{Y}^k, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} + \sum_{h=1}^{k} (\sigma^h - 1^{(n_h)}) \cdot \mathcal{Y}^h - \mathcal{W} \geq -1\Big\}\end{aligned}$$

and

$$\begin{aligned}&\Pi_2(\rho, \sigma^1, \dots, \sigma^k)\\ &= \Big\{(\mathcal{X}, \mathcal{Y}^1, \dots, \mathcal{Y}^k, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} + \sum_{h=1}^{k} (\sigma^h - 1^{(n_h)}) \cdot \mathcal{Y}^h - \mathcal{W} \geq 0\Big\},\end{aligned}$$

respectively, and define $\widetilde{S}_2$, $\widetilde{S}'_2$, $\widetilde{S}''_2$, S_2, S'_2 and S''_2 as follows:

$$\widetilde{S}_2 = \{(\rho, \sigma^1, \dots, \sigma^k) \in [1, +\infty)^d;\ N(P'_1) \subset \widetilde{\Pi}_2(\rho, \sigma^1, \dots, \sigma^k)\},$$

$$\begin{aligned}
\widetilde{S}_2' &= \{(\rho, \sigma^1, \ldots, \sigma^k) \in [1, +\infty)^d;\ N(P_1'') \subset \widetilde{\Pi}_2(\rho, \sigma^1, \ldots, \sigma^k)\},\\
\widetilde{S}_2'' &= \{(\rho, \sigma^1, \ldots, \sigma^k) \in [1, +\infty)^d;\ N(P_1'''') \subset \widetilde{\Pi}_2(\rho, \sigma^1, \ldots, \sigma^k)\},\\
S_2 &= \{(\rho, \sigma^1, \ldots, \sigma^k) \in [1, +\infty)^d;\ N(P_1''') \subset \Pi_2(\rho, \sigma^1, \ldots, \sigma^k)\},\\
S_2' &= \{(\rho, \sigma^1, \ldots, \sigma^k) \in [1, +\infty)^d;\ N(P_1'') \subset \Pi_2(\rho, \sigma^1, \ldots, \sigma^k)\},\\
S_2'' &= \{(\rho, \sigma^1, \ldots, \sigma^k) \in [1, +\infty)^d;\ N(P_1'''') \subset \Pi_2(\rho, \sigma^1, \ldots, \sigma^k)\}.
\end{aligned}$$

CASE (iii). We define $\widetilde{\Pi}_3(\rho, \tau)$ and $\Pi_3(\rho, \tau)$ $\big((\rho, \tau) \in [1, +\infty)^d\big)$ by

$$\widetilde{\Pi}_3(\rho, \tau) = \{(\mathcal{X}, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq -1\}$$

and

$$\Pi_3(\rho, \tau) = \{(\mathcal{X}, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq 0\},$$

respectively, and define $\widetilde{S}_3$, $\widetilde{S}_3'$, $\widetilde{S}_3''$, S_3, S_3' and S_3'' as follows:

$$\begin{aligned}
\widetilde{S}_3 &= \{(\rho, \tau) \in [1, +\infty)^d;\ N(P_1') \subset \widetilde{\Pi}_3(\rho, \tau)\},\\
\widetilde{S}_3' &= \{(\rho, \tau) \in [1, +\infty)^d;\ N(P_1'') \subset \widetilde{\Pi}_3(\rho, \tau)\},\\
\widetilde{S}_3'' &= \{(\rho, \tau) \in [1, +\infty)^d;\ N(P_1'''') \subset \widetilde{\Pi}_3(\rho, \tau)\},\\
S_3 &= \{(\rho, \tau) \in [1, +\infty)^d;\ N(P_1''') \subset \Pi_3(\rho, \tau)\},\\
S_3' &= \{(\rho, \tau) \in [1, +\infty)^d;\ N(P_1'') \subset \Pi_3(\rho, \tau)\},\\
S_3'' &= \{(\rho, \tau) \in [1, +\infty)^d;\ N(P_1'''') \subset \Pi_3(\rho, \tau)\}.
\end{aligned}$$

CASE (v). We define $\Pi_5(\sigma^1, \ldots, \sigma^k, \tau)$ $\big((\sigma^1, \ldots, \sigma^k, \tau) \in [1, +\infty)^d\big)$ by

$$\begin{aligned}
&\Pi_5(\sigma^1, \ldots, \sigma^k, \tau)\\
&= \left\{(\mathcal{Y}^1, \ldots, \mathcal{Y}^k, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ \sum_{h=1}^{k} (\sigma^h - 1^{(n_h)}) \cdot \mathcal{Y}^h + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq 0\right\},
\end{aligned}$$

and define S_5 by

$$S_5 = \{(\sigma^1, \ldots, \sigma^k, \tau) \in [1, +\infty)^d;\ N(P_1''') \subset \Pi_5(\sigma^1, \ldots, \sigma^k, \tau)\}.$$

CASE (vi). We define $\Pi_6(\sigma^1, \ldots, \sigma^k)$ $\big((\sigma^1, \ldots, \sigma^k) \in [1, +\infty)^d\big)$ by

$$\Pi_6(\sigma^1, \ldots, \sigma^k) = \left\{(\mathcal{Y}^1, \ldots, \mathcal{Y}^k, \mathcal{W}) \in \mathbf{R}^{d+1};\ \sum_{h=1}^{k} (\sigma^h - 1^{(n_h)}) \cdot \mathcal{Y}^h - \mathcal{W} \geq 0\right\},$$

and define S_6 by

$$S_6 = \{(\sigma^1, \ldots, \sigma^k) \in [1, +\infty)^d;\ N(P_1''') \subset \Pi_6(\sigma^1, \ldots, \sigma^k)\}.$$

CASE (vii). We define $\Pi_7(\tau)$ ($\tau \in [1, +\infty)^d$) by

$$\Pi_7(\tau) = \{(\mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq 0\},$$

and define S_7 by

$$S_7 = \{\tau \in [1, +\infty)^d;\ N(P_1''') \subset \Pi_7(\tau)\}.$$

Then we obtain the following theorem.

Theorem 2.1. *In* Case (i) (*resp.* (ii), (iii), (iv), (v), (vi) *and* (vii)), *under the condition* (Po2) *the equation* (2.1) (*resp.* (2.3), (2.4), (2.5), (2.7), (2.9) *and* (2.10)) *has a unique formal power series solution. Furthermore the formal solution belongs to* $G^{\{s\}}$ *if* s *satisfies the following condition*:

CASE (i). $P_1'''' = 0 \Rightarrow s = (\rho, \sigma^1, \dots, \sigma^k, \tau) \in \widetilde{S}_1 \cap S_1 \cap \widetilde{S}_1'$,
$P_1'' = 0 \Rightarrow s = (\rho, \sigma^1, \dots, \sigma^k, \tau) \in \widetilde{S}_1 \cap S_1 \cap \widetilde{S}_1''$,
$P_1'',\ P_1'''' \neq 0 \Rightarrow$
$s = (\rho, \sigma^1, \dots, \sigma^k, \tau) \in \widetilde{S}_1 \cap S_1 \cap \{(\widetilde{S}_1' \cap S_1'') \cup (S_1' \cap \widetilde{S}_1'')\}$,

CASE (ii). $P_1'''' = 0 \Rightarrow s = (\rho, \sigma^1, \dots, \sigma^k) \in \widetilde{S}_2 \cap S_2 \cap \widetilde{S}_2'$,
$P_1'' = 0 \Rightarrow s = (\rho, \sigma^1, \dots, \sigma^k) \in \widetilde{S}_2 \cap S_2 \cap \widetilde{S}_2''$,
$P_1'', P_1'''' \neq 0 \Rightarrow$
$s = (\rho, \sigma^1, \dots, \sigma^k) \in \widetilde{S}_2 \cap S_2 \cap \{(\widetilde{S}_2' \cap S_2'') \cup (S_2' \cap \widetilde{S}_2'')\}$,

CASE (iii). $P_1'''' = 0 \Rightarrow s = (\rho, \tau) \in \widetilde{S}_3 \cap S_3 \cap \widetilde{S}_3'$,
$P_1'' = 0 \Rightarrow s = (\rho, \tau) \in \widetilde{S}_3 \cap S_3 \cap \widetilde{S}_3''$,
$P_1'', P_1'''' \neq 0 \Rightarrow s = (\rho, \tau) \in \widetilde{S}_3 \cap S_3 \cap \{(\widetilde{S}_3' \cap S_3'') \cup (S_3' \cap \widetilde{S}_3'')\}$,

CASE (iv). $s = 1^{(d)}$,

CASE (v). $s = (\sigma^1, \dots, \sigma^k, \tau) \in S_5$,

CASE (vi). $s = (\sigma^1, \dots, \sigma^k) \in S_6$,

CASE (vii). $s = \tau \in S_7$.

On the concrete method of determining Gevrey orders see Hibino [2].

REMARK 2.1. In the case $m \geq 1$, the Gevrey orders given in Theorem 2.1 are more precise than those in Hibino [2]. In Case (i) (resp. Case (ii) and Case (iii)), when $P_1'',\ P_1'''' \neq 0$, Hibino [2] demands more strong condition $s = (\rho, \sigma^1, \dots, \sigma^k, \tau) \in \widetilde{S}_1 \cap S_1 \cap \widetilde{S}_1' \cap S_1''$ (resp. $s = (\rho, \sigma^1, \dots, \sigma^k) \in \widetilde{S}_2 \cap S_2 \cap \widetilde{S}_2' \cap S_2''$ and $s = (\rho, \tau) \in \widetilde{S}_3 \cap S_3 \cap \widetilde{S}_3' \cap S_3''$). For example, in Case (iii), let us consider the following linear partial differential operator:

$$P_1 = xD_x + 1 + x^2 D_z + z^2 D_x,$$

where $x,\ z \in \mathbf{C}$; $D_x = \partial/\partial x$, $D_z = \partial/\partial z$. Here $x^2 D_z$ and $z^2 D_x$ correspond to P_1'' and P_1'''', respectively. For this operator, we can easily prove that $(4/3, 5/3) \in \widetilde{S}_3 \cap S_3 \cap$

$\widetilde{S}_3' \cap S_3''$ and $(5/3, 4/3) \in \widetilde{S}_3 \cap S_3 \cap S_3' \cap \widetilde{S}_3''$. Therefore the formal solution $u(x,z)$ of the equation (2.4) belongs both to $G^{\{4/3,5/3\}}$ and to $G^{\{5/3,4/3\}}$. Hibino [2] proves only $u(x,z) \in G^{\{4/3,5/3\}}$.

REMARK 2.2. We can easily see that the following s_0 always satisfies the condition in Theorem 2.1 for each case:

CASE (i). $s_0 = (\rho_0, \sigma_0^1, \ldots, \sigma_0^k, \tau_0)$ (if $P_1'' \neq 0$),
$\quad = (1^{(m)}, \sigma_0^1, \ldots, \sigma_0^k, \tau_0)$ (if $P_1'' = 0$),
CASE (ii). $s_0 = (\rho_0, \sigma_0^1, \ldots, \sigma_0^k)$ (if $P_1'' \neq 0$), $= (1^{(m)}, \sigma_0^1, \ldots, \sigma_0^k)$ (if $P_1'' = 0$),
CASE (iii). $s_0 = (\rho_0, \tau_0)$ (if $P_1'' \neq 0$), $= (1^{(m)}, \tau_0)$ (if $P_1'' = 0$),
CASE (iv). $s_0 = 1^{(d)}$,
CASE (v). $s_0 = (\sigma_0^1, \ldots, \sigma_0^k, \tau_0)$,
CASE (vi). $s_0 = (\sigma_0^1, \ldots, \sigma_0^k)$,
CASE (vii). $s_0 = \tau_0$,

where $\rho_0 = (\overbrace{N + 1/2, \ldots, N + 1/2}^{m})$, $\sigma_0^h = (N+1, N+2, \ldots, N+n_h)$ $(h = 1, \ldots, k)$ and $\tau_0 = (\overbrace{N+1, \ldots, N+1}^{p})$.

Therefore by a linear transform of independent variables again we obtain Theorem 1.2 from Theorem 2.1 and the next Lemma 2.1. Thus the proof of Theorem 1.2 is reduced to that of Theorem 2.1.

Lemma 2.1 (Hibino [2]). *Let* $u(x) = \sum_{\alpha \in \mathbf{N}^d} u_\alpha x^\alpha \in G^{\{s,s,\ldots,s\}}$ $(s \geq 1)$. *Then for any linear transform* $L\colon \mathbf{C}^d \to \mathbf{C}^d$, *it holds that* $v(y) := u(Ly) \in G^{\{s,s,\ldots,s\}}$.

In the cases (i), (ii), (iii) and (iv) (that is, the case $m \geq 1$), Therorem 2.1 can be proved by a same method. On the other hand, in the cases (v), (vi) and (vii) (that is, the case $m = 0$), the theorem can be proved by a same method defferent from the one used in the cases (i)–(iv). Therefore we shall prove only the cases (i) and (v) in the following.

3. Banach spaces $G^{\{s\}}(R)$ and $\widetilde{G}^{\{s^1,s^2\}}(R^1, R^2)$

Theorem 2.1 is proved by a contraction mapping principle in Banach spaces which consist of formal power series. For this purpose we shall define two types of Banach spaces necessary in the proof, and we shall prove some lemmas needed later. These Banach spaces are originally introduced in Hibino [2] and some of lemmas in this section have been already proved there.

DEFINITION 3.1. (1) Let $s = (s_1, \ldots, s_d) \in {\mathbf{R}_+}^d$ $(\mathbf{R}_+ = \{r \in \mathbf{R};\ r \geq 0\})$, $(s^1, s^2) = (s_1^1, \ldots, s_{d_1}^1, s_1^2, \ldots, s_{d_2}^2) \in {\mathbf{R}_+}^{d_1+d_2}$, $R = (R_1, \ldots, R_d) \in (\mathbf{R}_+ \setminus \{0\})^d$ and $(R^1, R^2) = (R_1^1, \ldots, R_{d_1}^1, R_1^2, \ldots, R_{d_2}^2) \in (\mathbf{R}_+ \setminus \{0\})^{d_1+d_2}$. The spaces of formal power series $G^{\{s\}}(R)$ and $\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$ are defined as follows:

We say that $u(x)=\sum_{\alpha\in\mathbf{N}^d}u_\alpha x^\alpha$ *belongs to* $G^{\{s\}}(R)$ if

$$\|u\|_R^{\{s\}}:=\sum_{\alpha\in\mathbf{N}^d}|u_\alpha|\frac{|\alpha|!}{(s\cdot\alpha)!}R^\alpha<+\infty$$

($|\alpha|=\alpha_1+\cdots+\alpha_d$, $s\cdot\alpha=\sum_{i=1}^d s_i\alpha_i$).

We say that $u(x,y)=\sum_{(\alpha,\beta)\in\mathbf{N}^{d_1+d_2}}u_{\alpha\beta}x^\alpha y^\beta\in\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$ if

$$|||u|||_{R^1,R^2}^{\{s^1,s^2\}}:=\sum_{(\alpha,\beta)\in\mathbf{N}^{d_1+d_2}}|u_{\alpha\beta}|\frac{|\alpha|!|\beta|!}{(s^1\cdot\alpha+s^2\cdot\beta)!}(R^1)^\alpha(R^2)^\beta<+\infty$$

($|\alpha|=\alpha_1+\cdots+\alpha_{d_1}$, $|\beta|=\beta_1+\cdots+\beta_{d_2}$, $s^1\cdot\alpha=\sum_{i=1}^{d_1}s_i^1\alpha_i$, $s^2\cdot\beta=\sum_{j=1}^{d_2}s_j^2\beta_j$), where $k!=\Gamma(k+1)$, $k\geq 0$. Then $G^{\{s\}}(R)$ and $\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$ are Banach spaces equipped with the norms $\|\cdot\|_R^{\{s\}}$ and $|||\cdot|||_{R^1,R^2}^{\{s^1,s^2\}}$, respectively.

(2) We define the subspace $G_0^{\{s\}}(R)$ (resp. $\widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$) of the Banach space $G^{\{s\}}(R)$ (resp. $\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$) by

$$G_0^{\{s\}}(R):=\left\{u(x)=\sum_{\alpha\in\mathbf{N}^d}u_\alpha x^\alpha\in G^{\{s\}}(R);\ u_0(=u(0))=0\right\}$$

$$\Bigg(\text{resp.}\quad \widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$$

$$:=\left\{u(x,y)=\sum_{(\alpha,\beta)\in\mathbf{R}^{d_1+d_2}}u_{\alpha\beta}x^\alpha y^\beta\in\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2);\ u_{00}(=u(0,0))=0\right\}\Bigg).$$

Then $G_0^{\{s\}}(R)$ (resp. $\widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$) is also a Banach space as a closed linear subspace of $G^{\{s\}}(R)$ (resp. $\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$).

Lemma 3.1 (Hibino [2]). (1) *If $s_i\geq 1$ for all $i=1,\dots,d$, then*

$$G^{\{s\}}=\bigcup_{R\in(\mathbf{R}_+\setminus\{0\})^d}G^{\{s\}}(R).$$

(2) *If $s_i^1\geq 1$ and $s_j^2\geq 1$ for all $i=1,\dots,d_1$ and $j=1,\dots,d_2$, respectively, then*

$$G^{\{s^1,s^2\}}=\bigcup_{(R^1,R^2)\in(\mathbf{R}_+\setminus\{0\})^{d_1+d_2}}\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2).$$

Lemma 3.2 (Hibino [2]). *Let us fix $T=(T_1,\dots,T_d)\in(\mathbf{R}_+\setminus\{0\})^d$ and $(T^1,T^2)=(T_1^1,\dots,T_{d_1}^1,T_1^2,\dots,T_{d_2}^2)\in(\mathbf{R}_+\setminus\{0\})^{d_1+d_2}$, and let us assume that $a(x)=$*

$\sum_{\alpha\in\mathbf{N}^d} a_\alpha x^\alpha$ *and* $a(x,y) = \sum_{(\alpha,\beta)\in\mathbf{N}^{d_1+d_2}} a_{\alpha\beta}x^\alpha y^\beta$ *are holomorphic on* $\prod_{i=1}^d\{x_i \in \mathbf{C};\ |x_i| \le T_i\}$ *and* $\prod_{i=1}^{d_1}\{x_i \in \mathbf{C};\ |x_i| \le T_i^1\} \times \prod_{j=1}^{d_2}\{y_j \in \mathbf{C};\ |y_j| \le T_j^2\}$, *respectively.*

(1) *If* $0 < R_i \le T_i$ *for all* $i = 1, \dots, d$, *then the multiplication operator* $a(x)\cdot$ *is bounded on both* $G^{\{s\}}(R)$ *and* $G_0^{\{s\}}(R)$ *for all* $s \in [1, +\infty)^d$ *with the norm bounded by* $|a|(R)$, *where* $|a|(R) := \sum_{\alpha\in\mathbf{N}^d} |a_\alpha|R^\alpha$. *Especially the operator norm is bounded by* $|a|(T)$.

(2) *If* $0 < R_i^1 \le T_i^1$ *and* $0 < R_j^2 \le T_j^2$ *for all* $i = 1, \dots, d_1$ *and* $j = 1, \dots, d_2$, *respectively, then the multiplication operator* $a(x,y)\cdot$ *is bounded on both* $\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$ *and* $\widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$ *for all* $(s^1,s^2) \in [1,+\infty)^{d_1+d_2}$ *with the norm bounded by* $|a|(R^1,R^2)$, *where* $|a|(R^1,R^2) := \sum_{(\alpha,\beta)\in\mathbf{N}^{d_1+d_2}} |a_{\alpha\beta}|(R^1)^\alpha(R^2)^\beta$. *Especially the operator norm is bounded by* $|a|(T^1,T^2)$.

The following lemma will play a very important role when we deal with nonlinear terms.

Lemma 3.3. (1) *Let* $s \in [1,+\infty)^d$ *and assume that* $u(x)$ *and* $v(x)$ *belong to* $G^{\{s\}}(R)$ (*resp.* $G_0^{\{s\}}(R)$). *Then* $u(x)\cdot v(x)$ *also belongs to* $G^{\{s\}}(R)$ (*resp.* $G_0^{\{s\}}(R)$). *Furthermore for all* u *and* v *it holds that*

$$\|u\cdot v\|_R^{\{s\}} \le \mathsf{S}\|u\|_R^{\{s\}}\cdot\|v\|_R^{\{s\}}, \tag{3.1}$$

where $\mathsf{S} = \max\{s_i;\ i = 1, \dots, d\}$.

(2) *Let* $(s^1,s^2) \in [1,+\infty)^{d_1+d_2}$ *and let us assume that* $u(x,y)$ *and* $v(x,y)$ *belong to* $\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$ (*resp.* $\widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$). *Then it also holds that* $u(x,y)\cdot v(x,y) \in \widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$ (*resp.* $\in \widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$). *Furthermore for all* u *and* v *it holds that*

$$|||u\cdot v|||_{R^1,R^2}^{\{s^1,s^2\}} \le \widetilde{\mathsf{S}}|||u|||_{R^1,R^2}^{\{s^1,s^2\}}\cdot|||v|||_{R^1,R^2}^{\{s^1,s^2\}}, \tag{3.2}$$

where $\widetilde{\mathsf{S}} = \max\{s_i^1, s_j^2;\ i = 1, \dots, d_1;\ j = 1, \dots, d_2\}$.

Proof. First of all, we remark that in general the Beta function

$$B(k,l) = \int_0^1 t^{k-1}(1-t)^{l-1}\,dt$$

has the following property:

$$0 < k_1 < k_2,\ 0 < l_1 < l_2 \ \Rightarrow\ B(k_1,l_1) > B(k_2,l_2).$$

Moreover we remark that the following equality holds: For $k,\ l > 0$,

$$\frac{k!l!}{(k+l)!} = B(k+1,l+1)\cdot(k+l+1).$$

(1): Let $u(x)=\sum_{\alpha\in\mathbf{N}^d}u_\alpha x^\alpha$, $v(x)=\sum_{\beta\in\mathbf{N}^d}v_\beta x^\beta\in G^{\{s\}}(R)$. Then we have

$$\|u\cdot v\|_R^{\{s\}}=\sum_{\alpha,\beta\in\mathbf{N}^d}|u_\alpha v_\beta|\frac{|\alpha+\beta|!}{(s\cdot(\alpha+\beta))!}R^{\alpha+\beta}.$$

Here it follows from the above remarks that

$$\begin{aligned}\frac{(s\cdot\alpha)!(s\cdot\beta)!}{(s\cdot(\alpha+\beta))!}&=B(s\cdot\alpha+1,s\cdot\beta+1)\cdot(s\cdot\alpha+s\cdot\beta+1)\\&\le B(|\alpha|+1,|\beta|+1)\cdot(s\cdot\alpha+s\cdot\beta+1)\\&=\frac{|\alpha|!|\beta|!}{|\alpha+\beta|!}\cdot\frac{s\cdot\alpha+s\cdot\beta+1}{|\alpha|+|\beta|+1}\\&\le\mathsf{S}\cdot\frac{|\alpha|!|\beta|!}{|\alpha+\beta|!},\end{aligned}$$

which implies that

$$\frac{|\alpha+\beta|!}{(s\cdot(\alpha+\beta))!}\le\mathsf{S}\cdot\frac{|\alpha|!}{(s\cdot\alpha)!}\cdot\frac{|\beta|!}{(s\cdot\beta)!}.$$

Therefore we have obtained (3.1). It is clear that $u(x)\cdot v(x)\in G_0^{\{s\}}(R)$ for $u(x)$, $v(x)\in G_0^{\{s\}}(R)$.

(2): Let $u(x,y)=\sum_{(\alpha,\beta)\in\mathbf{N}^{d_1+d_2}}u_{\alpha\beta}x^\alpha y^\beta$ and $v(x,y)=\sum_{(\gamma,\delta)\in\mathbf{N}^{d_1+d_2}}v_{\gamma\delta}x^\gamma y^\delta$ be in $\widetilde{G}^{\{s^1,s^2\}}(R^1,R^2)$. Then we have

$$|||u\cdot v|||_{R^1,R^2}^{\{s^1,s^2\}}=\sum_{(\alpha,\beta),(\gamma,\delta)\in\mathbf{N}^{d_1+d_2}}|u_{\alpha\beta}v_{\gamma\delta}|\frac{|\alpha+\gamma|!|\beta+\delta|!}{(s^1\cdot(\alpha+\gamma)+s^2\cdot(\beta+\delta))!}(R^1)^{\alpha+\gamma}(R^2)^{\beta+\delta}.$$

Here it holds that

$$\begin{aligned}&\frac{(s^1\cdot\alpha+s^2\cdot\beta)!(s^1\cdot\gamma+s^2\cdot\delta)!}{(s^1\cdot(\alpha+\gamma)+s^2\cdot(\beta+\delta))!}\\&=B(s^1\cdot\alpha+s^2\cdot\beta+1,s^1\cdot\gamma+s^2\cdot\delta+1)\cdot\left(s^1\cdot(\alpha+\gamma)+s^2\cdot(\beta+\delta)+1\right)\\&\le B(|\alpha|+|\beta|+1,|\gamma|+|\delta|+1)\cdot\left(s^1\cdot(\alpha+\gamma)+s^2\cdot(\beta+\delta)+1\right)\\&=\frac{(|\alpha|+|\beta|)!(|\gamma|+|\delta|)!}{(|\alpha+\gamma|+|\beta+\delta|)!}\cdot\frac{s^1\cdot(\alpha+\gamma)+s^2\cdot(\beta+\delta)+1}{|\alpha+\gamma|+|\beta+\delta|+1}\\&\le\widetilde{\mathsf{S}}\cdot\frac{(|\alpha|+|\beta|)!(|\gamma|+|\delta|)!}{(|\alpha+\gamma|+|\beta+\delta|)!}.\end{aligned}$$

Moreover if we admit

$$\frac{(|\alpha|+|\beta|)!(|\gamma|+|\delta|)!}{(|\alpha+\gamma|+|\beta+\delta|)!}\le\frac{|\alpha|!|\gamma|!}{|\alpha+\gamma|!}\cdot\frac{|\beta|!|\delta|!}{|\beta+\delta|!},\tag{3.3}$$

then we obtain that

$$\frac{|\alpha+\gamma|!|\beta+\delta|!}{(s^1\cdot(\alpha+\gamma)+s^2\cdot(\beta+\delta))!} \le \widetilde{\mathsf{S}} \cdot \frac{|\alpha|!|\beta|!}{(s^1\cdot\alpha+s^2\cdot\beta)!}\cdot\frac{|\gamma|!|\delta|!}{(s^1\cdot\gamma+s^2\cdot\delta)!}.$$

Therefore we have obtained (3.2).

Let us prove (3.3). By putting $a := |\alpha|$, $b := |\beta|$, $c := |\gamma|$ and $d := |\delta|$, it is sufficient to prove the following inequality: For $a, b, c, d \ge 0$,

$$\frac{(a+b)!(c+d)!}{(a+b+c+d)!} \le \frac{a!c!}{(a+c)!}\cdot\frac{b!d!}{(b+d)!}. \tag{3.4}$$

Let us consider the equality

$$(\xi+\eta)^{a+b}\cdot(\xi+\eta)^{c+d} = (\xi+\eta)^{a+b+c+d},$$

and let us calculate the coefficients of $\xi^{a+c}\eta^{b+d}$ in both sides. Then we have

$$\sum_{\substack{1\le i\le a+b, 1\le j\le c+d\\ i+j=a+c}} \binom{a+b}{i}\cdot\binom{c+d}{j} = \binom{a+b+c+d}{a+c},$$

which implies that

$$\frac{(a+b)!}{a!b!}\cdot\frac{(c+d)!}{c!d!} = \binom{a+b}{a}\cdot\binom{c+d}{c} \le \binom{a+b+c+d}{a+c} = \frac{(a+b+c+d)!}{(a+c)!(b+d)!}.$$

Therefore (3.4) is proved and (3.2) is completely proved. It is clear that $u(x,y)\cdot v(x,y) \in \widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$ for $u(x,y),\ v(x,y) \in \widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$. □

4. Proof of Theorem 2.1 (when $m = 0$)

Let us start the proof of Theorem 2.1. We shall prove the unique existence of the formal solution in §6. So in this section and the next section, admitting the unique existence of the formal solution, we will prove its Gevrey order. In this section we study the case $m = 0$ (i.e., Cases (v), (vi) and (vii)). As mentioned in §2 we only consider Case (v), that is, we only consider the equation (2.7). Furthermore, for simplicity we assume $k = 1$. We write a formal power series solution as $u(y,z) = \sum_{(\beta,\gamma)\in\mathbf{N}^{n+p}, |\beta|+|\gamma|\ge 1} u_{\beta\gamma} y^\beta z^\gamma$ $(n+p=d)$ and use the Banach space $G_0^{\{\sigma,\tau\}}(Y,Z)$ instead of $G_0^{\{s\}}(R)$. Therefore $u(y,z)\in G_0^{\{\sigma,\tau\}}(Y,Z)$ means

$$\|u\|_{Y,Z}^{\{\sigma,\tau\}} := \sum_{\substack{(\beta,\gamma)\in\mathbf{N}^{n+p}\\ |\beta|+|\gamma|\ge 1}} |u_{\beta\gamma}| \frac{(|\beta|+|\gamma|)!}{(\sigma\cdot\beta+\tau\cdot\gamma)!} Y^\beta Z^\gamma < +\infty.$$

We recall that the equation (2.7) is written as follows:

$$P_1 u = g_0(y,z) + g\big(y,z,u(y,z)\big), \quad u(0,0) = 0, \tag{4.1}$$

where g_0 and g are holomorphic at the origin which satisfy $g_0(0,0) = 0$ and $g(y,z,0) \equiv g_u(y,z,0) \equiv 0$, respectively. Furthermore P_1 is a linear partial differential operator which has the following form: $P_1 = -f_u(0,0) + P_1''' + h$, where

$$\begin{aligned}
P_1''' &= \sum_{j=1}^{n-1} y_{j+1}\frac{\partial}{\partial y_j} + \sum_{j=1}^{n}\left(\sum_{|\beta|+|\gamma|\geq 2}^{\mathit{finite}} d_{j\beta\gamma}(y,z)y^\beta z^\gamma\right)\frac{\partial}{\partial y_j}\\
&\quad + \sum_{q=1}^{p}\left(\sum_{|\beta|+|\gamma|\geq 2}^{\mathit{finite}} e_{q\beta\gamma}(y,z)y^\beta z^\gamma\right)\frac{\partial}{\partial z_q},\\
h &= h(y,z)\\
&= \sum_{|\beta|+|\gamma|\geq 1}^{\mathit{finite}} h_{\beta\gamma}(y,z)y^\beta z^\gamma.
\end{aligned}$$

Here all coefficients $d_{j\beta\gamma}$, $e_{q\beta\gamma}$ and $h_{\beta\gamma}$ are holomorphic at the origin, and none of them vanish at the origin unless they vanish identically.

We assume that $s = (\sigma,\tau)$ satisfies the condition in Theorem 2.1, and prove that the formal solution of (4.1) belongs to $G^{\{\sigma,\tau\}}$.

Proof of Case (v) of Theorem 2.1. We may assume that $-f_u(0,0) = 1$ since $f_u(0,0) \neq 0$. Let us define the operator T by

$$Tu = -(P_1''' + h)u + g_0(y,z) + g\big(y,z,u(y,z)\big), \tag{4.2}$$

and let us write the ε-closed ball in $G_0^{\{\sigma,\tau\}}(Y,Z)$ as $G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon)$:

$$G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon) := \left\{u(y,z) = \sum_{\substack{(\beta,\gamma)\in\mathbf{N}^{n+p}\\ |\beta|+|\gamma|\geq 1}} u_{\beta\gamma}y^\beta z^\gamma \in G_0^{\{\sigma,\tau\}}(Y,Z);\ \|u\|_{Y,Z}^{\{\sigma,\tau\}} \leq \varepsilon\right\}.$$

We shall prove that T is well-defined as a mapping from $G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon)$ to itself by choosing Y, Z and ε suitably and that it becomes a contraction mapping there (note that $G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon)$ is a complete metric space as a closed subset of the Banach space $G_0^{\{\sigma,\tau\}}(Y,Z)$).

First we estimate the operator norms of $h\cdot$ and P_1''' on the space $G_0^{\{\sigma,\tau\}}(Y,Z)$.

It follows from Lemma 3.2, (1) that $h\cdot\colon G_0^{\{\sigma,\tau\}}(Y,Z) \to G_0^{\{\sigma,\tau\}}(Y,Z)$ is bounded for sufficiently small Y and Z with the estimate

$$\|h\cdot u\|_{Y,Z}^{\{\sigma,\tau\}} \leq A_1(Y,Z)\|u\|_{Y,Z}^{\{\sigma,\tau\}}, \tag{4.3}$$

where

$$A_1(Y,Z) = C_1\left\{\sum_{|\beta|+|\gamma|\geq 1}^{\mathit{finite}} Y^\beta Z^\gamma\right\}$$

for some constant C_1. Here and hereafter $Y = (Y_1, \dots, Y_n)$ and $Z = (Z_1, \dots, Z_p)$ are taken so small such that the coefficients of the operators $\partial/\partial y_j$, etc., are holomorphic on $\prod_{j=1}^{n}\{y_j \in \mathbf{C};\ |y_j| \le Y_j\} \times \prod_{q=1}^{p}\{z_q \in \mathbf{C};\ |z_q| \le Z_q\}$. In order to estimate the operator norm of P_1''' we need the following:

Lemma 4.1. *Let σ, τ, μ, ν, μ' and ν' satisfy*

(4.4)
$$\sigma_j, \tau_q \ge 1\ \ (j = 1, \dots, n;\ q = 1, \dots, p) \quad \textit{and} \quad \sigma \cdot (\mu - \mu') + \tau \cdot (\nu - \nu') \ge |\mu| + |\nu|.$$

Then $y^\mu z^\nu D_y^{\mu'} D_z^{\nu'}$ is a bounded operator on $G^{\{\sigma,\tau\}}(Y, Z)$ and the operator norm is bounded by $(Y^\mu Z^\nu)/(Y^{\mu'} Z^{\nu'})$. Furthermore if $|\mu| + |\nu| \ge 1$, the operator $y^\mu z^\nu D_y^{\mu'} D_z^{\nu'}$ is bounded on $G_0^{\{\sigma,\tau\}}(Y, Z)$ and the operator norm has the same estimate.

REMARK 4.1. Let us write the Newton polyhedron of the operator $y^\mu z^\nu D_y^{\mu'} D_z^{\nu'}$ as

$$N\left(y^\mu z^\nu D_y^{\mu'} D_z^{\nu'}\right) = \left\{ (\mathcal{Y}, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ \begin{array}{ll} \mathcal{Y}_j \ge \mu_j - \mu_j' & (j = 1, \dots, n), \\ \mathcal{Z}_q \ge \nu_q - \nu_q' & (q = 1, \dots, p), \\ \mathcal{W} \le |\mu'| + |\nu'| & \end{array} \right\}.$$

Furthermore we define $\Pi(\sigma, \tau)$ $((\sigma, \tau) \in [1, +\infty)^d)$ by

$$\Pi(\sigma, \tau) = \{(\mathcal{Y}, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\sigma - 1^{(n)}) \cdot \mathcal{Y} + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \ge 0\},$$

and define S by

$$S = \left\{ (\sigma, \tau) \in [1, +\infty)^d;\ N\left(y^\mu z^\nu D_y^{\mu'} D_z^{\nu'}\right) \subset \Pi(\sigma, \tau) \right\}.$$

Then the condition $(\sigma, \tau) \in S$ is equivalent to (4.4).

Proof of Lemma 4.1. It is similar to the proof of Lemma 4.1 in Hibino [2]. □

Proof of Case (v) of Theorem 2.1 (continued). By the assumption $(\sigma, \tau) \in S_5$, Lemma 3.2, (1) and Lemma 4.1, it holds that $P_1''': G_0^{\{\sigma,\tau\}}(Y, Z) \to G_0^{\{\sigma,\tau\}}(Y, Z)$ is bounded for sufficiently small Y and Z and that

$$\|P_1''' u\|_{Y,Z}^{\{\sigma,\tau\}} \le A_2(Y, Z) \|u\|_{Y,Z}^{\{\sigma,\tau\}}, \tag{4.5}$$

where

$$A_2(Y, Z) = C_2 \left\{ \sum_{j=1}^{n-1} \frac{Y_{j+1}}{Y_j} + \sum_{j=1}^{n} \left(\sum_{|\beta|+|\gamma| \ge 2}^{\textit{finite}} Y^\beta Z^\gamma \right) \frac{1}{Y_j} + \sum_{q=1}^{p} \left(\sum_{|\beta|+|\gamma| \ge 2}^{\textit{finite}} Y^\beta Z^\gamma \right) \frac{1}{Z_q} \right\}$$

for some constant C_2.

Next, in order to estimate nonlinear terms, we introduce some notations. Let

$$g(y, z, u) = \sum_{|\beta|+|\gamma|\geq 0, r\geq 2} g_{\beta\gamma r} y^\beta z^\gamma u^r$$

be the Taylor expansion of $g(y, z, u)$ (recall that $g(y, z, 0) \equiv g_u(y, z, 0) \equiv 0$). Furthermore let us define the formal power series $|g|(y, z, u)$ by

$$|g|(y, z, u) = \sum_{|\beta|+|\gamma|\geq 0, r\geq 2} |g_{\beta\gamma r}| y^\beta z^\gamma u^r.$$

We may assume that $|g|(y, z, u)$ converges in $\prod_{j=1}^n \{y_j \in \mathbf{C};\ |y_j| \leq L_j\} \times \prod_{q=1}^p \{z_q \in \mathbf{C};\ |z_q| \leq M_q\} \times \{u \in \mathbf{C};\ |u| \leq N\}$ for some positive constants L_j, M_q and N ($j = 1, \dots, n$; $q = 1, \dots, p$).

We remark the following: It holds that

$$g_u(y, z, u) = \sum_{|\beta|+|\gamma|\geq 0, r\geq 1} (r+1) g_{\beta\gamma, r+1} y^\beta z^\gamma u^r,$$

and that

$$|g_u|(y, z, u) := \sum_{|\beta|+|\gamma|\geq 0, r\geq 1} (r+1) |g_{\beta\gamma, r+1}| y^\beta z^\gamma u^r$$

converges in $\prod_{j=1}^n \{y_j \in \mathbf{C};\ |y_j| \leq L_j\} \times \prod_{q=1}^p \{z_q \in \mathbf{C};\ |z_q| \leq M_q\} \times \{u \in \mathbf{C};\ |u| \leq N\}$.

Now it follows from Lemma 3.3, (1) that if $Y_j \leq L_j$ ($j = 1, \dots, n$), $Z_q \leq M_q$ ($q = 1, \dots, p$), $u \in G_0^{\{\sigma,\tau\}}(Y, Z)$ and $\|u\|_{Y,Z}^{\{\sigma,\tau\}} \leq N/\mathsf{S}$, where $\mathsf{S} = \max\{\sigma_j, \tau_q;\ j = 1, \dots, n \text{ and } q = 1, \dots, p\}$, then $g(y, z, u(y, z))$ belongs to $G_0^{\{\sigma,\tau\}}(Y, Z)$. Moreover it holds that

$$\begin{aligned} \|g(y, z, u(y, z))\|_{Y,Z}^{\{\sigma,\tau\}} &\leq \frac{1}{\mathsf{S}} |g| \left(Y, Z, \mathsf{S} \|u\|_{Y,Z}^{\{\sigma,\tau\}} \right) \\ &\leq \frac{1}{\mathsf{S}} |g| \left(L, M, \mathsf{S} \|u\|_{Y,Z}^{\{\sigma,\tau\}} \right) < +\infty, \end{aligned} \tag{4.6}$$

where $L = (L_1, \dots, L_n)$ and $M = (M_1, \dots, M_p)$.

Next by noting

$$g(y, z, u) - g(y, z, v) = (u - v) \int_0^1 g_u\big(y, z, v + \theta(u - v)\big)\, d\theta,$$

we see that if $Y_j \leq L_j$ ($j = 1, \dots, n$), $Z_q \leq M_q$ ($q = 1, \dots, p$) and $\|u\|_{Y,Z}^{\{\sigma,\tau\}}$, $\|v\|_{Y,Z}^{\{\sigma,\tau\}} \leq N/2\mathsf{S}$, then we have

$$\|g\big(y, z, u(y, z)\big) - g\big(y, z, v(y, z)\big)\|_{Y,Z}^{\{\sigma,\tau\}} \tag{4.7}$$

$$\leq \|u-v\|_{Y,Z}^{\{\sigma,\tau\}} \times |g_u|\Big(Y, Z, \mathsf{S}\big(\|u\|_{Y,Z}^{\{\sigma,\tau\}} + \|v\|_{Y,Z}^{\{\sigma,\tau\}}\big)\Big)$$
$$\leq \|u-v\|_{Y,Z}^{\{\sigma,\tau\}} \times |g_u|\Big(L, M, \mathsf{S}\big(\|u\|_{Y,Z}^{\{\sigma,\tau\}} + \|v\|_{Y,Z}^{\{\sigma,\tau\}}\big)\Big).$$

Under the above preparations let us take $\varepsilon > 0$, Y and Z as follows: We take $\varepsilon > 0$ such that

$$\frac{1}{\mathsf{S}}|g|(L, M, \mathsf{S}\varepsilon) < \varepsilon \tag{4.8}$$

and

$$|g_u|(L, M, 2\mathsf{S}\varepsilon) < 1. \tag{4.9}$$

Since $|g|(y,z,u) = O(u^2)$ and $|g_u|(y,z,u) = O(u)$, we can take such $\varepsilon > 0$. Furthermore for this ε we take Y and Z such that

$$A(Y,Z)\varepsilon + \|g_0\|_{Y,Z}^{\{\sigma,\tau\}} + \frac{1}{\mathsf{S}}|g|(L, M, \mathsf{S}\varepsilon) \leq \varepsilon \tag{4.10}$$

and

$$A(Y,Z) + |g_u|(L, M, 2\mathsf{S}\varepsilon) < 1, \tag{4.11}$$

where

$$A(Y,Z) = A_1(Y,Z) + A_2(Y,Z).$$

We can take such Y and Z by the fact $g_0(0,0) = 0$ and the expression of $A(Y,Z)$.

It follows from (4.3), (4.5), (4.6) and (4.10) that $u \in G_0^{\{\sigma,\tau\}}(Y,Z)$ and $\|u\|_{Y,Z}^{\{\sigma,\tau\}} \leq \varepsilon$ imply $Tu \in G_0^{\{\sigma,\tau\}}(Y,Z)$ and $\|Tu\|_{Y,Z}^{\{\sigma,\tau\}} \leq \varepsilon$. Hence T is well-defined as a mapping from $G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon)$ to itself. Moreover by (4.3), (4.5), (4.7) and (4.11), we see that $T\colon G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon) \to G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon)$ is a contraction mapping. Therefore there exists a unique $u(y,z) \in G_0^{\{\sigma,\tau\}}(Y,Z;\varepsilon)$ which satisfies $Tu(y,z) = u(y,z)$. Lemma 3.1, (1) implies $u(y,z) \in G^{\{\sigma,\tau\}}$, and it is easy to see that this $u(y,z)$ is a solution of (4.1). Since we admit the unique existence of the formal solution, the proof is completed. □

5. Proof of Theorem 2.1 (when $m \geq 1$)

In this section we study the case $m \geq 1$ (i.e. Cases (i), (ii), (iii) and (iv)). We only consider Case (i). By the same reason as in the previous section we consider the case $k = 1$. We write a formal power series solution as $u(x,y,z) = \sum_{(\alpha,\beta,\gamma)\in\mathbf{N}^{m+n+p}, |\alpha|+|\beta|+|\gamma|\geq 1} u_{\alpha\beta\gamma} x^\alpha y^\beta z^\gamma$ $(m+n+p = d)$ and use the Banach space $\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ (resp. $G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$) instead of $\widetilde{G}_0^{\{s^1,s^2\}}(R^1,R^2)$ (resp. $G_0^{\{s^1,s^2\}}(R^1,R^2)$). Therefore $u(x,y,z)\in\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ (resp. $\in G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$) means

$$|||u|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} := \sum_{\substack{(\alpha,\beta,\gamma)\in\mathbf{N}^{m+n+p}\\ |\alpha|+|\beta|+|\gamma|\geq 1}} |u_{\alpha\beta\gamma}| \frac{|\alpha|!(|\beta|+|\gamma|)!}{(\rho\cdot\alpha+\sigma\cdot\beta+\tau\cdot\gamma)!} X^\alpha Y^\beta Z^\gamma < +\infty$$

$$\left(\text{resp.}\quad \|u\|^{\{\rho,\sigma,\tau\}}_{X,Y,Z} := \sum_{\substack{(\alpha,\beta,\gamma)\in\mathbf{N}^{m+n+p}\\ |\alpha|+|\beta|+|\gamma|\geq 1}} |u_{\alpha\beta\gamma}| \frac{(|\alpha|+|\beta|+|\gamma|)!}{(\rho\cdot\alpha+\sigma\cdot\beta+\tau\cdot\gamma)!} X^\alpha Y^\beta Z^\gamma < +\infty\right).$$

We recall that the equation (2.1) is written as follows:

$$P_1 u = g_0(x,y,z) + g\big(x,y,z,u(x,y,z)\big), \quad u(0,0,0)=0, \tag{5.1}$$

where g_0 and g are holomorphic at the origin which satisfy $g_0(0,0,0)=0$ and $g(x,y,z,0)\equiv g_u(x,y,z,0)\equiv 0$, respectively. Furthermore P_1 is a linear partial differential operator which has the following form: $P_1=\sum_{i=1}^m \lambda_i x_i(\partial/\partial x_i) - f_u(0,0) + P_1' + P_1'' + P_1''' + P_1'''' + h$, where

$$\begin{aligned}
P_1' &= \sum_{i=1}^{m-1}\delta_i x_{i+1}\frac{\partial}{\partial x_i} + \sum_{i=1}^{m}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} c_{i\alpha\beta\gamma}(x,y,z)x^\alpha y^\beta z^\gamma\Bigg)\frac{\partial}{\partial x_i},\\
P_1'' &= \sum_{j=1}^{n}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} d_{j\alpha\beta\gamma}(x,y,z)x^\alpha y^\beta z^\gamma\Bigg)\frac{\partial}{\partial y_j}\\
&\quad + \sum_{q=1}^{p}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} e_{q\alpha\beta\gamma}(x,y,z)x^\alpha y^\beta z^\gamma\Bigg)\frac{\partial}{\partial z_q},\\
P_1''' &= \sum_{j=1}^{n-1} y_{j+1}\frac{\partial}{\partial y_j} + \sum_{j=1}^{n}\Bigg(\sum_{|\beta|+|\gamma|\geq 2}^{\text{finite}} d_{j\beta\gamma}(x,y,z)y^\beta z^\gamma\Bigg)\frac{\partial}{\partial y_j}\\
&\quad + \sum_{q=1}^{p}\Bigg(\sum_{|\beta|+|\gamma|\geq 2}^{\text{finite}} e_{q\beta\gamma}(x,y,z)y^\beta z^\gamma\Bigg)\frac{\partial}{\partial z_q},\\
P_1'''' &= \sum_{i=1}^{m}\Bigg(\sum_{|\beta|+|\gamma|\geq 2}^{\text{finite}} c_{i\beta\gamma}(x,y,z)y^\beta z^\gamma\Bigg)\frac{\partial}{\partial x_i},\\
h &= h(x,y,z)\\
&= \sum_{|\alpha|+|\beta|+|\gamma|\geq 1}^{\text{finite}} h_{\alpha\beta\gamma}(x,y,z)x^\alpha y^\beta z^\gamma.
\end{aligned}$$

Here all coefficients $c_{i\alpha\beta\gamma}$, $c_{i\beta\gamma}$, $d_{j\alpha\beta\gamma}$, $d_{j\beta\gamma}$, $e_{q\alpha\beta\gamma}$, $e_{q\beta\gamma}$ and $h_{\alpha\beta\gamma}$ are holomorphic at the origin, and none of them vanish at the origin unless they vanish identically.

We assume that $s = (\rho, \sigma, \tau)$ satisfies the condition in Theorem 2.1, and prove that the formal solution of (5.1) belongs to $G^{\{\rho,\sigma,\tau\}}$. We remark that we admit the unique existence of the formal solution.

Proof of Case (i) of Theorem 2.1. First we define the operator $\Lambda\colon G^{\{\rho,\sigma,\tau\}} \to G^{\{\rho,\sigma,\tau\}}$ by

$$\Lambda = \sum_{i=1}^{m} \lambda_i x_i \frac{\partial}{\partial x_i} - f_u(0,0).$$

The condition (Po2) implies that $\lambda\cdot\alpha - f_u(0,0) \neq 0$ for all $\alpha \in \mathbf{N}^m$, where $\lambda\cdot\alpha = \sum_{i=1}^m \lambda_i\alpha_i$. Hence the operator Λ is bijective and Λ^{-1} is given by

$$\Lambda^{-1}\Bigg(\sum_{(\alpha,\beta,\gamma)\in\mathbf{N}^{m+n+p}} U_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma\Bigg) = \sum_{(\alpha,\beta,\gamma)\in\mathbf{N}^{m+n+p}} \frac{U_{\alpha\beta\gamma}}{\lambda\cdot\alpha - f_u(0,0)} x^\alpha y^\beta z^\gamma.$$

Now we introduce a new unknown function $U(x,y,z)$ by

$$U(x,y,z) = \Lambda u(x,y,z), \quad \text{that is,} \quad u(x,y,z) = \Lambda^{-1}U(x,y,z).$$

Then the equation (5.1) is equivalent to the following one:

$$\text{(5.2)} \qquad P_2 U = g_0(x,y,z) + g\big(x,y,z,\Lambda^{-1}U(x,y,z)\big), \quad U(0,0,0) = 0,$$

where

$$P_2 = I + (P_1' + P_1'' + P_1''' + P_1'''' + h)\Lambda^{-1}$$
$$(I:\ \text{identity mapping}).$$

Let us define the operator T by

(5.3)
$$TU = -(P_1' + P_1'' + P_1''' + P_1'''' + h)\Lambda^{-1}U + g_0(x,y,z) + g(x,y,z,\Lambda^{-1}U(x,y,z)),$$

and let us write the ε-closed ball in $\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ and $G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$ as

$$\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z);\varepsilon)$$
$$:= \Bigg\{U(x,y,z) = \sum_{\substack{(\alpha,\beta,\gamma)\in\mathbf{N}^{m+n+p}\\ |\alpha|+|\beta|+|\gamma|\geq 1}} U_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma \in \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z));\ |||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \leq \varepsilon\Bigg\}$$

and

$$G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z;\varepsilon)$$
$$:= \left\{ U(x,y,z) = \sum_{\substack{(\alpha,\beta,\gamma)\in\mathbf{N}^{m+n+p}\\ |\alpha|+|\beta|+|\gamma|\geq 1}} U_{\alpha\beta\gamma}x^\alpha y^\beta z^\gamma \in G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z);\ \|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \leq \varepsilon \right\},$$

respectively.

We shall prove that T is well-defined as a mapping from G to itself by choosing X, Y, Z and ε suitably and that it becomes a contraction mapping there, where

$$G = \begin{cases} \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z);\varepsilon) & \left(\begin{array}{l}\text{when } P_1'''' = 0 \text{ or “} P_1'', P_1'''' \neq 0 \text{ and}\\ s = (\rho,\sigma,\tau) \in \widetilde{S}_1 \cap S_1 \cap \widetilde{S}_1' \cap S_1'' \text{ ”}\end{array}\right), \\ G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z;\varepsilon) & \left(\begin{array}{l}\text{when } P_1'' = 0 \text{ or “} P_1'', P_1'''' \neq 0 \text{ and}\\ s = (\rho,\sigma,\tau) \in \widetilde{S}_1 \cap S_1 \cap S_1' \cap \widetilde{S}_1'' \text{ ”}\end{array}\right). \end{cases}$$

Let us estimate the operator norms of $(P_1' + P_1'' + P_1''' + P_1'''' + h)\Lambda^{-1}$ on the spaces $\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ and $G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$.

By the condition (Po2) there is some constant C such that $|1/(\lambda\cdot\alpha - f_u(0,0))| \leq C$ for all $\alpha \in \mathbf{N}^m$. Hence the operator $\Lambda^{-1}\colon \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z)) \to \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ (resp. $G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z) \to G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$) is bounded and we have

$$\begin{aligned} |||\Lambda^{-1}U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \leq{}& C|||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \\ & \left(\text{resp.}\quad \|\Lambda^{-1}U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \leq C\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\right). \end{aligned} \tag{5.4}$$

Therefore it follows from Lemma 3.2 that the operator $h\cdot\Lambda^{-1}\colon \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z)) \to \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ (resp. $G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z) \to G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$) is bounded and we have

$$\begin{aligned} |||h\cdot\Lambda^{-1}U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \leq{}& A_1(X,Y,Z)|||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \\ & \left(\text{resp.}\quad \|h\cdot\Lambda^{-1}U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \leq A_1(X,Y,Z)\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\right), \end{aligned} \tag{5.5}$$

where

$$A_1(X,Y,Z) = C_1\left(\sum_{|\alpha|+|\beta|+|\gamma|\geq 1}^{\text{finite}} X^\alpha Y^\beta Z^\gamma\right)$$

for some constant C_1. In order to estimate the operator norm of $(P_1' + P_1'' + P_1''' + P_1'''')\Lambda^{-1}$ we need the following lemma:

Lemma 5.1. (1) *Let ρ, σ, τ, μ, ν, ω, μ', ν', ω' satisfy*

$$\begin{aligned}&\rho_i, \sigma_j, \tau_q \geq 1 \ (i = 1, \ldots, m;\ j = 1, \ldots, n;\ q = 1, \ldots, p) \quad \text{and}\\ &\rho \cdot (\omega - \omega') + \sigma \cdot (\mu - \mu') + \tau \cdot (\nu - \nu') \geq |\omega| + |\mu| + |\nu|.\end{aligned} \tag{5.6}$$

Then the operator $x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'} \Lambda^{-1}$ is bounded both on $\widetilde{G}^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ and on $G^{\{\rho,\sigma,\tau\}}(X,Y,Z)$, and the operator norm is bounded by $C(X^\omega Y^\mu Z^\nu)/(X^{\omega'} Y^{\mu'} Z^{\nu'})$, where C is the same constant as in (5.4). *Furthermore if $|\omega|+|\mu|+|\nu| \geq 1$, the operator $x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'} \Lambda^{-1}$ is bounded both on $\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ and on $G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$, and the operator norm has the same estimate.*
(2) *If $|\omega| \geq 1$,*

$$\begin{aligned}&\rho_i, \sigma_j, \tau_q \geq 1 \ (i = 1, \ldots, m;\ j = 1, \ldots, n;\ q = 1, \ldots, p) \quad \text{and}\\ &\rho \cdot (\omega - \omega') + \sigma \cdot (\mu - \mu') + \tau \cdot (\nu - \nu') \geq |\omega| + |\mu| + |\nu| - 1,\end{aligned} \tag{5.7}$$

then the operator $x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'} \Lambda^{-1}$ is bounded both on $\widetilde{G}^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ and on $\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$, and the operator norm is bounded by $C_{\omega\omega'}(X^\omega Y^\mu Z^\nu)/(X^{\omega'} Y^{\mu'} Z^{\nu'})$ for some constant $C_{\omega\omega'}$.
(3) *If $|\omega'| \geq 1$ and* (5.7) *hold, then the operator $x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'} \Lambda^{-1}$ is bounded on $G^{\{\rho,\sigma,\tau\}}(X,Y,Z)$, and the operator norm is bounded by $C_{\omega\mu\nu\omega'\mu'\nu'}(X^\omega Y^\mu Z^\nu)/(X^{\omega'} Y^{\mu'} Z^{\nu'})$ for some constant $C_{\omega\mu\nu\omega'\mu'\nu'}$. Furthermore if $|\omega| + |\mu| + |\nu| \geq 1$, then $x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'} \Lambda^{-1}$ is bounded on $G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$ and the operator norm has the same estimate.*

REMARK 5.1. Let us write the Newton polyhedron of the operator $x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'}$ as

$$N\big(x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'}\big) = \left\{ (\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ \begin{array}{ll} \mathcal{X}_i \geq \omega_i - \omega_i' & (i = 1, \ldots, m), \\ \mathcal{Y}_j \geq \mu_j - \mu_j' & (j = 1, \ldots, n), \\ \mathcal{Z}_q \geq \nu_q - \nu_q' & (q = 1, \ldots, p), \\ \mathcal{W} \leq |\omega'| + |\mu'| + |\nu'| & \end{array} \right\}.$$

Furthermore we define $\widetilde{\Pi}(\rho,\sigma,\tau)$ and $\Pi(\rho,\sigma,\tau)$ ($(\rho,\sigma,\tau) \in [1,+\infty)^d$) by

$$\widetilde{\Pi}(\rho,\sigma,\tau) = \{(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} + (\sigma - 1^{(n)}) \cdot \mathcal{Y} + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq -1\}$$

and

$$\Pi(\rho,\sigma,\tau) = \{(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}) \in \mathbf{R}^{d+1};\ (\rho - 1^{(m)}) \cdot \mathcal{X} + (\sigma - 1^{(n)}) \cdot \mathcal{Y} + (\tau - 1^{(p)}) \cdot \mathcal{Z} - \mathcal{W} \geq 0\},$$

respectively, and define $\widetilde{S}$ and S as follows:

$$\widetilde{S} = \{(\rho,\sigma,\tau) \in [1,+\infty)^d;\ N(x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'}) \subset \widetilde{\Pi}(\rho,\sigma,\tau)\},$$

$$S = \{(\rho,\sigma,\tau) \in [1,+\infty)^d;\ N(x^\omega y^\mu z^\nu D_x^{\omega'} D_y^{\mu'} D_z^{\nu'}) \subset \Pi(\rho,\sigma,\tau)\}.$$

Then the conditions $(\rho,\sigma,\tau) \in \widetilde{S}$ and $(\rho,\sigma,\tau) \in S$ are equivalent to (5.7) and (5.6), respectively.

Proof of Lemma 5.1. It is similar to the proof of Lemma 5.1 in Hibino [2]. We remark that the condition (Po2) plays an important role in the proof. $\square$

Proof of Case (i) of Theorem 2.1 (continued). When $P_1'''' = 0$, it follows from the assumption $(\rho,\sigma,\tau) \in \widetilde{S}_1 \cap S_1 \cap \widetilde{S}_1'$, Lemma 3.2, (2), Lemma 5.1, (1) and (2) that the operator $(P_1' + P_1'' + P_1''')\Lambda^{-1}\colon \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z)) \to \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ is bounded for sufficiently small X, Y and Z. Moreover we have

$$|||(P_1' + P_1'' + P_1''')\Lambda^{-1}U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \le A_2(X,Y,Z)|||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}}, \tag{5.8}$$

where

$$\begin{aligned} A_2(X,Y,Z) = C_2\Bigg\{ &\sum_{i=1}^{m-1}\frac{X_{i+1}}{X_i} + \sum_{i=1}^{m}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\ge 2\\ |\alpha|\ge 1}}^{\text{finite}} X^\alpha Y^\beta Z^\gamma\Bigg)\frac{1}{X_i} \\ &+ \sum_{j=1}^{n}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\ge 2\\ |\alpha|\ge 1}}^{\text{finite}} X^\alpha Y^\beta Z^\gamma\Bigg)\frac{1}{Y_j} + \sum_{q=1}^{p}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\ge 2\\ |\alpha|\ge 1}}^{\text{finite}} X^\alpha Y^\beta Z^\gamma\Bigg)\frac{1}{Z_q} \\ &+ \sum_{j=1}^{n-1}\frac{Y_{j+1}}{Y_j} + \sum_{j=1}^{n}\Bigg(\sum_{|\beta|+|\gamma|\ge 2}^{\text{finite}} Y^\beta Z^\gamma\Bigg)\frac{1}{Y_j} + \sum_{q=1}^{p}\Bigg(\sum_{|\beta|+|\gamma|\ge 2}^{\text{finite}} Y^\beta Z^\gamma\Bigg)\frac{1}{Z_q}\Bigg\} \end{aligned}$$

for some constant C_2.

When $P_1'' = 0$, it follows from the assumption $(\rho,\sigma,\tau) \in \widetilde{S}_1 \cap S_1 \cap \widetilde{S}_1''$, Lemma 3.2, (1), Lemma 5.1, (1) and (3) that the operator $(P_1' + P_1''' + P_1'''')\Lambda^{-1}\colon G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z) \to G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$ is bounded. Moreover we have

$$\|(P_1' + P_1''' + P_1'''')\Lambda^{-1}U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \le A_3(X,Y,Z)\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}, \tag{5.9}$$

where

$$\begin{aligned} A_3(X,Y,Z) = C_3\Bigg\{ &\sum_{i=1}^{m-1}\frac{X_{i+1}}{X_i} + \sum_{i=1}^{m}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\ge 2\\ |\alpha|\ge 1}}^{\text{finite}} X^\alpha Y^\beta Z^\gamma\Bigg)\frac{1}{X_i} \\ &+ \sum_{j=1}^{n-1}\frac{Y_{j+1}}{Y_j} + \sum_{j=1}^{n}\Bigg(\sum_{|\beta|+|\gamma|\ge 2}^{\text{finite}} Y^\beta Z^\gamma\Bigg)\frac{1}{Y_j} + \sum_{q=1}^{p}\Bigg(\sum_{|\beta|+|\gamma|\ge 2}^{\text{finite}} Y^\beta Z^\gamma\Bigg)\frac{1}{Z_q} \end{aligned}$$

$$+\sum_{i=1}^{m}\Bigg(\sum_{|\beta|+|\gamma|\geq 2}^{\text{finite}} Y^{\beta}Z^{\gamma}\Bigg)\frac{1}{X_i}\Bigg\}$$

for some constant C_3.

When P_1'', $P_1''''\neq 0$ and $(\rho,\sigma,\tau)\in\widetilde{S}_1\cap S_1\cap\widetilde{S}_1'\cap S_1''$, it follows from Lemma 3.2, (2), Lemma 5.1, (1) and (2) that the operator $(P_1'+P_1''+P_1'''+P_1'''')\Lambda^{-1}\colon \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))\to\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ is bounded. Moreover we have

$$|||(P_1'+P_1''+P_1'''+P_1'''')\Lambda^{-1}U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}}\leq A_4(X,Y,Z)|||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}}, \tag{5.10}$$

where

$$\begin{aligned}A_4(X,Y,Z) = C_4\Bigg\{&\sum_{i=1}^{m-1}\frac{X_{i+1}}{X_i}+\sum_{i=1}^{m}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} X^{\alpha}Y^{\beta}Z^{\gamma}\Bigg)\frac{1}{X_i}\\
&+\sum_{j=1}^{n}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} X^{\alpha}Y^{\beta}Z^{\gamma}\Bigg)\frac{1}{Y_j}+\sum_{q=1}^{p}\Bigg(\sum_{\substack{|\alpha|+|\beta|+|\gamma|\geq 2\\ |\alpha|\geq 1}}^{\text{finite}} X^{\alpha}Y^{\beta}Z^{\gamma}\Bigg)\frac{1}{Z_q}\\
&+\sum_{j=1}^{n-1}\frac{Y_{j+1}}{Y_j}+\sum_{j=1}^{n}\Bigg(\sum_{|\beta|+|\gamma|\geq 2}^{\text{finite}} Y^{\beta}Z^{\gamma}\Bigg)\frac{1}{Y_j}+\sum_{q=1}^{p}\Bigg(\sum_{|\beta|+|\gamma|\geq 2}^{\text{finite}} Y^{\beta}Z^{\gamma}\Bigg)\frac{1}{Z_q}\\
&+\sum_{i=1}^{m}\Bigg(\sum_{|\beta|+|\gamma|\geq 2}^{\text{finite}} Y^{\beta}Z^{\gamma}\Bigg)\frac{1}{X_i}\Bigg\}\end{aligned}$$

for some constant C_4. When P_1'', $P_1''''\neq 0$ and $(\rho,\sigma,\tau)\in\widetilde{S}_1\cap S_1\cap S_1'\cap\widetilde{S}_1''$, it follows from Lemma 3.2, (1), Lemma 5.1, (1) and (3) that the operator $(P_1'+P_1''+P_1'''+P_1'''')\Lambda^{-1}\colon G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)\to G_0^{\{\rho,\sigma,\tau\}}(X,Y,Z)$ is bounded. Moreover we have

$$\|(P_1'+P_1''+P_1'''+P_1'''')\Lambda^{-1}U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\leq A_4(X,Y,Z)\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}. \tag{5.11}$$

Next let us estimate nonlinear terms. Let

$$g(x,y,z,u)=\sum_{|\alpha|+|\beta|+|\gamma|\geq 0, r\geq 2} g_{\alpha\beta\gamma r}x^{\alpha}y^{\beta}z^{\gamma}u^{r}$$

be the Taylor expansion of $g(x,y,z,u)$ (recall that $g(x,y,z,0)\equiv g_u(x,y,z,0)\equiv 0$). Furthermore let us define the formal power series $|g|(x,y,z,u)$ by

$$|g|(x,y,z,u)=\sum_{|\alpha|+|\beta|+|\gamma|\geq 0, r\geq 2} |g_{\alpha\beta\gamma r}|x^{\alpha}y^{\beta}z^{\gamma}u^{r}.$$

We may assume that $|g|(x,y,z,u)$ converges in $\prod_{i=1}^{m}\{x_i\in\mathbf{C};\ |x_i|\leq K_i\}\times\prod_{j=1}^{n}\{y_j\in$

$\mathbf{C};\ |y_j| \le L_j\} \times \prod_{q=1}^{p}\{z_q \in \mathbf{C};\ |z_q| \le M_q\} \times \{u \in \mathbf{C};\ |u| \le N\}$ for some positive constants K_i, L_j, M_q and N ($i = 1, \dots, m;\ j = 1, \dots, n;\ q = 1, \dots, p$).

We remark the following: It holds that

$$g_u(x, y, z, u) = \sum_{|\alpha|+|\beta|+|\gamma|\ge 0, r\ge 1} (r+1) g_{\alpha\beta\gamma,r+1} x^\alpha y^\beta z^\gamma u^r,$$

and that

$$|g_u|(x, y, z, u) := \sum_{|\alpha|+|\beta|+|\gamma|\ge 0, r\ge 1} (r+1) |g_{\alpha\beta\gamma,r+1}| x^\alpha y^\beta z^\gamma u^r$$

converges in $\prod_{i=1}^{m}\{x_i \in \mathbf{C};\ |x_i| \le K_i\} \times \prod_{j=1}^{n}\{y_j \in \mathbf{C};\ |y_j| \le L_j\} \times \prod_{q=1}^{p}\{z_q \in \mathbf{C};\ |z_q| \le M_q\} \times \{u \in \mathbf{C};\ |u| \le N\}$.

Now it follows from (5.4) and Lemma 3.3, (1) that if $X_i \le K_i$ ($i = 1, \dots, m$), $Y_j \le L_j$ ($j = 1, \dots, n$), $Z_q \le M_q$ ($q = 1, \dots, p$), $U \in G_0^{\{\rho,\sigma,\tau\}}(X, Y, Z)$ and $\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \le N/\widetilde{\mathsf{S}}C$, where $\widetilde{\mathsf{S}} = \max\{\rho_i, \sigma_j, \tau_q;\ i = 1, \dots, m$ and $j = 1, \dots, n$ and $q = 1, \dots, p\}$, then $g(x, y, z, \Lambda^{-1}U(x, y, z))$ belongs to $G_0^{\{\rho,\sigma,\tau\}}(X, Y, Z)$. Moreover it holds that

$$\begin{aligned}(5.12)\qquad \|g(x, y, z, \Lambda^{-1}U(x, y, z))\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} &\le \frac{1}{\widetilde{\mathsf{S}}}|g|\Big(X, Y, Z, \widetilde{\mathsf{S}}C\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\Big)\\ &\le \frac{1}{\widetilde{\mathsf{S}}}|g|\Big(K, L, M, \widetilde{\mathsf{S}}C\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\Big) < +\infty,\end{aligned}$$

where $K = (K_1, \dots, K_m)$, $L = (L_1, \dots, L_n)$, $M = (M_1, \dots, M_p)$.

Next by noting

$$g(x, y, z, u) - g(x, y, z, v) = (u - v)\int_0^1 g_u\big(x, y, z, v + \theta(u - v)\big)\, d\theta,$$

we see that if $X_i \le K_i$ ($i = 1, \dots, m$), $Y_j \le L_j$ ($j = 1, \dots, n$), $Z_q \le M_q$ ($q = 1, \dots, p$) and $\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}$, $\|V\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \le N/2\widetilde{\mathsf{S}}C$, then we have

$$\begin{aligned}(5.13)\qquad &\|g\big(x, y, z, \Lambda^{-1}U(x, y, z)\big) - g\big(x, y, z, \Lambda^{-1}V(x, y, z)\big)\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\\ &\le \|U - V\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \times C|g_u|\Big(X, Y, Z, \widetilde{\mathsf{S}}C\Big(\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} + \|V\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\Big)\Big)\\ &\le \|U - V\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} \times C|g_u|\Big(K, L, M, \widetilde{\mathsf{S}}C\Big(\|U\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} + \|V\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}}\Big)\Big).\end{aligned}$$

Similarly it follows from (5.4) and Lemma 3.3, (2) that if $U \in \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X, (Y, Z))$ and $|||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \le N/\widetilde{\mathsf{S}}C$, where X, Y, Z and $\widetilde{\mathsf{S}}$ are same as above, then we have $g(x, y, z, \Lambda^{-1}U(x, y, z)) \in \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X, (Y, Z))$, and that

$$(5.14)\qquad |||g\big(x, y, z, \Lambda^{-1}U(x, y, z)\big)|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \le \frac{1}{\widetilde{\mathsf{S}}}|g|\Big(X, Y, Z, \widetilde{\mathsf{S}}C|||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}}\Big)$$

$$\leq \frac{1}{\widetilde{S}}|g|\Big(K, L, M, \widetilde{S}C|||U|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)}\Big)$$
$$< +\infty.$$

Moreover if $|||U|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)}$, $|||V|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} \leq N/2\widetilde{S}C$, we have

$$\text{(5.15)} \quad |||g\big(x, y, z, \Lambda^{-1}U(x, y, z)\big) - g\big(x, y, z, \Lambda^{-1}V(x, y, z)\big)|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)}$$
$$\leq |||U - V|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} \times C|g_u|\Big(X, Y, Z, \widetilde{S}C\Big(|||U|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} + |||V|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)}\Big)\Big)$$
$$\leq |||U - V|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} \times C|g_u|\Big(K, L, M, \widetilde{S}C\Big(|||U|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} + |||V|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)}\Big)\Big).$$

Under the above preparations let us take $\varepsilon > 0$, X, Y and Z as follows: We take $\varepsilon > 0$ such that

$$\text{(5.16)} \qquad \frac{1}{\widetilde{S}}|g|(K, L, M, \widetilde{S}C\varepsilon) < \varepsilon$$

and

$$\text{(5.17)} \qquad |g_u|(K, L, M, 2\widetilde{S}C\varepsilon) < 1.$$

Since $|g|(x, y, z, u) = O(u^2)$ and $|g_u|(x, y, z, u) = O(u)$, we can take such $\varepsilon > 0$. Furthermore for this ε let us take X, Y and Z such that the followings hold:

In the case $P_1'''' = 0$:

$$\text{(5.18)} \qquad \{A_1(X, Y, Z) + A_2(X, Y, Z)\}\varepsilon + |||g_0|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} + \frac{1}{\widetilde{S}}|g|(K, L, M, \widetilde{S}C\varepsilon) \leq \varepsilon$$

and

$$\text{(5.19)} \qquad A_1(X, Y, Z) + A_2(X, Y, Z) + C|g_u|(K, L, M, 2\widetilde{S}C\varepsilon) < 1.$$

In the case $P_1'' = 0$:

$$\text{(5.20)} \qquad \{A_1(X, Y, Z) + A_3(X, Y, Z)\}\varepsilon + \|g_0\|^{\{\rho,\sigma,\tau\}}_{X,Y,Z} + \frac{1}{\widetilde{S}}|g|(K, L, M, \widetilde{S}C\varepsilon) \leq \varepsilon$$

and

$$\text{(5.21)} \qquad A_1(X, Y, Z) + A_3(X, Y, Z) + C|g_u|(K, L, M, 2\widetilde{S}C\varepsilon) < 1.$$

In the case P_1'', $P_1'''' \neq 0$ and $(\rho, \sigma, \tau) \in \widetilde{S}_1 \cap S_1 \cap \widetilde{S}_1' \cap S_1''$:

$$\text{(5.22)} \qquad \{A_1(X, Y, Z) + A_4(X, Y, Z)\}\varepsilon + |||g_0|||^{\{\rho,(\sigma,\tau)\}}_{X,(Y,Z)} + \frac{1}{\widetilde{S}}|g|(K, L, M, \widetilde{S}C\varepsilon) \leq \varepsilon$$

and

$$A_1(X,Y,Z) + A_4(X,Y,Z) + C|g_u|(K,L,M,2\widetilde{\mathsf{S}}C\varepsilon) < 1. \tag{5.23}$$

In the case P_1'', $P_1'''' \neq 0$ and $(\rho,\sigma,\tau) \in \widetilde{S}_1 \cap S_1 \cap S_1' \cap \widetilde{S}_1''$:

$$\{A_1(X,Y,Z) + A_4(X,Y,Z)\}\varepsilon + \|g_0\|_{X,Y,Z}^{\{\rho,\sigma,\tau\}} + \frac{1}{\widetilde{\mathsf{S}}}|g|(K,L,M,\widetilde{\mathsf{S}}C\varepsilon) \le \varepsilon \tag{5.24}$$

and (5.23).

We can take such X, Y and Z by the fact $g_0(0,0,0) = 0$ and the expressions of $A_1(X,Y,Z)$, $A_2(X,Y,Z)$, $A_3(X,Y,Z)$ and $A_4(X,Y,Z)$.

In the case $P_1'''' = 0$ we see that if $U \in \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ and $|||U|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \le \varepsilon$, then $TU \in \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z))$ and $|||TU|||_{X,(Y,Z)}^{\{\rho,(\sigma,\tau)\}} \le \varepsilon$ by (5.5), (5.8), (5.14) and (5.18). Hence T is well-defined as a mapping from $\widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z);\varepsilon)$ to itself. Moreover by (5.5), (5.8), (5.15) and (5.19), we see that $T\colon \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z);\varepsilon) \to \widetilde{G}_0^{\{\rho,(\sigma,\tau)\}}(X,(Y,Z);\varepsilon)$ is a contraction mapping. Similarly in other cases we can prove that $T\colon G \to G$ is well-defined and that it is a contraction mapping.

Therefore there exists a unique $U(x,y,z) \in G$ which satisfies $TU(x,y,z) = U(x,y,z)$. Lemma 3.1 implies $U(x,y,z) \in G^{\{\rho,\sigma,\tau\}}$. Hence $u(x,y,z) = \Lambda^{-1}U(x,y,z)$ also belongs to $G^{\{\rho,\sigma,\tau\}}$ and it is a solution of (5.1). The proof is completed. $\square$

6. Unique existence of formal solution

Here we shall prove the unique existence of the formal solution.

(I) Case $m = 0$

We only consider Case (v), and assume $k = 1$. Let us consider the equation (4.1). We may assume $-f_u(0,0) = 1$.

First we write the operator P_1 as $P_1 = Q_0 - Q_1$, where

$$Q_0 = 1 + \sum_{j=1}^{n-1} y_{j+1}\frac{\partial}{\partial y_j}, \quad Q_1 = Q_0 - P_1.$$

Let us define the vector space $H(y,z;l)$ which consists of homogeneous polynomials of degree l ($l \ge 0$) as follows:

$$H(y,z;l) = \left(\text{the vector space spanned by } \{y^\beta z^\gamma;\ (\beta,\gamma) \in \mathbf{N}^{n+p},\ |\beta| + |\gamma| = l\}\right).$$

Lemma 6.1. *For all $l \ge 0$ the linear operator*

$$Q_0\colon H(y,z;l) \to H(y,z;l)$$

is bijective.

Proof. Let us notice

$$Q_0(y^\beta z^\gamma) = y^\beta z^\gamma + \sum_{j=1}^{n} \beta_j y^{\beta - \mathrm{e}_j + \mathrm{e}_{j+1}} z^\gamma,$$

where $\mathrm{e}_j = (\delta_{j1}, \delta_{j2}, \dots, \delta_{jn})$ ($\delta_{jj'}$: Kronecker's delta) for $j = 1, \dots, n$.

Therefore by suitably arranging the basis of $H(y, z; l)$, the matrix representation of Q_0 becomes the following triangular matrix:

$$\underbrace{\begin{pmatrix} 1 & * & \cdots & * \\ & 1 & \cdots & * \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix}}_{\sharp\{(\beta,\gamma)\in\mathbf{N}^{n+p};\ |\beta|+|\gamma|=l\}} .$$

This completes the proof. □

Now in order to solve the equation (4.1) we set

$$u(y, z) = \sum_{l=1}^{\infty} u_l(y, z), \quad g_0(y, z) = \sum_{l=1}^{\infty} g_{0l}(y, z),$$

where $u_l(y, z)$, $g_{0l}(y, z) \in H(y, z; l)$. Then we have the following recursion formula for $\{u_l(y, z)\}_{l=1}^{\infty}$:

$$\begin{aligned}
Q_0 u_1(y, z) &= g_{01}(y, z), \\
Q_0 u_2(y, z) &= g_{02}(y, z) \\
&\quad + \text{(homogeneous part of degree 2 of } Q_1 u_1(y, z) + g(y, z, u_1(y, z))), \\
Q_0 u_3(y, z) &= g_{03}(y, z) + \text{(homogeneous part of degree 3 of} \\
&\qquad Q_1(u_1(y, z) + u_2(y, z)) + g(y, z, u_1(y, z) + u_2(y, z))), \\
&\cdots \\
Q_0 u_l(y, z) &= g_{0l}(y, z) \\
&\quad + \text{(homogeneous part of degree } l \text{ of} \\
&\qquad Q_1(u_1(y, z) + \cdots + u_{l-1}(y, z)) + g(y, z, u_1(y, z) + \cdots + u_{l-1}(y, z))), \\
&\cdots
\end{aligned}$$

Therefore by Lemma 6.1 we can obtain $\{u_l(y, z)\}_{l=1}^{\infty}$ inductively and uniquely. This completes the proof of the unique solvability for the equation (4.1).

(II) Case $m \geq 1$

We only consider Case (i). Similarly to the previous case, we assume $k = 1$. Let us consider the equation (5.1).

We write the operator P_1 as $P_1 = Q_0 - Q_1$, where

$$Q_0 = \sum_{i=1}^{m} \lambda_i x_i \frac{\partial}{\partial x_i} - f_u(0,0) + \sum_{i=1}^{m-1} \delta_i x_{i+1} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n-1} y_{j+1} \frac{\partial}{\partial y_j}, \quad Q_1 = Q_0 - P_1.$$

Let us define the vector space $H(x, y, z; l)$ which consists of homogeneous polynomials of degree l as follows:

$$\begin{aligned} &H(x, y, z; l) \\ &= (\text{the vector space spanned by } \{x^\alpha y^\beta z^\gamma;\ (\alpha, \beta, \gamma) \in \mathbf{N}^{m+n+p},\ |\alpha| + |\beta| + |\gamma| = l\}). \end{aligned}$$

Lemma 6.2. *For all $l \geq 0$ the linear operator*

$$Q_0 \colon H(x, y, z; l) \to H(x, y, z; l)$$

is bijective.

Proof. Let us notice

$$\begin{aligned} Q_0(x^\alpha y^\beta z^\gamma) &= \{\lambda \cdot \alpha - f_u(0,0)\} x^\alpha y^\beta z^\gamma \\ &\quad + \sum_{i=1}^{m-1} \delta_i \alpha_i x^{\alpha - \mathrm{e}_i^{(m)} + \mathrm{e}_{i+1}^{(m)}} y^\beta z^\gamma + \sum_{j=1}^{n-1} \beta_j x^\alpha y^{\beta - \mathrm{e}_j^{(n)} + \mathrm{e}_{j+1}^{(n)}} z^\gamma, \end{aligned}$$

where $\mathrm{e}_i^{(m)} = (\delta_{i1}, \delta_{i2}, \dots, \delta_{im})$ $(i = 1, \dots, m)$ and $\mathrm{e}_j^{(n)} = (\delta_{j1}, \delta_{j2}, \dots, \delta_{jn})$ $(j = 1, \dots, n)$. Therefore by suitably arranging the basis of $H(x, y, z; l)$, the matrix representation of Q_0 becomes the following triangular matrix:

$$\begin{pmatrix} \lambda \cdot \alpha^{(1)} - f_u(0,0) & * & \cdots & * \\ & \lambda \cdot \alpha^{(2)} - f_u(0,0) & \cdots & * \\ & & \ddots & \vdots \\ & & & \lambda \cdot \alpha^{(\kappa)} - f_u(0,0) \end{pmatrix},$$

where $\kappa = \sharp\{(\alpha, \beta, \gamma) \in \mathbf{N}^{m+n+p};\ |\alpha| + |\beta| + |\gamma| = l\}$. The condition (Po2) implies that this matrix is regular, which completes the proof. $\square$

Therefore similarly to the previous case, we can prove the unique solvability of the equation (5.1) by using Lemma 6.2.

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