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AN EXTENSION OF GLAESER INEQUALITY AND ITS APPLICATIONS

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1. Introduction

Let us consider the $m \times m$ first order system, with one space variable,

$$Lu \equiv D_t u + A(t,x) D_x u = f(t,x), \qquad (t,x) \in \mathbf{R} \times \mathbf{R}, \tag{1.1}$$

where

$$A(t,x) = \begin{pmatrix} a_{11}(t,x) & \cdots & a_{1m}(t,x) \\ \vdots & & \vdots \\ a_{m1}(t,x) & \cdots & a_{mm}(t,x) \end{pmatrix}.$$

If $A(t,x)$ is Hermitian, it is well known that the Cauchy problem for L is C^∞ well posed. On the other hand if $A(t,x)$ is triangular, say upper triangular, that is $a_{ij}(t,x) = 0$ for $i < j$ and $a_{ii}(t,x)$ are real valued, then it is clear that the Cauchy problem is C^∞ well posed. In [1], D'Ancona and Spagnolo introduced an interesting class of systems, they called *pseudosymmetric hyperbolic* systems, which includes both symmetric and triangular systems. Recall that the matrix $A(t,x)$ is called pseudosymmetric if the following conditions are fulfilled for all choices of the indices i, j, $j_1, \dots, j_\nu \in \{1, \dots, m\}$:

$$\begin{aligned} a_{ij} \cdot a_{ji} &\geq 0, \\ a_{j_1 j_2} \cdot a_{j_2 j_3} \cdots a_{j_\nu j_1} &= \overline{a_{j_1 j_\nu}} \cdots \overline{a_{j_3 j_2}} \cdot \overline{a_{j_2 j_1}}. \end{aligned}$$

It is quite natural to ask if the Cauchy problem for pseudosymmetric systems is well posed in some function spaces. As far as C^∞ well posedness is concerned, few results are known, mainly for the case of analytic coefficients $a_{ij}(t,x)$ (see [1], [4]).

In this note we are interested in the case of C^∞ coefficients, and, more precisely, to the 2×2 systems of the form

$$D_t u + A(x) D_x u = f(t,x), \qquad \text{with} \quad A(x) = \begin{pmatrix} d(x) & a(x) \\ b(x) & -d(x) \end{pmatrix}, \tag{1.2}$$

where $a(x)$, $b(x)$ are real valued functions on $\mathbf{R}$ and

$$D_t = \frac{\partial}{\partial t}, \quad D_x = \frac{\partial}{\partial x}.$$

Such a system is pseudosymmetric if and only if

$$a(x)b(x) \geq 0. \tag{1.3}$$

When $A(x)$ is real analytic, it was proved in [5] that the Cauchy problem for (1.2) is C^∞ well posed if and only if $A(x)$ is hyperbolic, that is

$$d(x)^2 + a(x)b(x) \geq 0.$$

To go further, let us assume that $d(x) = 0$, and that the coefficients $a(x)$, $b(x)$ are of class C^2. If $u = {}^t(v, w)$ verifies the system $Lu = 0$, then v satisfies the scalar equation

$$D_t^2 v - D_x[a(x)b(x)D_x v] + a'(x)b(x)D_x v = 0 \tag{1.4}$$

where $a'(x)$ denotes the derivative of $a(x)$. To get an apriori estimate for u, a natural question arises: can we estimate the function $a'(x)b(x)$ by constant times $\sqrt{a(x)b(x)}$, if (1.3) is verified? Actually this is the case, and we give two proofs in §2 and §3. This result implies the solvability of the Cauchy problem for (1.4) (Theorem 4.1 in §4).

Note the simplest version of Glaeser inequality (cf. Lemma 3.3 in §3) says that

$$|f'(x)| \leq \sqrt{2M}\,\sqrt{f(x)},$$

whenever $f(x) \geq 0$ and $f''(x) \leq M$ on $\mathbf{R}$. Thus, taking $f = ab$, we can estimate the sum $a'b + ab'$, but not the single summands.

In §5 we consider the Cauchy problem for any system of type (1.2)–(1.3), with indefinitely differentiable coefficients and data, and we derive an apriori estimate which leads to the C^∞ well posedness (Theorem 5.1 in §5). We also add to (1.2) a zero order term, and we find a sufficient Levi type condition on this term.

2. An extension of the Glaeser inequality

In this section we prove

Theorem 2.1. *Let f, $g \in C^2(\mathbf{R})$. Assume that $f^{(j)}(x)$, $g^{(j)}(x)$, $j = 0, 1, 2$, are bounded on $\mathbf{R}$, and*

$$f(x)g(x) \geq 0, \qquad \forall x \in \mathbf{R}.$$

Then we have

$$|f(x)g'(x)|, \quad |f'(x)g(x)| \leq M\sqrt{f(x)g(x)}, \qquad \forall x \in \mathbf{R} \tag{2.1}$$

for some constant M depending on $\sup\{|f^{(j)}(x)| + |g^{(j)}(x)|,\ x \in \mathbf{R},\ j = 0, 1, 2\}$.

Proof. Let us set

$$\Lambda = \sup_{x \in \mathbf{R}} |(fg)''(x)|.$$

If $\Lambda = 0$, then we have $fg \equiv C$ (a constant) on $\mathbf{R}$. If $C \neq 0$ the assertion (2.1) is trivial. On the other hand, if $fg \equiv 0$, at each point x where $f(x) \neq 0$ (resp. $g(x) \neq 0$) we have $g(x) = 0$ (resp. $f(x) = 0$), hence also $g'(x) = 0$ (resp. $f'(x) = 0$) since $f'g + fg' = 0$. This shows that $f(x)g'(x) = f'(x)g(x) = 0$. Then (2.1) holds.

Thus, we may assume $\Lambda \neq 0$. We set

$$\delta^{-1} = \sqrt{\frac{\Lambda}{2}}.$$

We first assume that f, g have compact support, say $f(x) = g(x) = 0$ for $|x| \geq r$. We start with

Lemma 2.2. *Let $|x| < r$ be such that $f(x)g(x) \neq 0$. Then, in the interval*

$$\left(x - \delta\sqrt{f(x)g(x)},\, x + \delta\sqrt{f(x)g(x)}\right)$$

we have $f(\xi)g(\xi) > 0$.

Proof. Since $\Omega = \{\xi\colon |\xi| < r,\ f(\xi)g(\xi) \neq 0\}$ is an open set, we can express

$$\Omega = \bigcup_{\nu=1}^{\infty} I_\nu, \qquad I_\nu = (a_\nu, b_\nu),$$

where I_ν are open intervals which are disjoint each other. Taking into account that $fg \geq 0$ in a neighborhood of a_ν and b_ν, we have

$$(fg)(a_\nu) = (fg)(b_\nu) = 0, \qquad (fg)'(a_\nu) = (fg)'(b_\nu) = 0.$$

Therefore, assuming that $x \in I_\nu$, we can write

$$(fg)(x) = \int_{a_\nu}^{x} dt \int_{a_\nu}^{t} (fg)''(s)\, ds,$$

and hence we get

$$|(fg)(x)| \leq \frac{\Lambda(x - a_\nu)^2}{2}.$$

From this it follows that

$$x - a_\nu \geq \delta\sqrt{f(x)g(x)}.$$

The same argument shows that

$$b_\nu - x \geq \delta\sqrt{f(x)g(x)}.$$

Thus we conclude that

$$\left(x - \delta\sqrt{f(x)g(x)}, x + \delta\sqrt{f(x)g(x)}\right) \subset I_\nu.$$

This proves the assertion, since $f(\xi)g(\xi)$ does not change sign in I_ν. $\square$

Proof of Theorem 2.1. Let $x \in \mathbf{R}$. If $f(x)g(x) = 0$, (2.1) holds as we observed above. Thus, we assume that $f(x)g(x) \neq 0$.

From Lemma 2.2, we may assume that we have either

$$f\left(x + s\sqrt{f(x)g(x)}\right) \geq 0 \quad \text{for} \quad |s| \leq \delta,$$

or

$$f\left(x + s\sqrt{f(x)g(x)}\right) \leq 0 \quad \text{for} \quad |s| \leq \delta.$$

We treat the first case. By the Taylor expansion at $s = 0$, we have

$$\begin{aligned}
0 \leq\ & f\left(x \pm \delta\sqrt{f(x)g(x)}\right) \\
=\ & f(x) \pm \delta f'(x)\sqrt{f(x)g(x)} + \frac{1}{2}\delta^2 f''\left(x \pm \theta^\pm\sqrt{f(x)g(x)}\right) f(x)g(x) \\
\leq\ & f(x) \pm \delta f'(x)\sqrt{f(x)g(x)} + \frac{1}{2}\delta^2 M_2 f(x)g(x),
\end{aligned}$$

where $0 < \theta^\pm < 1$ and $M_2 = \sup_{\mathbf{R}} |f''|$. This proves

$$\mp f'(x)\sqrt{f(x)g(x)} \leq \delta^{-1} f(x) + \frac{1}{2}\delta M_2 f(x)g(x). \tag{2.2}$$

Multiplying (2.2) by $\sqrt{g(x)/f(x)}$ we get

$$|f'(x)g(x)| \leq \delta^{-1}\sqrt{f(x)g(x)} + \frac{1}{2}\delta M_2 \sup_{\mathbf{R}}|g| \cdot \sqrt{f(x)g(x)}.$$

The same argument, exchanging $g(x)$ and $f(x)$, proves

$$|f(x)g'(x)| \leq \delta^{-1}\sqrt{f(x)g(x)} + \frac{1}{2}\delta M_2 \sup_{\mathbf{R}}|g| \cdot \sqrt{f(x)g(x)}$$

and hence the result.

If f, g are not compactly supported, for a fixed $x_0 \in \mathbf{R}$ we apply the above result to the functions χf and χg, where $\chi(x)$ is a cut-off function ≥ 0 such that $\chi \equiv 1$ on $\{|x-x_0| \leq 1\}$, $\chi \equiv 0$ on $\{|x-x_0| \geq 2\}$. Thus, (2.1) holds in the interval $[x_0-1, x_0+1]$ with a constant M independent of x_0, hence it holds in the whole $\mathbf{R}$. $\square$

3. A refinement of the inequality

In this section we prove Theorem 2.1 under less regular assumptions on $g(x)$. For a subset $S \subseteq \mathbf{R}$ and a function $\varphi(x)$ on $\mathbf{R}$, we define

$$\Lambda(\varphi, S) = \sup_S |\varphi(x)|.$$

Theorem 3.1. *Let $f \in C^2(\mathbf{R})$ and $g \in C^1(\mathbf{R})$ be such that*

$$f(x)g(x) \geq 0, \qquad \forall x \in \mathbf{R}.$$

Therefore, for every bounded $S \subset \mathbf{R}$ and every $r > 0$, we have

$$|f'(x)g(x)| \leq M(S,r)\sqrt{f(x)g(x)}, \qquad \forall x \in S, \tag{3.1}$$

with

$$M(S,r) = C\Big[\big(r^{-1}\Lambda(f,S_r)+\Lambda(f',S_r)+r\Lambda(f'',S_r)\big)\cdot\big(r^{-1}\Lambda(g,S_r)+\Lambda(g',S_r)\big)\Big]^{1/2} \tag{3.2}$$

where $S_r = \{x\colon \operatorname{dist}(x,S) \leq r\}$ and C denotes some universal constant.

REMARK. If the functions $f^{(j)}(x)$, $j = 0, 1, 2$, and $g^{(i)}(x)$, $i = 0, 1$, are bounded on the whole $\mathbf{R}$, then (3.1)–(3.2) imply

$$|f'(x)g(x)| \leq M\sqrt{f(x)g(x)}, \qquad \forall x \in \mathbf{R},$$

with

$$M = C\Big[\big(\Lambda(f,\mathbf{R}) + \Lambda(f',\mathbf{R}) + \Lambda(f'',\mathbf{R})\big)\cdot\big(\Lambda(g,\mathbf{R}) + \Lambda(g',\mathbf{R})\big)\Big]^{1/2}.$$

Theorem 3.1 will be proved as a consequence of the following

Lemma 3.2. i) *Given an open interval $I = (a,b)$, let $f \in C^2(\overline{I})$, $g \in C^1(\overline{I})$, with*

$$f(a) = f(b) = g(a) = g(b) = 0.$$

Assume that

$$f(x) \geq 0 \quad \textit{on } I, \tag{3.3}$$

and, for some positive constants M_j, N_1,

$$|f'(x)| \leq M_1, \quad f''(x) \leq M_2, \quad |g'(x)| \leq N_1, \qquad \forall x \in I. \tag{3.4}$$

Therefore, we have

$$|f'(x)g(x)| \leq 2\sqrt{(8M_1 + (b-a)M_2)N_1} \cdot \sqrt{f(x)|g(x)|}, \qquad \forall x \in I. \tag{3.5}$$

ii) *The same conclusion holds true if we replace the assumption* (3.3) *by*

$$f(x)g(x) \geq 0, \qquad \forall x \in I$$

and (3.4) *by*

$$|f'(x)| \leq M_1, \quad |f''(x)| \leq M_2, \quad |g'(x)| \leq N_1, \qquad \forall x \in I.$$

Proof of Lemma 3.2. We first derive part (ii) from (i). Consider the set

$$\Omega = \Big\{ x \in I : \text{there is a nbd. of } x \text{ where } f \text{ does not change sign} \Big\}.$$

In other words, $x \in I \setminus \Omega$ if and only if there are two sequences of points $\{x_j'\}$, $\{x_j''\}$, converging to x, for which

$$f(x_j') > 0, \quad f(x_j'') < 0, \qquad j = 1,\ 2,\ 3, \ldots \tag{3.6}$$

Clearly, Ω is an open set. Moreover, by (3.6), we see that $f(x) = 0$ for all $x \in I \setminus \Omega$. But $f(x)g(x) \geq 0$, hence on $I \setminus \Omega$ we have also $g(x) = 0$. Consequently, writing

$$\Omega = \bigcup_{k=1}^{\infty} I_k, \qquad I_k = (a_k, b_k),$$

we see that $f(x)$ does not change sign on each interval I_k, and

$$f(a_k) = f(b_k) = g(a_k) = g(b_k) = 0.$$

Considering $-f(x)$ instead of $f(x)$ we may assume that $f \geq 0$ on the interval I_k; hence we can apply part (i) on this interval.

In order to prove the part (i) of Lemma 3.2, we need three auxiliary lemmas. The first lemma is a version of the Glaeser inequality, where less regularity of the function is required: it is assumed only to be a C^1 function with absolutely continuous first derivative. The second and the third lemmas provide some comparisons of a given function with the reference function $(x-a)(b-x)$, on the interval $\{a < x < b\}$.

Lemma 3.3. *For every* $\psi \in W^{2,1}_{loc}(\mathbf{R})$ *satisfying*

$$\psi(x) \geq 0, \qquad \psi''(x) \leq M, \qquad \textit{a.e. on } \mathbf{R},$$

we have

$$|\psi'(x)| \leq \sqrt{2M}\,\sqrt{\psi(x)}, \qquad \forall x \in \mathbf{R}. \tag{3.7}$$

Proof. We can write

$$\begin{aligned}\psi(x+h) - \psi(x) - h\psi'(x) &= \int_x^{x+h} \big(\psi'(\xi) - \psi'(x)\big)\,d\xi \\ = \int_x^{x+h}\int_x^{\xi} \psi''(\eta)\,d\eta\,d\xi &= \int_{x+h}^{x}\int_{\xi}^{x} \psi''(\eta)\,d\eta\,d\xi.\end{aligned}$$

Thus, regardless on the sign of h, we have

$$0 \leq \psi(x+h) \leq \psi(x) + h\psi'(x) + \frac{Mh^2}{2}, \qquad \forall h,$$

and hence (3.7) follows immediately. □

Lemma 3.4. *Let* $I = (a,b)$*, and let* $\varphi(x) \in C^1(I) \cap C^0(\overline{I})$ *be a function such that*

$$\varphi(a) = \varphi(b) = 0, \qquad |\varphi'(x)| \leq K < \infty, \quad \forall x \in I.$$

Then, we have

$$|\varphi(x)| \leq \frac{2K}{b-a}(x-a)(b-x), \qquad \forall x \in I. \tag{3.8}$$

Proof. It is sufficient to remark that:

$$\begin{aligned}|\varphi(x)| &= \left|\int_a^x \varphi'(\xi)\,d\xi\right| \leq K(x-a), \qquad \frac{1}{b-x} \leq \frac{2}{b-a}, \qquad \text{if} \quad x \in \left[a, \frac{a+b}{2}\right], \\ |\varphi(x)| &= \left|\int_x^b \varphi'(\xi)\,d\xi\right| \leq K(b-x), \qquad \frac{1}{x-a} \leq \frac{2}{b-a}, \qquad \text{if} \quad x \in \left[\frac{a+b}{2}, b\right].\end{aligned}$$ □

Lemma 3.5. *Let* $I = (a,b)$ *and* $f \in C^2(\overline{I})$ *be such that* $f(a) = f(b) = 0$*. Assume*

$$f(x) \geq 0, \qquad |f'(x)| \leq M_1, \qquad f''(x) \leq M_2, \qquad \forall x \in I, \tag{3.9}$$

for some positive constants M_j. Therefore, putting $d = b - a$, we have

$$|f'(x)| \le 2\sqrt{d}\,\sqrt{\frac{d}{2}M_2 + 4M_1}\,\frac{\sqrt{f(x)}}{\sqrt{(x-a)(b-x)}}, \qquad \forall x \in I. \tag{3.10}$$

Proof. Let $\chi_I(x)$ be the characteristic function of $I = [a, b]$. Then, the function

$$\psi(x) = (x-a)(b-x)f(x)\chi_I(x) \tag{3.11}$$

is of class C^1 on the whole real line, with first derivative

$$\psi'(x) = \big[(x-a)(b-x)f'(x) + (a+b-2x)f(x)\big]\chi_I(x). \tag{3.12}$$

Note that $\psi'(x)$ is Lipschitz continuous on $\mathbf{R}$, and its distributional derivative is the L^∞ function

$$\psi''(x) = \big[(x-a)(b-x)f''(x) + 2(a+b-2x)f'(x) - 2f(x)\big]\chi_I(x).$$

Observing that $0 \le (x-a)(b-x) \le d^2/4$ and $|a+b-2x| \le d$, for $x \in I$, we get, by (3.9),

$$\psi''(x) \le \frac{d^2}{4}M_2 + 2dM_1 - 2f(x)\chi_I(x) \le d\Big(\frac{d}{4}M_2 + 2M_1\Big) \quad \text{a.e. on } \mathbf{R}. \tag{3.13}$$

Now, we apply Lemma 3.3 to the function (3.11): by (3.7) we get

$$|\psi'(x)| \le C_0\sqrt{d}\,\sqrt{\psi(x)}, \qquad \forall x \in \mathbf{R},$$

with

$$C_0 = \sqrt{\frac{d}{2}M_2 + 4M_1}. \tag{3.14}$$

Recalling (3.12), this yields

$$\begin{aligned}(x-a)(b-x)|f'(x)| &\le C_0\sqrt{d}\,\sqrt{(x-a)(b-x)}\,\sqrt{f(x)} + |a+b-2x|f(x)\\ &\le C_0\sqrt{d}\,\sqrt{(x-a)(b-x)}\,\sqrt{f(x)} + df(x),\end{aligned}$$

and hence

$$|f'(x)| \le \sqrt{d}\,\sqrt{\frac{f(x)}{(x-a)(b-x)}}\left(C_0 + \sqrt{\frac{df(x)}{(x-a)(b-x)}}\right). \tag{3.15}$$

To conclude the proof of Lemma 3.4, let us apply (3.8) with $\varphi(x) = f(x)$, $K =$

M_1. Thus, recalling (3.14), we find

$$C_0 + \sqrt{\frac{df(x)}{(x-a)(b-x)}} \le C_0 + \sqrt{2M_1} \le 2C_0, \tag{3.16}$$

so that the desired estimate (3.10) follows from (3.15)–(3.16). □

Conclusion of the proof of Lemma 3.2. We apply Lemma 3.4 to the function $g(x)$. By (3.8) with $\varphi = g$ and $K = N_1$, together with (3.10), we get

$$\begin{aligned} |f'(x)|\sqrt{|g(x)|} &\le 2\sqrt{d}\sqrt{\frac{d}{2}M_2 + 4M_1}\,\frac{\sqrt{f(x)}}{\sqrt{(x-a)(b-x)}} \cdot \sqrt{\frac{2N_1}{d}(x-a)(b-x)} \\ &\le 2\sqrt{\big(dM_2 + 8M_1\big)N_1} \cdot \sqrt{f(x)}, \end{aligned}$$

that is, (3.5). This completes the proof of Lemma 3.2. □

Conclusion of the proof of Theorem 3.1. If $f(x)$ and $g(x)$ are as in Theorem 3.1, the estimates (3.1)–(3.2) can be derived from Lemma 3.2 in the following way:

Given $x_0 \in \mathbf{R}$, let $I = (x_0 - r, x_0 + r)$. Take a cut-off function $\chi(x)$, equal to 1 in a neigborhood of x_0 and vanishing at the endpoints of I, so that $0 \le \chi(x) \le 1$, and $|\chi^{(j)}(x)| \le Cr^{-j}$ for $j = 1, 2$. We apply Lemma 3.2, part (ii), to the functions $\widetilde{f}(x) = \chi(x)f(x)$ and $\widetilde{g}(x) = \chi(x)g(x)$. Assume that

$$|f(x)| \le M_0, \quad |f'(x)| \le M_1, \quad f''(x) \le M_2, \quad |g(x)| \le N_0, \quad |g'(x)| \le N_1, \quad \forall x \in I,$$

therefore $\widetilde{f}(x)$ and $\widetilde{g}(x)$ fulfil (3.4) with constants

$$\widetilde{M}_1 = C\left(\frac{M_0}{r} + M_1\right), \qquad \widetilde{M}_2 = C\left(\frac{M_0}{r^2} + \frac{2M_1}{r} + M_0\right), \qquad \widetilde{N}_1 = C\left(\frac{N_0}{r} + N_1\right).$$

Hence (3.5) gives

$$|f'(x_0)g(x_0)| \le 2\sqrt{10}\,C\left[\left(\frac{M_0}{r} + 2M_1 + rM_2\right) \cdot \left(\frac{N_0}{r} + N_1\right)\right]^{1/2} \sqrt{f(x_0)g(x_0)},$$

which implies (3.1)–(3.2), by the arbitrariness of x_0. □

4. An application to the Cauchy problem

In this section we return to the equation (1.4) which motivated us to extend the Glaeser inequality. Let us consider the Cauchy problem

$$D_t^2 v - D_x[a(x)b(x)D_x v] + a'(x)b(x)D_x v = 0, \tag{4.1}$$

$$v(0,x) = v_0(x), \quad D_t v(0,x) = v_1(x), \tag{4.2}$$

where, for simplicity, we assume that $a(x)$, $b(x)$ are C^∞ functions with bounded derivatives of all orders on $\mathbf{R}$ and

$$a(x)b(x) \geq 0 \quad \text{on } \mathbf{R}.$$

We define the energy function

$$E(t) = \frac{1}{2}\int \left(|v_t|^2 + a(x)b(x)|v_x|^2 + |v|^2\right)\, dx,$$

where the integral is extended to $\mathbf{R}$.

Integrating by parts, we get

$$E'(t) = -\operatorname{Re}\int a'(x)b(x)v_x\bar{v}_t\, dx + \operatorname{Re}\int v\bar{v}_t\, dx$$

and hence, using Theorem 3.1, we find the apriori estimate

$$E'(t) \leq C_1 E(t).$$

Thus we get

$$\|v(t)\|_{L^2} + \|v_t(t)\|_{L^2} \leq C(\tau)\left(\|v_0\|_{H^1} + \|v_1\|_{L^2}\right), \qquad 0 \leq t \leq \tau.$$

To get an estimate of the H^k norm of the solution we differentiate (4.1) with respect to x to obtain an equation for $w = D_x v$:

$$\begin{aligned} D_t^2 w - D_x[a(x)b(x)D_x w] + \left[a'(x)b(x) - (ab)'(x)\right] D_x w& \\ + \left[(a'b)'(x) - (ab)''(x)\right] w = 0.& \end{aligned}$$

By iterating the same procedure we find the estimates

$$\|v(t)\|_{H^k} + \|v_t(t)\|_{H^k} \leq C_k(\tau)\left(\|v_0\|_{H^{k+1}} + \|v_1\|_{H^k}\right), \qquad 0 \leq t \leq \tau$$

for all k which proves the C^∞ well-posedness.

Theorem 4.1. *Under the assumptions as above, the Cauchy Problem* (4.1)–(4.2) *is C^∞ well posed.*

5. Further results of well-posedness

Here we consider the more general class of systems

$$\begin{cases} Lu \equiv [D_t + A(x)D_x + B(x)]u = f(t,x), & 0 \leq t \leq \tau, \\ u(0,x) = 0, \end{cases} \tag{5.1}$$

where

$$A(x) = \begin{pmatrix} d(x) & a(x) \\ b(x) & -d(x) \end{pmatrix}, \qquad B(x) = \begin{pmatrix} \delta_1(x) & \alpha(x) \\ \beta(x) & \delta_2(x) \end{pmatrix}. \tag{5.2}$$

For the sake of simplicity, we assume that the coefficients of these matrices are C^∞ functions, with bounded derivatives of all orders on $\mathbf{R}$. The coefficients of $A(x)$ are real, those of $B(x)$ may be complex. We assume, as always,

$$a(x)b(x) \geq 0. \tag{5.3}$$

As for the lower order term $B(x)$ we assume that, for some positive constant C,

$$|a(x)\beta(x)| \leq C\sqrt{a(x)b(x)}, \qquad |b(x)\alpha(x)| \leq C\sqrt{a(x)b(x)}. \tag{5.4}$$

Theorem 5.1. *Under the assumptions* (5.3)–(5.4), *the Cauchy problem* (5.1)–(5.2) *is well posed in* C^∞.

Proof. In order to find an apriori estimate for (5.1), we consider $M \circ L$ with an operator M such as

$$M = D_t - A(x)D_x + \widetilde{B}(x), \tag{5.5}$$

where $\widetilde{B}(x)$ will be chosen in a suitable way. The matrix $A(x)$ enjoys a very good property: its square is Hermitian; more precisely, we have

$$A^2(x) = h(x)I, \qquad \text{with} \qquad h(x) = d^2(x) + a(x)b(x) \geq 0. \tag{5.6}$$

After some computations, we see that

$$M \circ L = \left(D_t^2 - A^2(x)D_x^2\right) + K(x)D_x + T_1(x)D_t + T_0(x)$$

with $T_1 = B + \widetilde{B}$, $T_0 = \widetilde{B}B - AB_x$, and

$$K(x) = \widetilde{B}A - AA_x - AB. \tag{5.7}$$

Thus, each smooth solution $u(t,x)$ to (5.1) solves also the second order system

$$\left[D_t^2 - h(x)D_x^2 + K(x)D_x + T_1(x)D_t + T_0(x)\right] u = g \equiv Mf. \tag{5.8}$$

The natural energy for such a system is given by

$$E(t,u) = \frac{1}{2}\int \left(|u_t|^2 + h(x)|D_x u|^2 + |u|^2\right)\, dx,$$

hence, if we multiply each term of (5.8) by $\overline{D_t u}$, we need an estimate like:

$$\int |(K(x)D_x u, D_t u)|\, dx \le C \int \sqrt{h(x)}\, |D_x u||D_t u|\, dx \le CE(t,u) \tag{5.9}$$

i.e.

$$\|K(x)\| \le C\sqrt{h(x)}. \tag{5.10}$$

In view of (5.10), let us choose the matrix $\widetilde{B}(x)$ in (5.5) of the form

$$\widetilde{B}(x) = -A_x(x) + X(x), \qquad \text{with} \quad X = \begin{pmatrix} \varphi(x) & 0 \\ 0 & \psi(x) \end{pmatrix}, \tag{5.11}$$

so that, by (5.7),

$$K(x) = -(AA_x + A_x A) + (XA - AB).$$

Now, the matrix $AA_x + A_x A = (A^2)_x = h'(x)I$ has a norm which can be estimated by $C\sqrt{h(x)}$, by the (classical) Glaeser inequality. Thus, we must choose the functions $\varphi(x)$, $\psi(x)$ in (5.11) in such a way that

$$\|(XA - AB)(x)\| \le C\sqrt{h(x)}. \tag{5.12}$$

We compute:

$$XA - AB = \begin{pmatrix} d(\varphi - \delta_1) - a\beta & a(\varphi - \delta_2) - d\alpha \\ b(\psi - \delta_1) + d\beta & -d(\psi - \delta_2) - b\alpha \end{pmatrix}.$$

Hence, the choice

$$\varphi(x) = \delta_2(x), \qquad \psi(x) = \delta_1(x),$$

produces

$$XA - AB = d(x)T(x) - \begin{pmatrix} a(x)\beta(x) & 0 \\ 0 & b(x)\alpha(x) \end{pmatrix},$$

for some (bounded) matrix $T(x)$. By (5.3) and (5.6), we know that $|d(x)| \le \sqrt{h(x)}$, while, by (5.4), $a(x)\beta(x)$ and $b(x)\alpha(x)$ are estimated by $\sqrt{a(x)b(x)}$. In conclusion we get (5.12), hence also (5.10) and (5.9).

We are now in the position to prove an energy estimate for the solutions to (5.8),

$$E'(t,u) \le C\big(E(t,u) + \|g(t)\|^2_{L^2}\big), \qquad 0 \le t \le \tau,$$

whence, by (5.1),

$$\|u(t)\|_{L^2} + \|D_t u(t)\|_{L^2} \leq C \int_0^\tau \|g(s)\|_{L^2}\, ds.$$

Note that the terms $T_1(x)D_t u$ and $T_0(x)u$ do not give any trouble.

To get an estimate of the H^1 norm of the solution, we differentiate each term of (5.8) with respect to x, thus obtaining an equation in the unknown $v = D_x u$:

$$D_t^2 v - h(x)D_x^2 v + \big(K(x) - h'(x)I\big)D_x v + T_1(x)D_t v + \widetilde{T}_0(x)v \\ = D_x g - T_1'(x)D_t u + T_2(x)u.$$

By iterating this procedure, and going back to (5.1), we find the apriori estimates

$$\|u(t)\|_{H^k} + \|D_t u(t)\|_{H^k} \leq C \int_0^\tau \|g(s)\|_{H^k}\, ds \leq C' \int_0^\tau \|f(s)\|_{H^{k+1}}\, ds,$$

for all integers k, which ensure the well-posedness in C^∞. □

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