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A HURWITZ-LIKE CLASSIFICATION OF THURSTON COMBINATORIAL CLASSES

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1. Introduction

In the mid-1980s Thurston showed that certain holomorphic dynamical systems could be classified by purely topological data [3]. Specifically, let $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map from the complex projective line to itself of degree $d > 1$. By the Riemann-Hurwitz formula, such a map has $2d - 2$ critical points, counted with multiplicity. Letting C_f denote the set of critical points, assume that the *postcritical set* $P_f = \bigcup_{n>0} f^{\circ n}(C_f)$ is finite.

This classification takes the following form. Suppose more generally that f is an orientation-preserving branched covering of the two-sphere to itself. Then f has a postcritical set P_f which we assume is finite. Given two such maps f, g we say f and g are *Thurston* equivalent if there is a homeomorphism $\phi_0\colon (S^2, P_f) \to (S^2, P_g)$ conjugating f to g *up to isotopy relative to* P_f. Thurston then characterized those branched coverings which are equivalent to rational maps. He also proved that, apart from a precisely described set of exceptions known as *integral Lattès examples*, that a rational map is determined by its combinatorial class. Since then, this rigidity has been employed to give, for example, combinatorial descriptions of various parameter spaces and combination procedures for rational maps, viewed as dynamical systems.

In practice, it can be difficult to determine when two explicitly given rational maps, or more generally two branched coverings, are combinatorially equivalent. It also of interest to consider similar notions of equivalence, where the postcritical set is enlarged to include other points, remaining finite and forward-invariant.

Since the notion of combinatorial equivalence is given in terms of topological data up to isotopy, it is tempting to formulate an algebraic notion of this equivalence, so that one might reduce certain computations to purely algebraic problems, a realm where invariants might be easier to define and compute.

Let $G = \pi_1(S^2 - P_f)$. In [4] Kameyama showed that associated to a degree d branched covering f is a certain homomorphism

$$f_\dagger\colon G \to G^d \rtimes \mathcal{S}_d,$$

where $\mathcal{S}_d$ is the symmetric group on d letters, and obtained necessary and sufficient conditions in terms of $f_\dagger$, $g_\dagger$ for f and g to be combinatorially equivalent.

In this work, we give a reformulation of Kameyama's result by using the classical non-dynamical notion of *Hurwitz equivalence* of planar coverings and elementary algebraic topology. The main result, Theorem 4.1, is essentially identical to Kameyama's: to f, we associate a certain homomorphism

$$\hat{\rho}_f : G \to G^d \rtimes \mathcal{S}_d$$

and give necessary and sufficient conditions in terms of $\hat{\rho}_f$, $\hat{\rho}_g$ for f and g to be combinatorially equivalent. It seems highly probable that in fact $f_\dagger = \widehat{\rho}_f$.

In §2, we define Hurwitz and Thurston equivalence in a more general setting. In §3, we define, after some algebraic preliminaries, the homomorphism $\hat{\rho}_f$. In §4, we analyze the dependence of $\hat{\rho}_f$ and prove our main result.

Notation and conventions.

- Branched coverings are continuous and orientation-preserving.
- $\mathcal{S}_X$: the symmetric group on a set X.
- Let $d \geq 1$ be an integer. If $\varphi\colon X \to Y$ is a function, then the induced "diagonal" map $x \mapsto (\varphi(x), \ldots, \varphi(x)) \in Y^d$ will be denoted φ^d.

2. Branched covers

2.1. Branched covers. An orientation-preserving, continuous map $f\colon S^2 \to S^2$ is a *branched covering* if can be written in the form $z \mapsto z^k$ in local complex coordinates. The integer k is the *local degree* of f at the origin x and is denoted $\deg(f, x)$. Points x where $k \geq 1$ are called *critical points* of f; their images are *critical values*.

Fix A_0, $A_1 \subset S^2$ with A_1 finite and $A_0 \subset A_1$. Put

$$X_i = S^2 - A_i, \quad i = 1,\ 2.$$

DEFINITION 2.1. Let $\mathcal{B}(A_1, A_0, d)$ denote the space of smooth branched covers $f\colon S^2 \to S^2$ such that the critical points of f lie in A_1, $\deg(f) = d$, and $A_1 = f^{-1}(A_0)$.

2.2. Hurwitz, covering, and Thurston equivalence. Fix A_0, A_1, and d; write $\mathcal{B} = \mathcal{B}(A_0, A_1, d)$. On $\mathcal{B}$ we consider three equivalence relations:

DEFINITION 2.2. Let $f,\ g \in \mathcal{B}$. Suppose there exist orientation-preserving homeomorphisms $\phi_0, \phi_1 \colon S^2 \to S^2$ such that $\phi_i(A_i) = A_i$, $i = 1, 2$ and

$$\begin{array}{ccc} (S^2, A_1) & \xrightarrow{\phi_1} & (S^2, A_1) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ (S^2, A_0) & \xrightarrow[\phi_0]{} & (S^2, A_0) \end{array}$$

commutes. Then we say $f,\ g$ are *Hurwitz equivalent*.

If $\phi_0 = \mathrm{id}$ we say $f,\ g$ are *covering equivalent*.

If ϕ_0 and ϕ_1 are isotopic through maps agreeing on A_0 we say $f,\ g$ are *Thurston equivalent*. (In particular, ϕ_1 sends A_0 to A_0, and the restriction $\phi_1|A_0$ agrees with $\phi_1|A_0$).

Thus,

$$\text{covering eq.} \implies \text{Hurwitz eq.} \impliedby \text{Thurston eq.}$$

The converses, generally, fail to hold. Note that pre- and/or post-composing $f \in \mathcal{B}$ by homeomorphisms preserving A_0, A_1 leaves the Hurwitz class unchanged.

2.3. Covering classification. The general theory of covering spaces and their relation to fundamental groups (see e.g. [5]) implies that covering equivalence classes of elements of $\mathcal{B}$ are in one-to-one correspondence with conjugacy classes of index d subgroups of $\pi_1(X_0, \cdot)$.

The correspondence is induced by the assignment

$$f \mapsto f_*\pi_1(X_1, \cdot) = H_f < \pi_1(X_0, \cdot)$$

followed by recording the conjugacy class of the image H_f. Thus, the invariant classifying covering equivalence classes is completely algebraic in nature.

2.4. Monodromy and Hurwitz classification. Suppose $f \in \mathcal{B}$. The general theory of covering spaces implies that the monodromy (right) action of $\pi_1(X_0, b_0)$ on the fiber $f^{-1}(b_0)$ obtained by lifting loops based at b_0 is isomorphic to the *right regular action* of G on the set of cosets $H \setminus G$ by right multiplication, where $H = f_*\pi_1(X_1, \tilde{b})$, $\tilde{b} \in f^{-1}(b)$.

More precisely, choose coset representatives to write

$$G = Hg_1 \sqcup \cdots \sqcup Hg_d$$

with $g_1 = 1_G$. Given $g \in G$ and $i \in \{1, \ldots, d\}$ there is a unique j for which

$$Hg_i g = Hg_j.$$

Therefore g determines a permutation $\sigma(g) \in \mathcal{S}_d$, and we write the image as $j = i.\sigma(g)$. One finds readily that for $g,\ g' \in G$, this satisfies $i.\sigma(gg') = (i.\sigma(g)).\sigma(g')$. Therefore $\sigma(g) \in \mathcal{S}_d$ satisfies $\sigma(gg') = \sigma(g)\sigma(g')$. Hence one has a representation $\sigma\colon G \to \mathcal{S}_d$. A different choice of coset representatives, indices, or conjugacy class of subgroup H changes this representation by postcomposition by an inner automorphism of $\mathcal{S}_n$. Write $\sigma = \sigma_f$.

An immediate consequence of the Hurwitz Classification Theorem for planar branched covers (see e.g. [1]) is

Theorem 2.1. *Two branched coverings $f,\ g \in \mathcal{B}$ are Hurwitz equivalent if and only if there is a homeomorphism $\phi_0\colon (S^2, A_0) \to (S^2, A_0)$ such that $\sigma_f = \sigma_g \circ \phi_{0*}$ up to inner automorphisms.*

One may interpret this in the following way. The group of automorphisms of $\pi_1(X_0, b)$ generated by such maps ϕ_{0*} and by inner automorphisms acts on the set of subgroups H arising as images under induced homomorphisms f_* of planar, connected coverings. The orbits under this action then classify the Hurwitz classes.

A priori the Hurwitz criterion is a mixture of topological and algebraic conditions. However, if one presents $\pi_1(X_0, \cdot)$ as e.g. $\left\langle g_1, \ldots, g_{n_0} \mid \prod_k g_k = 1 \right\rangle$ where the generator g_k is freely homotopic in X_0 to a simple oriented loop around $a_k \in A_0$, $k = 1, \ldots, n_0 = \#A_0$, then there are algebraic conditions for an isomorphism on fundamental group to be induced by a homeomorphism; see ([7], Thm. 5.7.1).[1]

3. Definition of $\hat{\rho}$

As before, fix A_0, A_1, and d and write $\mathcal{B} = \mathcal{B}(A_0, A_1, d)$. The main result of this section is

Proposition 3.1. *Given $f \in \mathcal{B}$, there is a natural induced homomorphism*

$$\hat{\rho}_f\colon G_0 \to G_0^d \rtimes \mathcal{S}_d$$

well-defined up to composition by inner automorphisms.

Composition of $\hat{\rho}$ with projection onto the second factor is just the Hurwitz homomorphism σ_f. However, whereas σ_f depends only on the image group $H = f_*\pi_1(X_1, \cdot)$, the homomorphism $\hat{\rho}$ uses crucially the *map* f_* and not merely its image. This is unavoidable: by the proof of ([2]), Prop. 2.12), it is possible for a given Hurwitz class to contain infinitely many maps in distinct Thurston classes.

[1] I am grateful to Ryan Budney for the timely provision of this reference.

3.1. Algebraic preliminaries. We begin by refining the analysis of the right regular representation in §2.4. We avoid the language of wreath products, and certain conventions of notation which, although familiar to algebraists, might cause confusion in other readers.

Let G be any group and H a subgroup of finite index d. Fix a choice of coset representatives $\{g_i\}_{i=1}^d$ of H in G. Given i and $g \in G$, there is a unique element $h(g,i)$ of H such that

$$g_i g = h(g,i) g_j. \tag{1}$$

It is easily verified that

$$h(gg', i) = h(g,i) h(g', i.\sigma(g)). \tag{2}$$

Thus one obtains a homomorphism

$$\rho\colon G \to H^d \rtimes \mathcal{S}_d < G^d \rtimes \mathcal{S}_d$$

where the product in the latter group is written

$$[(x_1, x_2, \ldots, x_d), \sigma] \cdot [(x'_1, x'_2, \ldots, x'_d), \sigma'] = [(x_1 x'_{1.\sigma}, x_2 x'_{2.\sigma}, \ldots, x_d x'_{d.\sigma}), \sigma\sigma'].$$

In particular,

$$[(x_1, \ldots, x_d), \sigma]^{-1} = [(x^{-1}_{1.\sigma^{-1}}, \ldots, x^{-1}_{d.\sigma^{-1}}), \sigma^{-1}].$$

Given a different choice of coset representatives $g'_i = \hat{h}_i g_i$, one has that the corresponding homomorphism ρ' is given by

$$\rho'(g) = [(\hat{h}_1, \ldots, \hat{h}_d), 1_{\mathcal{S}_d}] \cdot \rho(g) \cdot [(\hat{h}_1^{-1}, \ldots, \hat{h}_d^{-1}), 1_{\mathcal{S}_d}]$$

i.e. differs from ρ by postcomposition by an inner automorphism of $H^d \rtimes \mathcal{S}_d$ in which the projection to $\mathcal{S}_d$ of the conjugating element is trivial.

Suppose a conjugate $H' = xHx^{-1}$ of H is employed in the above construction. Let $c_x\colon G \to G$ be given by $c_x(g) = x^{-1}gx$. Given coset representatives g_i for $H \setminus G$, put $g'_i = c_x(g_i)$. If $\sigma, \sigma'\colon G \to \mathcal{S}_d$ are as above then one finds that as in §2.4,

$$\sigma(g) = \sigma'(c_x(g)). \tag{3}$$

Define as above $h'(g,j)$ to be the unique element of H' satisfying $g'_j g = h'(g,j) g_{j.\sigma'(g)}$. Applying the isomorphism c_x to both sides of the defining equation (1) for the h's

$$g_i g = h(g,i) g_{i.\sigma}$$

yields, using the definition of the g_i's,

$$g_i' c_x(g) = c_x(h(g,i)) g_{i.\sigma}'.$$

Since the element $c_x(h(g,i))$ lies in H', by uniqueness we have

$$h'(c_x(g), i) = c_x(h(g,i)). \tag{4}$$

Using the definition of ρ' we have using Equations (3), (4)

$$\begin{aligned} \rho'(c_x(g)) &= [(h'(c_x(g),1), \ldots, h'(c_x(g),d)), \sigma'(c_x(g))] \\ &= [(c_x(h(g,1)), \ldots, c_x(h(g,d)), \sigma(g)] \\ &= [(x^{-1}, \ldots, x^{-1}), 1_{\mathcal{S}_d}] \cdot \rho(g) \cdot [(x, \ldots, x), 1_{\mathcal{S}_d}]. \end{aligned}$$

This may be summarized in the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\rho} & H^d \rtimes \mathcal{S}_d \\ {\scriptstyle c_x}\downarrow & & \downarrow{\scriptstyle c_{[(x,\ldots,x),1]}} \\ G & \xrightarrow[\rho']{} & H'^d \rtimes \mathcal{S}_d \end{array} \tag{5}$$

Applying this to $g = c_x^{-1}(\overline{g})$ and then writing g for $\overline{g}$ yields

$$\rho'(g) = [(x^{-1}, \ldots, x^{-1}), 1_{\mathcal{S}_d}] \cdot \rho(x) \cdot \rho(g) \cdot \rho(x)^{-1} \cdot [(x, \ldots, x), 1_{\mathcal{S}_d}]. \tag{6}$$

That is,

$$\rho' = c_X \circ \rho \tag{7}$$

where $X = \rho(x)^{-1}[(x, \ldots, x), 1_{\mathcal{S}_d}]$ and the conjugation takes place in $G^d \rtimes \mathcal{S}_d$. Summarizing still further:

$$\rho = \rho' \text{ up to inner automorphisms.} \tag{8}$$

If $N \lhd H$ is any normal subgroup and $q\colon H \to N \setminus H$ is the projection then composing ρ with the homomorphism $[(q, \ldots, q), \mathrm{id}_{\mathcal{S}_d}]\colon H^d \rtimes \mathcal{S}_d \to (N \setminus H)^d \rtimes \mathcal{S}_d$ induces a homomorphism $\overline{\rho}\colon G \to (N \setminus H)^d \rtimes \mathcal{S}_d$. Considering (5), if $H' = c_x(H)$ and $N' = c_x(N)$ then the induced isomorphism $\overline{c}_x\colon N \setminus H \to N' \setminus H'$ makes the diagram below commute:

$$\begin{array}{ccc} G & \xrightarrow{\overline{\rho}} & (N \setminus H)^d \rtimes \mathcal{S}_d \\ {\scriptstyle c_x}\downarrow & & \downarrow{\scriptstyle [\overline{c}_x^d, \mathrm{id}_{\mathcal{S}_d}]} \\ G & \xrightarrow[\overline{\rho'}]{} & (N' \setminus H')^d \rtimes \mathcal{S}_d \end{array} \tag{9}$$

Equivalently,

$$\overline{\rho}' = [\overline{c}_x^d, \mathrm{id}_{\mathcal{S}_d}] \circ \overline{\rho} \circ c_{x^{-1}}. \tag{10}$$

Summarizing still further:

$$\overline{\rho}' = [\overline{c}_x^d, \mathrm{id}_{\mathcal{S}_d}] \circ \overline{\rho} \text{ up to inner automorphisms.} \tag{11}$$

REMARK. The induced homomorphism $\overline{\rho}$ arises from the right action of G on the cosets $N \setminus G$ as follows. Write $G = \bigsqcup_{i=1}^d Hg_i$ and $H = \bigsqcup_{\lambda\in\Lambda} Nh_\lambda$. Define an action of $(N \setminus H)^d \rtimes \mathcal{S}_d$ on the set of cosets $N \setminus G$ by

$$Nh_\lambda g_i.[(Nh_1, \ldots, Nh_d), \tau)] = Nh_\lambda h_i g_{i.\tau}. \tag{12}$$

It is easily seen that this is a well-defined right action, and that the induced homomorphism from $(N \setminus H)^d \rtimes \mathcal{S}_d$ to the symmetric group $\mathcal{S}_{N\setminus G}$ is injective. Summarizing, the usual right action of G on the cosets $N \setminus G$ induces a map $G \to \mathcal{S}_{N\setminus G}$ which factors through $\overline{\rho}$, i.e. the diagram

$$\begin{array}{ccc} G & \xrightarrow{\overline{\rho}} & (N \setminus H)^d \rtimes \mathcal{S}_d \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow \\ G & \longrightarrow & \mathcal{S}_{N\setminus G} \end{array} \tag{13}$$

commutes. Thus

$$\ker(\overline{\rho}) = \bigcap_{x\in G} x^{-1}Nx,$$

the *normal core* of N in G (see [6], §1.6). Conjugate groups N, N' yield the same cores. Thus $\ker(\overline{\rho})$ depends only on the conjugacy class of N, and one has since $N \lhd H$ that

$$\ker(\overline{\rho}) = \bigcap_{i=1}^d g_i^{-1}Ng_i.$$

3.2. Application to branched covers. Fix now $b \in X_1$, thought of as also lying in X_0. Write $G_i = \pi_1(X_i, b)$, $i = 1$, 2, and let $\iota_*\colon G_1 \to G_0$ be the homomorphism induced by the inclusion $\iota\colon X_1 \hookrightarrow X_0$; note that ι_* is surjective. Let $N_1 \lhd G_1$ denote the kernel of ι_*.

Fix $f \in \mathcal{B}$. We expand on the discussion at the start of §2.4. Recall that a choice of homotopy class of path $\alpha_0\colon ([0,1], 0, 1) \to (X_1, f(b), b)$ in X_0 yields an isomorphism $\alpha_{0*}\colon \pi_1(X_0, f(b)) \to \pi_1(X_0, b)$. (Here, we have used $A_1 = f^{-1}(A_0)$, so that $f(b) \in X_0$.) A pair α_0, α_0' of paths yields a pair of isomorphisms differing by an inner automorphism of $\pi_1(X_0, b)$. Abusing notation, we write $f_*\colon G_1 \to G_0$ for the composition of the map on fundamental groups induced by f with α_{0*}. Thus f_* and its image $H = f_*(G_1) < G_0$ are defined only up to postcomposition by an inner automorphism and conjugation, respectively. Put $N = f_*(N_1)$. Then $N \lhd H$ and the map f_* induces an isomorphism $\overline{f}_*\colon G_0 \to N \setminus H$ since one has

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N_1 & \longrightarrow & G_1 & \xrightarrow{\iota_*} & G_0 & \longrightarrow & 1 \\ \big\downarrow & & {\scriptstyle f_*}\big\downarrow & & {\scriptstyle f_*}\big\downarrow & & {\scriptstyle \overline{f}_*}\big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & N & \longrightarrow & H & \xrightarrow[q]{} & N \setminus H & \longrightarrow & 1 \end{array} \tag{14}$$

The dependence is clear: if $x \in G_0$ and $f'_* = c_x \circ f_*$ then

$$\overline{f}'_* = c_{\overline{x}} \circ \overline{f}_* \tag{15}$$

where $\overline{x} = q(x)$.

We now apply the construction in the preceding section to obtain a homomorphism

$$\overline{\rho}_f\colon G_0 \to (N \setminus H)^d \rtimes \mathcal{S}_d$$

well-defined up to the kind of equivalence expressed in §2.1, Equations (10), (11). The map $\overline{f}_*\colon G_0 \to N \setminus H$ induces an isomorphism

$$[\overline{f}^d_*, \mathrm{id}_{\mathcal{S}_d}]\colon G_0^d \rtimes \mathcal{S}_d \to (N \setminus H)^d \rtimes \mathcal{S}_d.$$

By composing $\overline{\rho}_f$ with the inverse of this map we obtain a homomorphism

$$\hat{\rho}_f\colon G_0 \to G_0^d \rtimes \mathcal{S}_d.$$

Equation (15) implies that the diagram

$$\begin{array}{ccc} (N \setminus H)^d \rtimes \mathcal{S}_d & \xleftarrow{[\overline{f}^d_*, \mathrm{id}_{\mathcal{S}_d}]} & G_0^d \rtimes \mathcal{S}_d \\ {\scriptstyle [\overline{c}^d_x, \mathrm{id}_{\mathcal{S}_d})]}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ (N' \setminus H')^d \rtimes \mathcal{S}_d & \xleftarrow{[\overline{f}'^d_*, \mathrm{id}_{\mathcal{S}_d}]} & G_0^d \rtimes \mathcal{S}_d \end{array} \tag{16}$$

commutes. Concatenation of this diagram with the diagram in (9) proves Proposition 3.1 □

We note here some properties of the homomorphism $\hat{\rho}$.

1. Composition of $\hat{\rho}$ with projection onto the ith coordinate in the first factor of $G_0^d \rtimes \mathcal{S}_d$ (not a homomorphism) is surjective, since given any h, i, j putting $g = g_i^{-1} h g_j$ makes $h(g, i) = h$.

2. The covering space of X_1 corresponding to $N_1 \lhd G_1$ is just the restriction p_1 of the universal covering $p_0\colon U_0 \to X_0$ to the connected subspace $U_1 = p_0^{-1}(X_1)$ (see [5], Prop. 11.1). Note that this covering is regular with deck group $N_1 \setminus G_1$, which is isomorphic via ι_* to G_0.
If $f \in \mathcal{B}$, the composition

$$f \circ p_1\colon U_1 \to X_0$$

is the covering corresponding to the conjugacy class of subgroup $f_*(N_1) = N < G_0$. Let $\{b_1, \ldots, b_d\}$ be a fiber of f above some point. Then each fiber $p_1^{-1}(b_i)$ is the orbit of a point under the deck group $N_1 \setminus G_1 \overset{\iota_*}{\simeq} G_0$. The monodromy action of G_0 on the fibers of $f \circ p_1$ is imprimitive, since it preserves the d blocks $p_1^{-1}(b_i)$. The discussion of monodromy in §2.4 and that at the end of §3.1 together then make this observation manifest algebraically by showing that this action factors through $\overline{\rho}_f$. Recalling that $H = f_*(G_1)$, the map $\overline{f}_*$ induced by f identifies $N \setminus H$ with G_0. Thus, the map $\hat{\rho}$ may be viewed as simply recording the fact that the monodromy action of G_0 on the fibers of $f \circ p_1$ factors through $G_0^d \rtimes \mathcal{S}_d$.

4. Dependence of $\hat{\rho}_f$

As before, fix A_0, A_1, and d and write $\mathcal{B} = \mathcal{B}(A_0, A_1, d)$. Fix $b \in X_1$ and let G_i be as in the preceding section.

The main result of this section is

Theorem 4.1 (Thurston classification). *Two elements f, $g \in \mathcal{B}$ are Thurston equivalent via a pair ϕ_0, ϕ_1 if and only if up to inner automorphisms $\hat{\rho}_g \circ \phi_{0*} = [\phi_{0*}^d, \mathrm{id}_{\mathcal{S}_d}] \circ \hat{\rho}_f$, i.e.*

$$\begin{array}{ccc} G_0 & \xrightarrow{\hat{\rho}_f} & G_0^d \rtimes \mathcal{S}_d \\ {\scriptstyle \phi_{0*}}\downarrow & & \downarrow{\scriptstyle [\phi_{0*}^d, \mathrm{id}_{\mathcal{S}_d}]} \\ G_0 & \xrightarrow[\hat{\rho}_g]{} & G_0^d \rtimes \mathcal{S}_d \end{array}$$

commutes.

An homeomorphism $\phi_1\colon (S^2, A_1) \to (S^2, A_1)$, upon restricting to $X_1 = S^2 - A_1$, induces (by choice of a path α_1 as in §2.2) an induced map $\phi_{1*}\colon G_1 \to G_1$ whose image is determined up to conjugacy in G_1.

Lemma 4.1. *Suppose ϕ_1 sends A_i to itself, $i = 0, 1$. Then ϕ_{1*} sends the normal subgroup $N_1 = \ker(\iota_*)$ to itself.*

Proof. Let $\gamma\colon (S^1, 1) \to (X_1, b)$ represent an element of N_1. Then there is a homotopy in X_0 from γ to the constant map at b. Since ϕ sends A_i to itself, $i = 0, 1$ it sends X_0 to itself. Hence composition of the homotopy with ϕ yields a homotopy in X_0 of $\phi \circ \gamma$ with the constant map at $\phi(b)$. □

The lemma implies that there is an induced isomorphism $\overline{\phi}_{1*}\colon G_0 \to G_0$ since one has

(17)
$$\begin{array}{ccccccccc}
1 & \longrightarrow & N_1 & \longrightarrow & G_1 & \xrightarrow{\iota_*} & G_0 & \longrightarrow & 1 \\
\big\downarrow & & \phi_{1*}\big\downarrow & & \phi_{1*}\big\downarrow & & \overline{\phi}_{1*}\big\downarrow & & \big\downarrow \\
1 & \longrightarrow & N_1 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & 1
\end{array}$$

Since the middle vertical map is determined only up to inner automorphisms of G_1 (which preserve N_1), the induced map $\overline{\phi}_{1*}$ is determined only up to inner automorphisms of G_0.

Lemma 4.2. *Two maps f, $g \in \mathcal{B}$ are Thurston equivalent via a pair ϕ_0, ϕ_1 if and only if* 1. *the diagram*

$$\begin{array}{ccc}
G_1 & \xrightarrow{\phi_{1*}} & G_1 \\
f_*\big\downarrow & & \big\downarrow g_* \\
G_0 & \xrightarrow[\phi_{0*}]{} & G_0
\end{array}$$

commutes up to pre- and postcomposition by inner automorphisms, and 2. $\overline{\phi}_{1*} = \phi_{0*}$, *up to inner automorphisms. The three inner automorphisms are allowed to vary independently.*

Proof. Necessity is clear, by the preceding discussion.

To prove sufficiency, recall that two self-homeomorphisms of a surface (orientable, homeomorphic to a compact surface minus a finite set of points) are isotopic if and only if they induce the same map on fundamental group, up to inner automorphisms (combine [7], Thms. 3.7.3 and 5.13.1). Suppose ϕ_0, ϕ_1 are orientation-preserving homeomorphisms for which (1) and (2) hold. Condition (2) implies that ϕ_0 is isotopic to ϕ_1 through homeomorphisms agreeing on P_f. Condition (1) implies that there exists a lift $\tilde{\phi}_0$ such that $\phi_0 \circ f = g \circ \tilde{\phi}_0$. Passing to maps on fundamental groups, and using the fact that g_* is injective, we have $\tilde{\phi}_{0*} = g_*^{-1} \circ \phi_{0*} \circ f_*$. By condition (2), this in turn is ϕ_{1*}. So ϕ_1 is isotopic to $\tilde{\phi}_0$ through maps agreeing on P_f. Hence, $(\phi_0, \tilde{\phi}_0)$ is a pair of maps yielding a Thurston equivalence between f and g. □

Lemma 4.3. *Two branched covers f, g are Hurwitz equivalent via a pair ϕ_0, ϕ_1 if and only if, up to inner automorphisms, $[\phi^d_{1*}, \mathrm{id}_{\mathcal{S}_d}] \circ \hat\rho_f = \hat\rho_g \circ \phi_{0*}$, i.e.*

$$\begin{array}{ccc} G_0 & \xrightarrow{\hat\rho_f} & G_0^d \rtimes \mathcal{S}_d \\ {\scriptstyle \phi_{0*}}\big\downarrow & & \big\downarrow{\scriptstyle [\overline{\phi}^d_{1*}, \mathrm{id}_{\mathcal{S}_d}]} \\ G_0 & \xrightarrow[\hat\rho_g]{} & G_0^d \rtimes \mathcal{S}_d \end{array}$$

commutes.

Proof. It will be notationally convenient to put $g = f'$.

The sufficiency is clear: the assumption $[\phi^d_{1*}, \mathrm{id}_{\mathcal{S}_d}] \circ \hat\rho_f = \hat\rho_{f'} \circ \phi_{0*}$ implies $\sigma_f = \sigma_{f'} \circ \phi_{0*}$, and therefore the hypothesis of Theorem 2.1 is satisfied.

To prove the necessity, arrange by suitable conjugations for the diagram in Lemma 4.2 to commute. Let $H = f_*(G_1)$, $H' = f'_*(G_1)$, $N = f_*(N_1)$, and $N' = f'_*(N_1)$. Since $\phi_{1*}(N_1) = N_1$ (Lemma 4.1) we have $N' = \phi_{0*}(N)$. Thus there is an induced map $\hat\phi_{0*} \colon N \setminus H \to N' \setminus H'$ making the diagrams

$$\begin{array}{ccc} H & \xrightarrow{\phi_{0*}|H} & H' \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q'} \\ N \setminus H & \xrightarrow[\hat\phi_{0*}]{} & N' \setminus H' \end{array} \tag{18}$$

and

$$\begin{array}{ccc} G_0 & \xrightarrow{\overline{\phi}_{1*}} & G_0 \\ {\scriptstyle \overline{f}_*}\big\downarrow & & \big\downarrow{\scriptstyle \overline{f'}_*} \\ N \setminus H & \xrightarrow[\hat\phi_{0*}]{} & N' \setminus H' \end{array} \tag{19}$$

commute (erect diagram (17) on (14)).

Pick coset representatives g_i of H in G_0, and define coset representatives g'_i of H' in G_0 by $g'_i = \phi_{0*}(g_i)$. Applying ϕ_{0*} to the equation

$$g_i g = h(g, i) g_{i.\sigma(g)},$$

substituting $g = \phi_{0*}^{-1}(\overline{g})$, and then writing g for $\overline{g}$ yields

$$g'_i g = \phi_{0*}(h(\phi_{0*}^{-1}(g), i)) g'_{i.\sigma(\phi_{0*}^{-1}(g))} = h'(g, i) g'_{i.\sigma'(g)}.$$

Thus by uniqueness and the definition of $h'(g, i)$ we have

$$h'(g, i) = \phi_{0*} h(\phi_{0*}^{-1}(g), i), \quad \sigma'(g) = \sigma \circ \phi_{0*}^{-1}(g).$$

It follows from the definitions of ρ, ρ' that

$$\rho'(g) = [\phi_{0*}^d, \mathrm{id}_{\mathcal{S}_d}] \circ \rho(g) \circ \phi_{0*}^{-1}(g) \tag{20}$$

i.e. $\rho' \circ \phi_{0*} = [\phi_{0*}^d, \mathrm{id}_{\mathcal{S}_d}] \circ \rho$.

Equations (18), (19), and (20) show that

$$\begin{array}{ccccccc} G_0 & \xrightarrow{\rho} & H^d \rtimes \mathcal{S}_d & \xrightarrow{[q^d,\mathrm{id}]} & (N \setminus H)^d \rtimes \mathcal{S}_d & \xleftarrow{[\overline{f}_*^d,\mathrm{id}]} & G_0^d \rtimes \mathcal{S}_d \\ \phi_{0*}\downarrow & & [\phi_{0*}^d,\mathrm{id}]\downarrow & & [\hat{\phi}_{0*}^d,\mathrm{id}]\downarrow & & [\overline{\phi}_{1*}^d,\mathrm{id}]\downarrow \\ G_0 & \xrightarrow{\rho'} & H'^d \rtimes \mathcal{S}_d & \xrightarrow{[q'^d,\mathrm{id}]} & (N' \setminus H')^d \rtimes \mathcal{S}_d & \xleftarrow{[\overline{f'}_*^d,\mathrm{id}]} & G_0^d \rtimes \mathcal{S}_d \end{array}$$

commutes. Applying the definition of $\hat{\rho}$, $\hat{\rho}'$ as the composition from left to right of the top and bottom, respectively, completes the proof. □

Proof of Theorem 4.1. As before it is convenient to set $f' = g$.

The necessity follows immediately from Lemmas 4.2 and 4.3.

We now prove sufficiency. The assumption $\hat{\rho}_{f'} \circ \phi_{0*} = [\phi_{0*}^d, \mathrm{id}_{\mathcal{S}_d}] \circ \hat{\rho}_f$ implies that $\sigma_f = \sigma_{f'} \circ \phi_{0*}$. Theorem 2.1 then implies that there exists a homeomorphism ϕ_1 for which ϕ_0, ϕ_1 yields a Hurwitz equivalence between f and f'.

By Lemma 4.2 it is enough to show $\overline{\phi}_{1*} = \phi_{0*}$. Let $x \in G_0$ be arbitrary. Then by surjectivity (see the note at the end of §3) there exists $g \in G_0$ such that

$$\hat{\rho}(g) = [(x, \ldots), \cdot].$$

Focusing just on the first coordinate in the G_0^d factor, by hypothesis we have

$$\hat{\rho}' \circ \phi_{0*}(g) = [(\phi_{0*}(x), \ldots), \cdot]$$

and by Lemma 4.3 we have

$$\hat{\rho}' \circ \phi_{0*}(g) = [(\overline{\phi}_{1*}(x), \ldots), \cdot].$$

Equating the two we have $\phi_{0*}(x) = \overline{\phi}_{1*}(x)$ for all $x \in G_0$, and the theorem is proved. □

References

[1] I. Berstein and A.L. Edmonds: *On the construction of branched coverings of low-dimensional manifolds*, Trans. Amer. Math. Soc. **247** (1979), 87–124.

[2] E. Brezin, R. Byrne, J. Levy, K. Pilgrim and K. Plummer: *A census of rational maps*, Conform. Geom. Dyn. **4** (2000), 35–74.

[3] A. Douady and J. Hubbard: *A proof of Thurston's topological characterization of rational functions*, Acta. Math. **171** (1993), 263–297.

[4] A. Kameyama: *The Thurston equivalence for postcritically finite branched coverings*, Osaka J. Math. **38** (2001), 565–610.

[5] W. Massey: Algebraic Topology: An Introduction, Springer Verlag, Graduate Texts in Mathematics, New York, 1987.

[6] D.J.S. Robinson: A course in the theory of groups, Springer Verlag, New York, second edition, 1996.

[7] H. Zieschang, E. Vogt and H.-D. Coldewey: Surfaces and planar discontinuous groups, Springer Verlag, Berlin, 1980.

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