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THE 3-MOVE AND KNOTTED 4-VALENT GRAPHS IN 3-SPACE

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1. Introduction

A topological graph is a one-dimensional complex consisting of finitely many 0-cells (vertices) and finitely many 1-cells (edges and loops). In [7], Kauffman proved that piecewise linear ambient isotopy of a piecewise linear embedding of a topological graph in Euclidean 3-space $\mathbb{R}^3$ or 3-sphere S^3, referred simply a *knotted graph*, is generated by a set of diagrammatic local moves (see Fig. 1) that generalize the Reidemeister moves for diagrams of classical links. This gives a complete combinatorial description of the topology of graphs in three dimensional space. Throughout this paper, all spaces and maps are in piecewise linear category and we speak of 3-space in referring to either $\mathbb{R}^3$ or $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

A method for producing invariants of knotted graphs in 3-space is to associate a collection of links to the knotted graph [7, 13] and also a polynomial invariant for knotted graphs is developed [16]. On the other hand, ambient isotopy of knotted graphs is rather complicated by the fact that the generalized Reidemeister move (V) (see Fig. 1) creates or destroys arbitrary braiding at a vertex and so it is not easy to define non trivial invariants of the braiding move (V). For this reason, many authors turned their attention to restrict the valency of vertices and the allowed movement in the neighborhoods of vertices. This makes the construction of invariants of such graphs rather easier [1, 5, 7, 8, 13, 14, 15, 18].

The purpose of this paper is to introduce a method for obtaining invariants of the braiding move (V) and consequently producing invariants of knotted 4-valent graphs, by using the 3-move for knots and links.

This paper is organized as follows. Section 2 contains fundamental concepts for graph embeddings in 3-space. In Section 3 we associate a collection of knots and links to a knotted 4-valent graph in 3-space and show that the 3-equivalent class of the collection is an invariant of the knotted 4-valent graphs. In Section 4 we construct new 3-move invariants by using Kauffman bracket polynomial and show that this 3-move invariant gives a useful way to distinguish knotted 4-valent graphs in 3-space.

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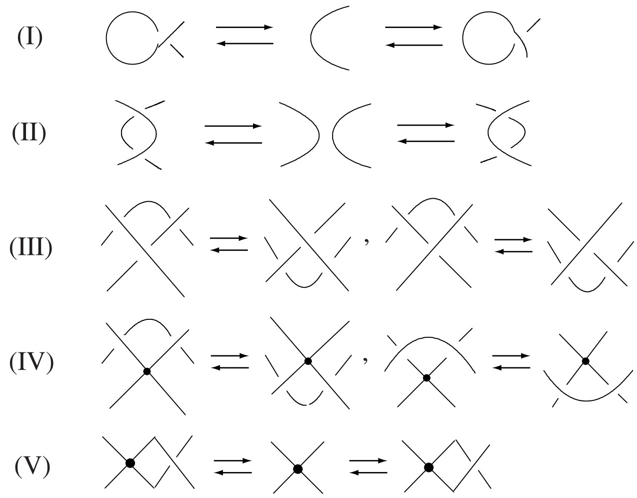


Fig. 1. The Reidemeister moves for knotted 4-valent graphs

2. Knotted graphs in 3-space

A *topological graph* is a 1-dimensional cell complex consisting of finitely many 0-cells (*vertices*) and finitely many 1-cells (*edges or loops*). Each edge is homeomorphic to a closed line segment, and its ends are vertices in the graph. A topological graph G is said to be *k-valent* if the number of arcs incident with each vertex is equal to k. Throughout this paper, a graph means a 4-valent topological graph and a *knotted graph* means an embedding of a 4-valent topological graph into $\mathbb{R}^3$ otherwise specified.

Two knotted graphs $\mathcal{G}$ and $\mathcal{G}'$ are said to be *equivalent* (or *ambient isotopic*) if there exists an orientation preserving homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(\mathcal{G}) = \mathcal{G}'$. Then it is well known that two knotted graphs are equivalent if and only if their graph diagrams can be transformed to each other by a finite sequence of the Reidemeister moves (I), (II), (III), (IV) and (V) as shown in Fig. 1 [5, 7].

A *rigid vertex* 4*-valent graph* (briefly, $RV4$ graph) is a 4-valent graph whose vertices are replaced by rigid 2-disks or 3-balls. Each disk or ball has four strands attached to it. A *knotted* $RV4$ *graph* means an embedding of a $RV4$ graph into $\mathbb{R}^3$. A rigid vertex ambient isotopy of a knotted $RV4$ graph $\mathcal{G}$ is a combination of topological ambient isotopies of the strands corresponding to the edges of $\mathcal{G}$ relative to the end points on the rigid disks, coupled with affine motions of the disks carrying along the strands in ambient isotopy. Two knotted $RV4$ graphs are *RV equivalent* (or *RV ambient isotopic*) if their graph diagrams are transformed to each other by a finite sequence of Reidemeister moves (I), (II), (III), (IV) of Fig. 1 and the move (V*) as shown in Fig. 2 [5, 7].

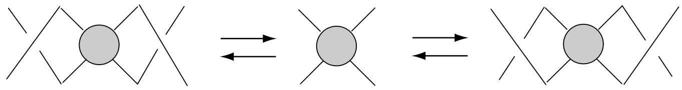
(V*) 

Fig. 2. Braiding move for knotted $RV4$ graphs

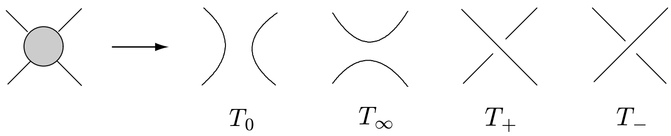


Fig. 3. Rigid vertex connection replacements

3. The 3-equivalent class of links in a knotted graph

In [7], Kauffman associated a collection $C(\mathcal{G})$ of links to each knotted $RV4$ graph $\mathcal{G}$ and showed that the ambient isotopy class of $C(\mathcal{G})$ is an invariant of the RV equivalence of the graph $\mathcal{G}$. An element of $C(\mathcal{G})$ is obtained by making a connection at each vertex, replacing the vertex locally by a configuration that connects the four edges in pairs. There are four ways to do this as shown in Fig. 3. In practice, the ambient isotopy class of $C(\mathcal{G})$ is very useful to distinguish knotted $RV4$ graphs in 3-space.

In the case of a topological vertex graph $\mathcal{G}$, however, the ambient isotopy class of $C(\mathcal{G})$ is not an invariant of the (topological vertex) equivalence of the graph $\mathcal{G}$ because the braiding move (V) may change the ambient isotopy type of a link in $C(\mathcal{G})$. This section is devoted to show that if we take the 3-equivalence class of $C(\mathcal{G})$, then it is an invariant of the knotted graph $\mathcal{G}$.

Let $\mathcal{G}$ be a knotted graph with the vertex set $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$ $(n \geq 0)$ and let D be a diagram of $\mathcal{G}$. Let $\mathcal{T} = \{T_0, T_\infty, T_+, T_-\}$, where T_0, T_∞, T_+ and T_- are 4-tangle diagrams as shown in Fig. 3, and let $f\colon V(\mathcal{G}) \to \mathcal{T}$ be an assignment of a member $f(v_j)$ in $\mathcal{T}$ for each vertex v_j of $\mathcal{G}$. Note that there are 4^n assignments of $\mathcal{G}$. We denote all such assignments of $\mathcal{G}$ by $f_1, f_2, \dots, f_{4^n}$ and let $F(\mathcal{G}) = \{f_1, f_2, \dots, f_{4^n}\}$. For each assignment $f_j \in F(\mathcal{G})$, let (D, f_j) denote the knot or link diagram obtained from D by replacing all vertices of $\mathcal{G}$ as shown in Fig. 4 in accordance with the assignment f_j.

Let $\mathcal{C}(D)$ denote the collection of all 4^n link diagrams (D, f_j) associated to D, i.e., $\mathcal{C}(D) = \{(D, f_j) \mid 1 \leq j \leq 4^n\}$. If $|V(\mathcal{G})| = 0$, then we define $\mathcal{C}(D) = \{D\}$.

Let L be a link diagram. Then the +3-*move* and the -3-*move* are local changes in L as shown in the following Figure:

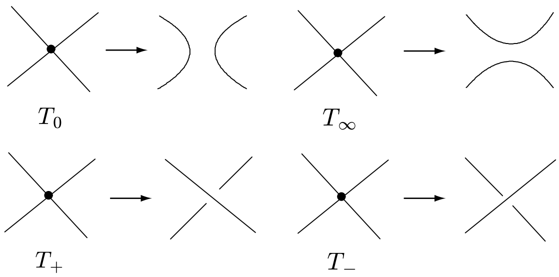


Fig. 4. Vertex connection replacements

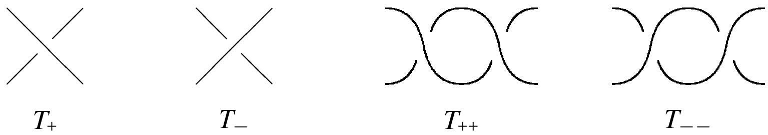


Fig. 5.

DEFINITION 3.1. Two links l and l' are said to be 3-*equivalent* if their diagrams can be transformed to each other by a finite sequence of Reidemeister moves (I), (II), (III) of Fig. 1, the +3-move, the -3-move and their inverses.

Then we have the following easy lemma.

Lemma 3.2. *Let T_+, T_-, T_{++} and T_{--} be four link diagrams that are identical except a small neighborhood where they are as shown in* Fig. 5. *Then T_+ and T_- are* 3-*equivalent to T_{--} and T_{++}, respectively.*

Theorem 3.3. *Let $\mathcal{G}$ be a knotted graph and let D and D' be any two diagrams of $\mathcal{G}$. Then there exists a permutation σ on the set $\{1, 2, \dots, 4^n\}$ such that the link (D, f_j) is* 3-*equivalent to the link $(D', f_{\sigma(j)})$ for each $j = 1, 2, \dots, 4^n$.*

Proof. Let $D = D_0, D_1, \dots, D_{m-1}, D_m = D'$ be a sequence of graph diagrams connecting D and D', where D_i is obtained from D_{i-1} by applying exactly one of the moves (I), (II), (III), (IV) and (V). Let $v_1, v_2, \dots, v_n$ be the vertices of $\mathcal{G}$ and let $F(\mathcal{G}) = \{f_j \mid 1 \leq j \leq 4^n\}$ as above. For each pair (i, j), $1 \leq i \leq m$, $1 \leq j \leq 4^n$, we denote by D_{ij} the knot or link (D_i, f_j). For each $i = 1, 2, \dots, m$, define a permutation σ_i on $\{1, 2, \dots, 4^n\}$ such that the links D_{i-1j} and $D_{i\sigma_i(j)}$ are 3-equivalent for each $j = 1, 2, \dots, 4^n$ as follows:

CASE I. D_i is obtained from D_{i-1} by applying the Reidemeister move (I), (II), (III), or (IV). Then it is clear that the moves (I), (II), and (III) do not affect vertex connection replacements. So D_{i-1j} and D_{ij} are ambient isotopic for each j =

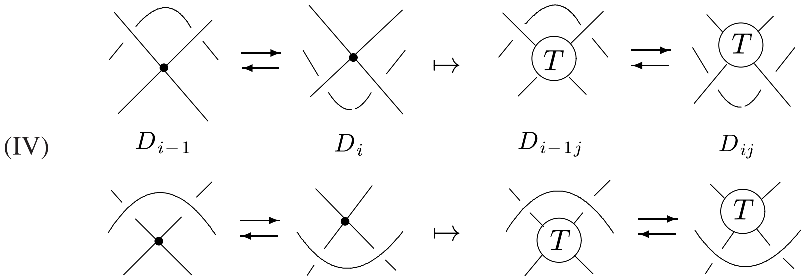


Fig. 6. Reidemeister move (IV)

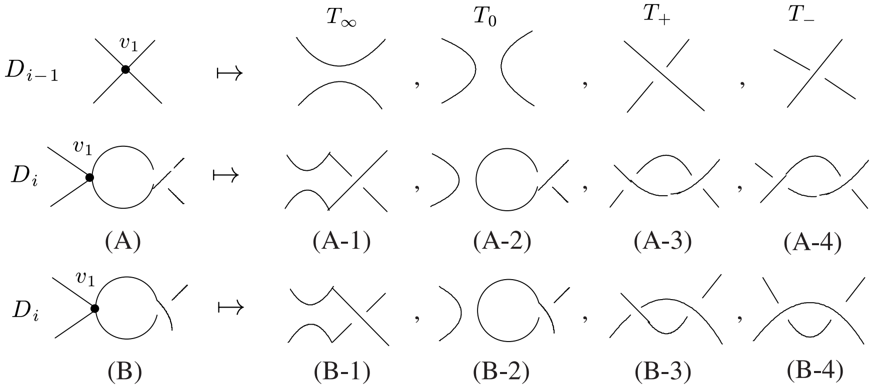


Fig. 7. Reidemeister move (V)

$1, 2, \ldots, 4^n$. On the other hand, the Fig. 6 illustrates a vertex connection replacement at a vertex by a tangle $T \in \mathcal{T}$ and the effect of the move (IV). This shows that the links D_{i-1j} and D_{ij} are ambient isotopic for each $j = 1, 2, \ldots, 4^n$. In this case, we define σ_i to be the identity permutation.

CASE II. D_i is obtained from D_{i-1} by applying the Reidemeister move (V). We may assume that the move (V) is accomplished at the vertex v_1 without loss of generality. Fig. 7 shows all possible vertex connection replacements in the diagram D_{i-1} and the corresponding replacements in the diagram D_i at the vertex v_1.

For the type (A) of Reidemeister move (V) in Fig. 7, we observe that T_∞ and (A-3) are ambient isotopic by Reidemeister move (II), T_0 and (A-2) are ambient isotopic by Reidemeister move (I), T_- and (A-1) are plane isotopic, and T_+ is 3-equivalent to (A-4) by Lemma 3.2. For the type (B), T_∞ and (B-4) are ambient isotopic by Reidemeister move (II), T_0 and (B-2) are ambient isotopic by Reidemeister move (I), T_+ and (B-1) are plane isotopic, and T_- is 3-equivalent to (B-3) by Lemma 3.2.

Now for each $f_j \in F(\mathcal{G})$, let $f'_j \colon V(\mathcal{G}) \to \mathcal{T}$ be an assignment of $\mathcal{G}$ defined by

$f'_j(v_k) = f_j(v_k)$ for $2 \leq k \leq n$ and

$$f'_j(v_1) = \begin{cases} T_+ & \text{if } f_j(v_1) = T_\infty, \\ T_0 & \text{if } f_j(v_1) = T_0, \\ T_- & \text{if } f_j(v_1) = T_+, \\ T_\infty & \text{if } f_j(v_1) = T_-. \end{cases}$$

Then $f'_j \in F(\mathcal{G})$ and it follows from the above observation that the mapping $g\colon F(\mathcal{G}) \to F(\mathcal{G})$ defined by $g(f_j) = f'_j$ for all $f_j \in F(\mathcal{G})$ is bijective and so it induces the desired permutation σ_i on $\{1, 2, \dots, 4^n\}$. Similarly, we can obtain a permutation σ_i for the type (B).

Finally, define $\sigma = \sigma_m \sigma_{m-1} \cdots \sigma_1$. Then $(D, f_j) = (D_0, f_j)$ is 3-equivalent to $(D_m, f_{\sigma(j)}) = (D', f_{\sigma(j)})$ for each $j = 1, 2, \dots, 4^n$. This completes the proof. $\square$

Two collections X_1 and X_2 of links are said to be 3-*equivalent* if every member of X_1 is 3-equivalent to some member of X_2 and vice versa. The following corollary is an immediate consequence of Theorem 3.3.

Corollary 3.4. *Let $\mathcal{G}$ be a knotted graph and let D be a diagram of $\mathcal{G}$. Then the* 3-*equivalent class $\mathcal{C}_3(\mathcal{G})$ of the collection $\mathcal{C}(\mathcal{D})$ is an invariant of $\mathcal{G}$.*

EXAMPLE 3.5. Let $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ be knotted graphs as shown in Fig. 8. Then

$$\mathcal{C}_3(\mathcal{G}_1) = \{U_1, U_2\},\ \ \mathcal{C}_3(\mathcal{G}_2) = \{U_2, U_3\},\ \ \mathcal{C}_3(\mathcal{G}_3) = \{U_1, U_2\},\ \ \mathcal{C}_3(\mathcal{G}_4) = \{U_1, U_2, U_3\},$$

where U_n denotes the unlink with n trivial components. Since $\mathcal{C}_3(\mathcal{G}_1)$ and $\mathcal{C}_3(\mathcal{G}_2)$ are not 3-equivalent, $\mathcal{G}_1$ and $\mathcal{G}_2$ are not equivalent and hence $\mathcal{G}_2$ is knotted. Similarly, $\mathcal{G}_3$ and $\mathcal{G}_4$ are not equivalent.

4. 3-move invariants and invariants of knotted graphs

An invariant $\mathcal{I}$ of links is called a 3-*move invariant* if $\mathcal{I}(L) = \mathcal{I}(L')$ for any two 3-equivalent knots or links L and L'. Now let I_3 be a numerical or more generally commutative ring valued 3-move invariant of links. Then it may be extended to an invariant of a knotted graph $\mathcal{G}$ by taking a suitable summation in terms of all values of links associated to the graph $\mathcal{G}$. The simplest such an example can be obtained by the way:

Let $\mathcal{G}$ be a knotted graph and let D be a diagram of $\mathcal{G}$. Then it follows from Theorem 3.3 that the value $I_3(\mathcal{G})$ defined by

$$I_3(\mathcal{G}) = \sum_{j=1}^{4^n} I_3((D, f_j))$$

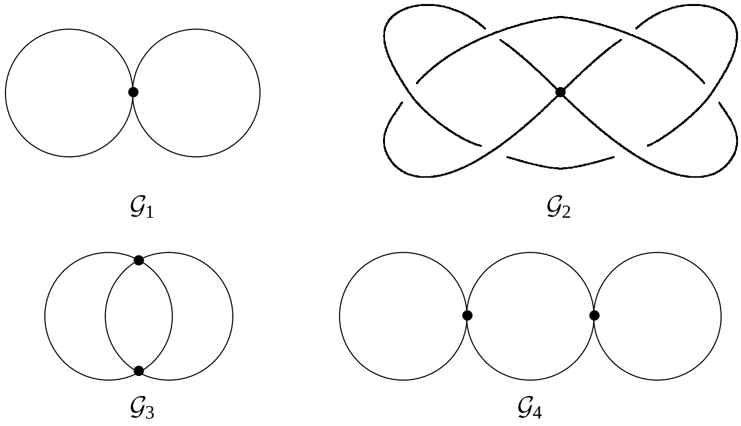


Fig. 8.

is an invariant of the knotted graph $\mathcal{G}$. This invariant is more useful than the 3-equivalence class $\mathcal{C}_3(\mathcal{G})$. In this section we shall discuss two examples of this type.

Let l be a link in S^3. Let $\mathcal{M}_n(l)$ denote the n-fold cyclic branched cover of S^3 branched along l and $H_1(\mathcal{M}_n(l); G)$ the first homology group of $\mathcal{M}_n(l)$ with coefficients in an Abelian group G.

DEFINITION 4.1. Let $\mathcal{G}$ be a knotted graph with n vertices, let D be a diagram of $\mathcal{G}$, and let $F(\mathcal{G}) = \{f_1, f_2, \dots, f_{4^n}\}$ be the set of all assignments of $\mathcal{G}$. Let l_j denote the link in S^3 represented by the diagram (D, f_j). Then we define two integers $\rho_1(\mathcal{G})$ and $\rho_2(\mathcal{G})$ for $\mathcal{G}$ by

$$\rho_1(\mathcal{G}) = \sum_{j=1}^{4^n} \operatorname{Dim} H_1(\mathcal{M}_2(l_j); \mathbb{Z}_3), \quad \rho_2(\mathcal{G}) = \sum_{j=1}^{4^n} \operatorname{Dim} H_1(\mathcal{M}_3(l_j); \mathbb{Z}_2).$$

Let l be any unoriented link in S^3 of μ components and let $\overline{l}$ denote an oriented link with underlying unoriented link l. Let $V_{\overline{l}}(t)$ and $P_{\overline{l}}(a, z)$ denote the Jones polynomial [4] and the skein polynomial [3] of $\overline{l}$, respectively, and $Q_l(t)$ the Q-polynomial invariant of the unoriented link l [2].

Theorem 4.2. *Let $\mathcal{G}$ be a knotted graph and let D be a diagram of $\mathcal{G}$. Then $\rho_1(\mathcal{G})$ and $\rho_2(\mathcal{G})$ are invariants of $\mathcal{G}$ and*

$$\rho_1(\mathcal{G}) = 2\log_3\left(\prod_{j=1}^{4^n}\left|P_{\overline{l}_j}(e^{\pi i/6}, 1)\right|\right) = 2\log_3\left(\prod_{j=1}^{4^n}\left|V_{\overline{l}_j}(e^{\pi i/3})\right|\right) \tag{4.1}$$

$$= \log_3 \left(\prod_{j=1}^{4^n} \left| Q_{l_j}(-1) \right| \right), \tag{4.2}$$

$$\rho_2(\mathcal{G}) = 2 \log_2 \left(\prod_{j=1}^{4^n} \left| P_{\overline{l}_j}(1,1) \right| \right), \tag{4.3}$$

where $\overline{l_j}$ denotes an oriented link with underlying unoriented link l_j represented by the diagram (D, f_j) and $i = \sqrt{-1}$.

Proof. It is well known that the 3-moves preserve the groups $H_1(\mathcal{M}_2(l); \mathbb{Z}_3)$ and $H_1(\mathcal{M}_3(l); \mathbb{Z}_2)$ [10, 11, 12]. Therefore the dimensions $\mathrm{Dim}\, H_1(\mathcal{M}_2(l); \mathbb{Z}_3)$ and $\mathrm{Dim}\, H_1(\mathcal{M}_3(l); \mathbb{Z}_2)$ are 3-move invariants. It follows immediately from Theorem 3.3 that $\rho_1(\mathcal{G})$ and $\rho_2(\mathcal{G})$ are invariants of $\mathcal{G}$.

To prove (4.1), let $\overline{l}$ be an oriented link in S^3 of μ components. By [10],

$$P_{\overline{l}}(e^{\pi i/6}, 1) = V_{\overline{l}}(e^{\pi i/3}) = \pm i^{\mu-1} (i\sqrt{3})^{\mathrm{Dim}\, H_1(\mathcal{M}_2(l); \mathbb{Z}_3)}.$$

So $\mathrm{Dim}\, H_1(\mathcal{M}_2(l); \mathbb{Z}_3) = 2 \log_3 \left| P_{\overline{l}}(e^{\pi i/6}, 1) \right| = 2 \log_3 \left| V_{\overline{l}}(e^{\pi i/3}) \right|$. Hence

$$\begin{aligned} \rho_1(\mathcal{G}) &= \sum_{j=1}^{4^n} \mathrm{Dim}\, H_1(\mathcal{M}_2(l_j); \mathbb{Z}_3) = \sum_{j=1}^{4^n} 2 \log_3 \left| P_{\overline{l}_j}(e^{\pi i/6}, 1) \right| \\ &= 2 \log_3 \prod_{j=1}^{4^n} \left| P_{\overline{l}_j}(e^{\pi i/6}, 1) \right| = 2 \log_3 \prod_{j=1}^{4^n} \left| V_{\overline{l}_j}(e^{\pi i/3}) \right|. \end{aligned}$$

For (4.2), let l be an unoriented link in S^3. It is known that $Q_l(-1) = (-3)^{\mathrm{Dim}\, H_1(\mathcal{M}_2(l); \mathbb{Z}_3)}$ [2]. So $\mathrm{Dim}\, H_1(\mathcal{M}_2(l); \mathbb{Z}_3) = \log_3 |Q_l(-1)|$. Hence

$$\rho_1(\mathcal{G}) = \sum_{j=1}^{4^n} \mathrm{Dim} H_1(\mathcal{M}_2(l_j); \mathbb{Z}_3) = \sum_{j=1}^{4^n} \log_3 |Q_{l_j}(-1)| = \log_3 \prod_{j=1}^{4^n} |Q_{l_j}(-1)|.$$

To prove (4.3) let $\overline{l}$ be an oriented link in S^3 of μ components. By [10], $P_{\overline{l}}(1,1) = (-2)^{(1/2)\, \mathrm{Dim}\, H_1(\mathcal{M}_3(l); \mathbb{Z}_2)}$. So $\mathrm{Dim}\, H_1(\mathcal{M}_3(l);\, \mathbb{Z}_2) = 2 \log_2 |P_{\overline{l}}(1,1)|$. Hence

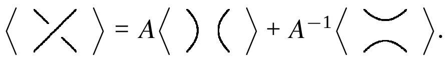
$$\rho_2(\mathcal{G}) = \sum_{j=1}^{4^n} \mathrm{Dim}\, H_1(\mathcal{M}_3(l_j); \mathbb{Z}_2) = \sum_{j=1}^{4^n} 2 \log_2 \left| P_{\overline{l}_j}(1,1) \right| = 2 \log_2 \prod_{j=1}^{4^n} \left| P_{\overline{l}_j}(1,1) \right|. \quad \square$$

Now we will construct new 3-move invariants of links by using Kauffman bracket polynomial and consequently give another numerical invariants of knotted graphs.

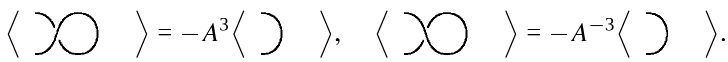
Let l be a link and let L be a diagram of l. The Kauffman bracket polynomial of L [6] is the Laurent polynomial $\langle L \rangle = \langle L \rangle(A) \in \mathbb{Z}[A, A^{-1}]$ defined by the following rules:

(i) $\langle \bigcirc \rangle = 1$,

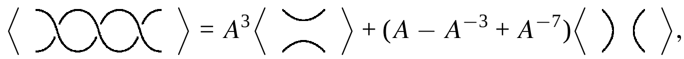
(ii) $\langle K \quad \bigcirc \rangle = (-A^2 - A^{-2})\langle K \rangle$,

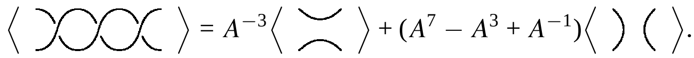
(iii) 

Note that the Kauffman bracket polynomial is a regular isotopy invariant and



So it is not an ambient isotopy invariant. Also, it is not invariant under the 3-moves since

(4.4) 

(4.5) 

Let $z_k = \cos(k\pi/12) + i\sin(k\pi/12)$, where $k = 1, 5, 7, 11, 13, 17, 19, 23$ and $i = \sqrt{-1}$. Then each z_k is a nonzero common root of the two equations $A - A^{-3} + A^{-7} = 0$ and $A^7 - A^3 + A^{-1} = 0$ or equivalently, $A^8 - A^4 + 1 = 0$. Substituting z_k in the Kauffman bracket polynomial $\langle L \rangle$, we get a regular isotopy invariant $\langle L \rangle_k$ of L:

$$\langle L \rangle_k = \langle L \rangle|_{A=z_k}.$$

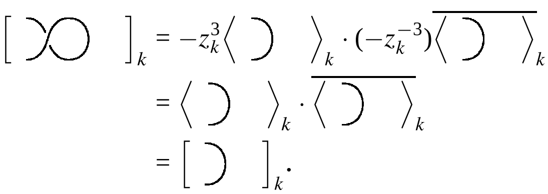
DEFINITION 4.3. Let L be a link diagram. For each $k = 1, 5, 7, 11, 13, 17, 19$, and 23, we define a real number $[L]_k \in \mathbb{R}$ by

$$[L]_k = \langle L \rangle_k \overline{\langle L \rangle}_k \tag{4.6}$$

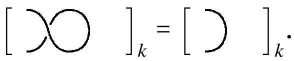
where $\overline{\langle L \rangle} = \langle L \rangle|_{A=A^{-1}}$ is a polynomial obtained from $\langle L \rangle(A)$ by interchanging A and A^{-1}.

Theorem 4.4. *Let L be a link diagram. Then for each $k = 1, 5, 7, 11, 13, 17$, 19, and 23, the real number $[L]_k$ is a 3-move invariant of knots and links.*

Proof. It is obvious that $[L]_k$ is a regular isotopy invariant. We observe that

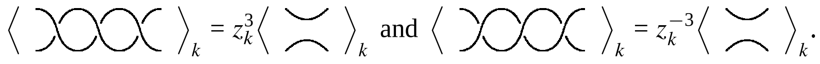


Similarly,

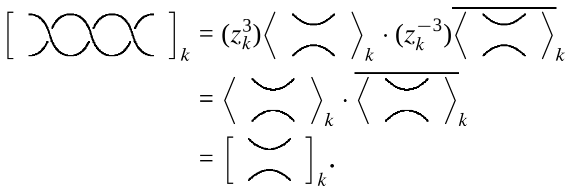


So $[L]_k$ is an ambient isotopy invariant.

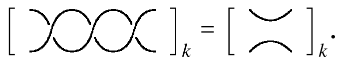
Since $z_k - z_k^{-3} + z_k^{-7} = 0$ and $z_k^7 - z_k^3 + z_k^{-1} = 0$, it follows from (4.4) and (4.5) that



So



Similarly,



Therefore $[L]_k$ is invariant under the 3-moves. This completes the proof. □

From Theorem 3.3 and Theorem 4.4, we obtain immediately the following numerical invariant of knotted graphs:

Theorem 4.5. *Let $\mathcal{G}$ be a knotted graph with n vertices and let D be a diagram of $\mathcal{G}$. For each $k = 1, 5, 7, 11, 13, 17, 19,$ and 23, define a real number $[\mathcal{G}]_k$ by*

$$[\mathcal{G}]_k = \sum_{j=1}^{4^n} [(D, f_j)]_k.$$

Then $[\mathcal{G}]_k$ is an invariant of $\mathcal{G}$ for each k.

EXAMPLE 4.6. Let $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$ be knotted graphs of Example 3.5. For $k = 1$, i.e., $z_1 = \cos(\pi/12) + i \sin(\pi/12)$, we obtain that

$$\begin{aligned}
&\rho_1(\mathcal{G}_1) = 1,\ \ \rho_2(\mathcal{G}_1) = 2,\ \ [\mathcal{G}_1]_1 = 6,\\
&\rho_1(\mathcal{G}_2) = 5,\ \ \rho_2(\mathcal{G}_2) = 10,\ \ [\mathcal{G}_2]_1 = 27,\\
&\rho_1(\mathcal{G}_3) = 4,\ \ \rho_2(\mathcal{G}_3) = 8,\ \ [\mathcal{G}_3]_1 = 24,\\
&\rho_1(\mathcal{G}_4) = 8,\ \ \rho_2(\mathcal{G}_4) = 16,\ \ [\mathcal{G}_4]_1 = 36.
\end{aligned}$$

This shows that the invariants ρ_1, ρ_2 and $[\ \ ]_1$ distinguish all graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_3$ and $\mathcal{G}_4$.

Final remarks. (1) Let D and D' be knotted graph diagrams with m, n vertices, respectively. Then $D \sqcup D'$ denotes the disjoint union of D and D' and $D \sharp D'$ denotes a connected sum of D and D' obtained by removing a small arc, not including vertices, from each diagram and then connecting the four endpoints by two new arcs without further crossing as a connected sum of two link diagrams. Connected sum of two knotted graphs is not well defined in general. By the properties of the Jones polynomial and the Kauffman bracket polynomial for $L \sqcup L'$ and $L\sharp L'$ of two links L and L', we have the following formulas:

$$\begin{aligned}
&\rho_1(D \sqcup D') = 4^n \rho_1(D) + 4^m \rho_1(D') + 4^{m+n},\\
&\rho_1(D\sharp D') = 4^n \rho_1(D) + 4^m \rho_1(D'),\\
&\rho_2(D \sqcup D') = 4^n \rho_2(D) + 4^m \rho_2(D') + 4^{m+n},\\
&\rho_2(D\sharp D') = 4^n \rho_2(D) + 4^m \rho_2(D'),\\
&[D \sqcup D']_k = 3[D]_k[D']_k,\\
&[D\sharp D']_k = [D]_k[D']_k.
\end{aligned}$$

(2) A knotted surface is a closed and locally flat surface embedded in the Euclidean 4-space $\mathbb{R}^4$ or the 4-sphere S^4. In 1994, Yoshikawa [17] represents a knotted surface in 4-space by a knotted graph diagram with 4-valent labelled vertices, called a surface diagram, and introduces equivalence of surface diagrams. In [9], Lee defined three variable state-sum polynomial invariants of equivalent surface diagrams by using the invariants of Definition 4.3 for $A = \exp(k\pi\sqrt{-1}/6)$ for $k = 1, 2, 4, 5, 7, 8, 10, 11$ as state evaluation, which are modifications of the graph invariants $[\mathcal{G}]_k$ of Theorem 4.5. This shows that the complex number $(1/4)^{|V(\mathcal{G})|}[\mathcal{G}]_k$ evaluated at $A = \exp(k\pi\sqrt{-1}/6)$ is an ambient isotopy invariant of a knotted surface in 4-space $\mathbb{R}^4$ or S^4 represented by $\mathcal{G}$, where $|V(\mathcal{G})|$ denotes the number of the vertices of $\mathcal{G}$ [9].

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