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INTEGRAL GEOMETRY ON SUBMANIFOLDS OF DIMENSION ONE AND CODIMENSION ONE IN THE PRODUCT OF SPHERES

HONG JAE KANG

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1. Introduction

Let G be a Lie group and K a closed subgroup of G. Consider two submanifolds in a Riemannian homogeneous space G/K, one fixed and the other moving under g in G. Let the fixed one be M and the moving one be gN and let μ_G be the invariant measure on G. By taking the geometric invariant $\mathrm{vol}(M \cap gN)$, volume of the submanifold $M \cap gN$, and integrating with respect to $d\mu_G(g)$, we get so called the Poincaré formula. This can be briefly stated as follows.

Let M^p and N^q be submanifolds of dimensions p and q respectively, in a Riemannian homogeneous space G/K. Then many works in integral geometry have been concerned with computing integrals of the following form

$$\int_G \mathrm{vol}(M \cap gN)\, d\mu_G(g).$$

The Poincaré formula means equalities which represent the above integral by some geometric invariants of submanifolds M and N of G/K. For example in the case that G is the group of isometries of Euclidean space $\mathbf{R}^n$ and M and N are submanifolds of $\mathbf{R}^n$ then the result of above integral leads to formulas of Poincaré, Crofton and other integral geometers (see [6]). Especially R. Howard [1] obtained a Poincaré formula for Riemannian homogeneous spaces as follows:

Let M and N be submanifolds of G/K with $\dim M + \dim N = \dim(G/K)$. Assume that G is unimodular. Then

$$\int_G \sharp(M \cap gN)\, d\mu_G(g) = \iint_{M \times N} \sigma_K(T_x^\perp M, T_y^\perp N)\, d\mu_{M\times N}(x, y), \tag{1.1}$$

where $\sharp(X)$ denotes the number of elements in a set X and $\sigma_K(T_x^\perp M, T_y^\perp N)$ is defined by (2.1) in Section 2.

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The formula (1.1) holds under the general situation. However, it is difficult to give an explicit description through the concrete computation of $\sigma_K(T_x^\perp M, T_y^\perp N)$, and only a little is known about it ([2], [3], [7]). In the present paper, we attempt to explicitly describe this formula for submanifolds of dimension and codimension one in the product of arbitrary dimensional unit spheres. More precisely,

Theorem 1.1. *Let M be a submanifold of $S^{m+1} \times S^{n+1}$ of dimension 1 and N a submanifold of codimension 1. Assume that for almost all $g \in G$, M and gN intersect transversely. For any point $x \in M$* (resp. $y \in N$)*, $\sin\theta_x$ and $\cos\theta_x$* (resp. $\sin\tau_y, \cos\tau_y$) *denote length of the first and second component of unit vector $u_x = (u_1, u_2)$* (resp. $v_y = (v_1, v_2)$) *of T_xM* (resp. $T_y^\perp N$)*, respectively. Then we have*

$$\begin{aligned}&\int_{SO(m+2)\times SO(n+2)} \sharp(M \cap gN)\, d\mu_{SO(m+2)\times SO(n+2)}(g)\\ &= 2\,\mathrm{vol}\big(SO(m+1)\times SO(n+1)\big) \iint_{M\times N} \sigma(x,y)\, d\mu_{M\times N}(x,y)\end{aligned}$$

where

$$\sigma(x,y) = \begin{cases} \dfrac{c_{xy}\mathrm{vol}(S^{n-1})}{n\mathrm{vol}(S^n)} F\left(-\dfrac{1}{2}, -\dfrac{n}{2}, \dfrac{1+m}{2}, \dfrac{s_{xy}^2}{c_{xy}^2}\right), & \text{if } s_{xy} \le c_{xy};\\[2ex] \dfrac{s_{xy}\mathrm{vol}(S^{m-1})}{m\mathrm{vol}(S^m)} F\left(-\dfrac{1}{2}, -\dfrac{m}{2}, \dfrac{1+n}{2}, \dfrac{c_{xy}^2}{s_{xy}^2}\right), & \text{if } s_{xy} \ge c_{xy}.\end{cases}$$

Here $F(a,b,c;x)$ is the Gauss hypergeometric function, and $s_{xy} = \sin\theta_x \sin\tau_y$ and $c_{xy} = \cos\theta_x \cos\tau_y$.

2. Preliminaries

In this section we shall review the Poincaré formula on Riemannian homogeneous spaces given by R. Howard [1] and recall the Gauss hypergeometric function.

Let E be a finite dimensional real vector space with an inner product. For vector subspaces V and W with orthonormal bases $v_1, \dots, v_p$ and $w_1, \dots, w_q$ respectively, we define $\sigma(V, W)$ by

$$\sigma(V,W) = |v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q|.$$

This definition is independent of the choice of orthonormal bases. Furthermore, if $p + q = \dim E$ then

$$\sigma(V,W) = \sigma(V^\perp, W^\perp).$$

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also invariant under the right actions of elements of K. This metric induces a G-invariant Riemannian metric on G/K. We denote by o the origin of G/K. If $x,\ y \in G/K$ and V is a vector subspace of $T_x(G/K)$ and W is a vector subspace of $T_y(G/K)$ then define $\sigma_K(V, W)$ by

$$\sigma_K(V, W) = \int_K \sigma\big((dg_x)_o^{-1}V, dk_o^{-1}(dg_y)_o^{-1}W\big)\, d\mu_K(k) \tag{2.1}$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y in G such that $g_x o = x$ and $g_y o = y$.

We list here the basic properties of the Gauss hypergeometric function that are needed in this paper only. For further details see [4].

The Gauss hypergeometric series, convergent for $|z| < 1$, is given by the power series

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{z^n}{n!}, \tag{2.2}$$

where Γ is the gamma function. By analytic continuation $F(a, b, c; z)$ can be extended to define a function analytic and single-valued in the complex z plane cut along the positive real axis from 1 to ∞. We remark that above series reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n = 0,\ 1,\ 2, \ldots)$. The series (2.2) is not defined when c is equal to $-m$, $(m = 0,\ 1,\ 2, \ldots)$, provided a or b is not a negative integer n with $n < m$. The hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + \big(c - (a+b+1)z\big)\frac{du}{dz} - abu = 0$$

has the solution $u = F(a, b, c; z)$.

The six functions $F(a \pm 1, b, c; z)$, $F(a, b \pm 1, c; z)$ and $F(a, b, c \pm 1; z)$ are called contiguous to $F(a, b, c; z)$. Relations between $F(a, b, c; z)$ and any two contiguous functions have been given by Gauss. By repeated application of these relations the function $F(a+m, b+n, c+l; z)$ with integer $m,\ n,\ l$ can be expressed as a linear combination of $F(a, b, c; z)$ and one of its contiguous functions with coefficients which are rational functions of $a,\ b,\ c,\ z$. For examples,

$$\begin{aligned} azF(a+1, b+1, c+1; z) &= c\,[F(a, b+1, c; z) - F(a, b, c; z)]\,, \\ (c-1)F(a, b, c-1; z) &= (c-a-1)F(a, b, c; z) + aF(a+1, b, c; z). \end{aligned} \tag{2.3}$$

Among the special cases are

$$(1-z)^t = F(-t, b, b; z), \tag{2.4}$$

$$\arcsin x = xF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2\right). \tag{2.5}$$

Furthermore C.F. Gauss evaluated, for $\Re(c-a-b)>0$,

$$F(a,b,c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.6}$$

In this paper, we may consider only when z is a real number.

3. Proof of Theorem 1.1

Let S^m be the unit sphere of dimension m. The special orthogonal group $SO(m+1)$ acts transitively on S^m. The isotropy subgroup of $SO(m+1)$ at a point in S^m is $SO(m)$. Thus $S^{m+1}\times S^{n+1}$ can be realized as a homogeneous space $(SO(m+2)\times SO(m+2))/(SO(m+1)\times SO(m+1))$. We have set, to simplify notation,

$$G=SO(m+2)\times SO(m+2),\ \ K=SO(m+1)\times SO(m+1).$$

Let $\mathfrak{g}=\mathfrak{so}(m+2)\times\mathfrak{so}(m+2)$ be the Lie algebra of G. Define an inner product on $\mathfrak{g}$ by

$$(X,Y)=-\frac{1}{2}\operatorname{Trace}(XY)\qquad (X,\ Y\in\mathfrak{g}).$$

We extend this inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ to the left invariant Riemannian metric on G. Then we obtain a bi-invariant Riemannian metric on G. This bi-invariant Riemannian metric on G induces a G-invariant Riemannian metric on $G/K=S^{m+1}\times S^{n+1}$.

Let M be a submanifold of $S^{m+1}\times S^{n+1}$ of dimension one and N a submanifold of codimension one. By the formula (1.1), we have

$$\int_G \sharp(M\cap gN)\,d\mu_G(g)=\iint_{M\times N}\sigma_K(T_xM,T_yN)\,d\mu_{M\times N}(x,y). \tag{3.1}$$

Let $u_x=(u_1,u_2)$ and $v_y=(v_1,v_2)$ be unit vectors of T_xM and $T_y^{\perp}N$ respectively. By the action of K, we can transport u_x to $((\sin\theta_x,0,\dots,0),\ (\cos\theta_x,0,\dots,0))$ and v_y to $((\sin\tau_y,0,\dots,0),(\cos\tau_y,0,\dots,0))$ respectively. Let $e_1,\dots,e_{m+n+2}$ be the standard orthonormal basis of $\mathbf{R}^{m+n+2}$. Thus we can take

$$(-\cos\tau_y e_1+\sin\tau_y e_{m+2}),e_2,\dots,e_{m+1},e_{m+3},\dots,e_{m+n+2}$$

as an orthonormal basis of T_yN. We can simply write

$$\sigma(\theta_x,\tau_y)=\sigma_K(T_xM,T_yN),$$

since $\sigma_K(T_xM,T_yN)$ is dependent only on θ_x and τ_y. Then we have

$$\sigma(k^{-1}T_xM,T_yN)$$

$$
\begin{aligned}
&= |(\sin\theta_x e_1 + \cos\theta_x e_{m+2})k \wedge (-\cos\tau_y e_1 + \sin\tau_y e_{m+2}) \wedge \hat{e}_{1,m+2}| \\
&= |\sin\theta_x \sin\tau_x k_{11} + \cos\theta_y \cos\tau_y k_{m+2m+2}|,
\end{aligned}
$$

where

$$\hat{e}_{1,m+2} = e_2 \wedge \cdots \wedge e_{m+1} \wedge e_{m+3} \wedge \cdots \wedge e_{m+n+2},$$

and

$$k = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \in SO(m+1) \times SO(n+1).$$

For simplicity of writing we put $\sin\theta\sin\tau = s$ and $\cos\theta\cos\tau = c$. Then we get

$$\sigma(\theta,\tau) = \iint_K |sk_{11} + ck_{m+2m+2}|\, d\mu_K(k)$$

We now have to compute following:

$$\int_{SO(m+1)} \int_{SO(n+1)} |tk_{11} + g_{11}|\ d\mu_{SO(n+1)}(g)\, d\mu_{SO(m+1)}(k) \tag{3.2}$$

We here give the following lemma to compute the above integral.

Lemma 3.1. *If* $|\alpha| \leq 1$ *then*

$$\int_{S^n} |\alpha + x_1|\, d\mu_{S^n}(x) = 2\mathrm{vol}(S^{n-1}) \left\{ \frac{1}{n}\sqrt{1-\alpha^2}^{\,n} + \alpha^2 F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3}{2}; \alpha^2\right) \right\}.$$

Proof. Define a mapping ϕ: $(-1,1) \times S^{n-1} \to S^n$ by

$$\big(t, x = (x_1, x_2, \ldots, x_n)\big) \mapsto \left(t, \sqrt{1-t^2}\, x\right).$$

Using ϕ as a variable transformation we have

$$
\begin{aligned}
\int_{S^n} |\alpha - x_1|\, d\mu_{S^n}(x) &= \int_{-1}^{1} \int_{S^{n-1}} |\alpha - t| \sqrt{1-t^2}^{\,n-2}\, d\mu_{S^{n-1}}(x)\, dt \\
&= \mathrm{vol}(S^{n-1}) \int_{-1}^{1} |\alpha - t| \sqrt{1-t^2}^{\,n-2}\, dt.
\end{aligned}
$$

Here $\mathrm{vol}(S^n)$ is the surface area of the n-dimensional unit sphere and its value is

$$\mathrm{vol}(S^n) = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)},$$

where Γ denotes the gamma function. By a simple calculation, we have

$$\int \sqrt{1-x^2}^{\,n}\,dx = xF\left(\frac{1}{2}, -\frac{n}{2}, \frac{3}{2}; x^2\right).$$

Hence we obtain

$$\int_{-1}^{1} |\alpha - t| \sqrt{1-t^2}^{\,n-2}\,dt = \frac{2}{n}\sqrt{1-\alpha^2}^{\,n} + 2\alpha^2 F\left(\frac{1}{2}, 1-\frac{n}{2}; \frac{3}{2}; \alpha^2\right),$$

which implies Lemma 3.1. □

We first consider the case where $0 \le t \le 1$ in (3.2). Then we have

$$\begin{aligned}
&\int_{SO(n+1)} |tk_{11} + g_{11}|\,d\mu_{SO(n+1)}(g) \\
&= \mathrm{vol}(SO(n)) \int_{S^n} |tk_{11} + x_1|\,d\mu_{S^n}(x) && (3.3)\\
&= 2\mathrm{vol}(SO(n))\mathrm{vol}(S^{n-1}) \\
&\quad \times \left\{ \frac{1}{n}\sqrt{1-t^2(k_{11})^2}^{\,n} + t^2(k_{11})^2 F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3}{2}, t^2(k_{11})^2\right)\right\}. && (3.4)
\end{aligned}$$

Equality (3.3) follows from the fibering of $SO(n+1)$ over S^n with the fiber $SO(n)$, and (3.4) follows from Lemma 3.1.

Notice

$$\int_0^{\pi} \sin x\,dx \int_0^{\pi} \sin^2 x\,dx \cdots \int_0^{\pi} \sin^m x\,dx = \frac{\mathrm{vol}(S^{m+1})}{2\pi} = \frac{\mathrm{vol}(S^{m-1})}{m}, \tag{3.5}$$

then, using spherical coordinate transformation, the integral of the first term in (3.4) over $SO(m+1)$ is as follows:

$$\begin{aligned}
&\int_{SO(m+1)} \sqrt{1-t^2(k_{11})^2}^{\,n}\,d\mu_{SO(m+1)}(k) \\
&= \mathrm{vol}(SO(m)) \int_{S^m} \sqrt{1-t^2(x_1)^2}^{\,n}\,d\mu_{S^m}(x) \\
&= 2\,\mathrm{vol}(SO(m))\,\mathrm{vol}(S^{m-1}) \int_0^{\pi/2} \sqrt{1-t^2\cos^2\theta_1}^{\,n} \cdot \sin^{m-1}\theta_1\,d\theta_1 \\
&= 2\,\mathrm{vol}(SO(m))\,\mathrm{vol}(S^{m-1}) \int_0^{1} \sqrt{1-t^2x^2}^{\,n} \cdot \sqrt{1-x^2}^{\,m-2}\,dx.
\end{aligned}$$

In the last integral,

$$\int_0^1 \sqrt{1-x^2}^{\,m-2}\,dx$$

is an Euler beta function. Hence it can be evaluated in terms of gamma function, and we find that

$$\int_0^1 \sqrt{1-x^2}^{\,m-2}\,dx = \frac{\mathrm{vol}(S^m)}{2\,\mathrm{vol}(S^{m-1})}. \tag{3.6}$$

By (3.6) and integration by part, for any even number l we obtain

$$\int_0^1 x^l \sqrt{1-x^2}^{\,m-2}\,dx = \frac{(l-1)!!(m-1)!!}{(m+l-1)!!}\cdot\frac{\mathrm{vol}(S^m)}{2\,\mathrm{vol}(S^{m-1})},$$

where

$$m!! = \begin{cases} m(m-2)\cdots 4\cdot 2, & m:\text{ even};\\ m(m-2)\cdots 3\cdot 1, & m:\text{ odd}.\end{cases}$$

And by (2.4) we have

$$\sqrt{1-t^2x^2}^{\,n} = F\left(-\frac{n}{2},1,1;t^2x^2\right).$$

So we obtain

$$\int_0^1 \sqrt{1-t^2x^2}^{\,n}\sqrt{1-x^2}^{\,m-2}\,dx = \frac{\mathrm{vol}(S^m)}{2\,\mathrm{vol}(S^{m-1})}\cdot F\left(\frac{1}{2},-\frac{n}{2},\frac{1+m}{2};t^2\right).$$

Hence we have

$$\begin{aligned}&\int_{SO(m+1)} \sqrt{1-t^2(k_{11})^2}^{\,n}\,d\mu_{SO(m+1)}(k)\\ &= \mathrm{vol}(SO(m))\,\mathrm{vol}(S^m)F\left(\frac{1}{2},-\frac{n}{2},\frac{1+m}{2};t^2\right).\end{aligned} \tag{3.7}$$

On the other hand, the integral of the second term in (3.4) on $SO(m+1)$ is as follows:

$$\begin{aligned}&\int_{SO(m+1)} (k_{11})^2 F\left(\frac{1}{2},1-\frac{n}{2},\frac{3}{2};t^2(k_{11})^2\right)\,d\mu_{SO(m+1)}(k)\\ &= \mathrm{vol}(SO(m))\int_{S^m}(x_1)^2F\left(\frac{1}{2},1-\frac{n}{2},\frac{3}{2};t^2(x_1)^2\right)\,d\mu_{S^m}(x).\end{aligned}$$

Using again spherical coordinate transformation we get

$$\begin{aligned}&\int_{S^m}(x_1)^2F\left(\frac{1}{2},1-\frac{n}{2},\frac{3}{2};t^2(x_1)^2\right)\,d\mu_{S^m}(x)\\ &= 2\,\mathrm{vol}(S^{m-1})\int_0^1 x^2\sqrt{1-x^2}^{\,m-2}F\left(\frac{1}{2},1-\frac{n}{2},\frac{3}{2};t^2x^2\right)\,dx.\end{aligned}$$

Let $a_n x^{2n}$ be a general term of the series $F(1/2, 1-n/2, 3/2; t^2x^2)$. Then we arrive at the relation

$$F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3}{2}; t^2x^2\right) = 1 + b_1 \cdot \frac{t^2x^2}{1!} + b_2 \cdot \frac{t^4x^4}{2!} + \cdots + b_n \cdot \frac{t^{2n}x^{2n}}{n!} + \cdots,$$

where $b_n = n!a_n$. Since

$$\int_0^1 x^2 \sqrt{1-x^2}^{\,m-2}\, dx = \frac{1}{m+1} \cdot \frac{\operatorname{vol}(S^m)}{2\operatorname{vol}(S^{m-1})},$$

we have

$$\begin{aligned}&\int_0^1 x^2 \sqrt{1-x^2}^{\,m-2} F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3}{2}; t^2x^2\right) dx \\ &= \frac{\operatorname{vol}(S^m)}{2(m+1)\operatorname{vol}(S^{m-1})} \cdot F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3+m}{2}; t^2\right).\end{aligned}$$

So we obtain

$$\begin{aligned}&\int_{SO(m+1)} (k_{11})^2 F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3}{2}; t^2(k_{11})^2\right) d\mu_{SO(m+1)}(k) \\ &= \operatorname{vol}(SO(m))\operatorname{vol}(S^m) \cdot \frac{1}{m+1} \cdot F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3+m}{2}; t^2\right).\end{aligned} \tag{3.8}$$

A simple calculation shows that

$$\begin{aligned}&F\left(\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right) + \frac{nt^2}{m+1} F\left(\frac{1}{2}, 1-\frac{n}{2}, \frac{3+m}{2}; t^2\right) \\ &= F\left(-\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right).\end{aligned}$$

As the result, from (3.7), (3.8) and the last equality, we have

$$\begin{aligned}&\int_{SO(m+1)} \int_{SO(n+1)} |tk_{11} + g_{11}|\, d\mu_{SO(n+1)}(g)\, d\mu_{SO(m+1)}(k) \\ &= \frac{2\operatorname{vol}(K)\operatorname{vol}(S^{n-1})}{n\operatorname{vol}(S^n)} F\left(-\frac{1}{2}, -\frac{n}{2}, \frac{1+m}{2}; t^2\right).\end{aligned}$$

It remains to compute the case where $t \geq 1$ of (3.2). In this case we may compute the following:

$$t \int_{SO(m+1)} \int_{SO(n+1)} \left| k_{11} + \frac{1}{t} g_{11} \right| d\mu_{SO(n+1)}(g)\, d\mu_{SO(m+1)}(k).$$

This integration is nothing but (3.2) which replace m with n. Hence we immediately obtain

$$t \cdot \frac{2\,\mathrm{vol}(K)\mathrm{vol}(S^{m-1})}{m\,\mathrm{vol}(S^m)} F\left(-\frac{1}{2}, -\frac{m}{2}, \frac{1+n}{2}; \frac{1}{t^2}\right).$$

Therefore we have

$$\sigma(\theta,\tau) = \begin{cases} 2\,\mathrm{vol}(K)c \cdot \dfrac{\mathrm{vol}(S^{n-1})}{n\,\mathrm{vol}(S^n)} F\left(-\dfrac{1}{2}, -\dfrac{n}{2}, \dfrac{1+m}{2}, \dfrac{s^2}{c^2}\right), & \text{if } s \le c; \\ 2\,\mathrm{vol}(K)s \cdot \dfrac{\mathrm{vol}(S^{m-1})}{m\,\mathrm{vol}(S^m)} F\left(-\dfrac{1}{2}, -\dfrac{m}{2}, \dfrac{1+n}{2}, \dfrac{c^2}{s^2}\right), & \text{if } s \ge c, \end{cases}$$

which implies Theorem 1.1.

Up to this point, we unrestrainedly used the notation $\mathrm{vol}(SO(n+1))$ the volume of the special orthogonal group $SO(n+1)$. This value is given by

$$\mathrm{vol}\big(SO(n+1)\big) = \mathrm{vol}\big(SO(n)\big) \cdot \mathrm{vol}(S^n).$$

EXAMPLE. The case where $m = n = 1$ in (3.2).
It is well known that

$$K(k) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1; k^2\right), \quad E(k) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1; k^2\right),$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind respectively. By a simple calculation we have

$$K(k) - E(k) = \frac{\pi}{4} k^2 F\left(\frac{1}{2}, \frac{3}{2}, 2; k^2\right).$$

Hence, for $0 < t < 1$, we have

$$\begin{aligned} \sigma(\theta,\tau) &= 2c\ \mathrm{vol}\big(SO(2)\big)^2 \cdot \frac{\mathrm{vol}(S^0)}{\mathrm{vol}(S^1)} \cdot F\left(-\frac{1}{2}, -\frac{1}{2}, 1; t^2\right) \\ &= 16c\big(2E(t) - (1-t^2)K(t)\big). \end{aligned}$$

REMARK. Let $M = S^1$ and $N = S^m \times S^{n+1}$ in Theorem 1.1. Then, for almost all $g \in G = SO(m+2) \times SO(n+2)$, we have $\sharp(M \cap gN) = 2$. Thus we have

$$\int_G \sharp(M \cap gN)\, d\mu_G(g) = 2\,\mathrm{vol}(G).$$

Finally we can give the following inequalities as an application of the integral formula in Theorem 1.1.

Corollary 3.2. *Put $G = SO(m+2) \times SO(n+2)$. Under the hypothesis of* Theorem 1.1:
(1) *If $N = S^m \times S^{n+1}$ then we have*

$$\frac{1}{\mathrm{vol}(G)} \int_G \sharp(M \cap gN)\, d\mu_G(g) \leq 2 \cdot \frac{\mathrm{vol}(M)}{\mathrm{vol}(S^1)}.$$

The inequality becomes an equality if and only if M is a curve in S^{m+1}.
(2) *If $M = S^1 (\subset S^{m+1})$ then we have*

$$\frac{1}{\mathrm{vol}(G)} \int_G \sharp(M \cap gN)\, d\mu_G(g) \leq 2 \cdot \frac{\mathrm{vol}(N)}{\mathrm{vol}(S^m \times S^{n+1})}.$$

The equality holds if and only if N is a submanifold of $L \times S^{n+1}$. Here L is a submanifold in S^{m+1}.

Proof. (1) In this case we can take $\sin\theta_x e_1 + \cos\theta_x e_{m+2}$ and $e_2, \dots, e_{m+n+2}$ as an orthonormal basis of $T_x M$ and $T_y N$ respectively. Here $e_1, \dots, e_{m+n+2}$ is the standard orthonormal basis of $\mathbf{R}^{m+n+2}$. Hence we obtain

$$\sigma(x, y) = \frac{\mathrm{vol}(S^{m-1})}{m\, \mathrm{vol}(S^m)} \sin\theta_x.$$

We therefore have

$$\begin{aligned} \int_G \sharp(M \cap gN)\, d\mu_G(g) &= 2\,\mathrm{vol}(K) \frac{\mathrm{vol}(S^{m-1})}{m\, \mathrm{vol}(S^m)} \mathrm{vol}(N) \int_M \sin\theta_x \, d\mu_M(x) \\ &= \mathrm{vol}(G) \cdot \frac{1}{\pi} \int_M \sin\theta_x \, d\mu_M(x). \end{aligned}$$

(2) In this case we can obtain

$$\sigma(x, y) = \frac{\mathrm{vol}(S^{m-1})}{m\, \mathrm{vol}(S^m)} \sin\tau_y.$$

This, by a computation similar to that in (1), completes the proof. □

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Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki
305-8571, Japan
e-mail: kang@math.tsukuba.ac.jp

Current address:
Major in Mathematics
Dong-A University
Saha-ku, Busan
604-714, Korea