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NOETHER INEQUALITY ON A THREEFOLDS WITH ONE-DIMENSIONAL CANONICAL IMAGE

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Throughout this paper, we are working over the complex number field $\mathbb{C}$.

On a projective minimal surface S of general type, Noether inequality

$$p_g(S) = h^0(S, \mathcal{O}_S(K_S)) \leq \frac{1}{2}K_S{}^2 + 2,$$

holds where K_S is the canonical divisor. Noether inequality made an important contribution to understanding the geography of surfaces of general type. Unfortunately, we can not extend Noether inequality to a threefold of general type. A threefold version of Noether inequality is the following.

$$p_g(X) = h^0(X, \mathcal{O}_X(K_X)) \leq \frac{1}{2}K_X{}^3 + \dim \operatorname{Im} \Phi_{|K_X|},$$

where X is a minimal threefold of general type. This version of Noether inequality holds true when $\dim \operatorname{Im} \Phi_{|K_X|} = 3$. But when $\dim \operatorname{Im} \Phi_{|K_X|} = 2$, M. Kobayashi showed the existence of a counter example in M. Kobayashi [3, Proposition (3.2)]. When $\dim \operatorname{Im} \Phi_{|K_X|} = 1$, M. Kobayashi described the possible exceptional cases assuming that X is factorial. When $\dim \operatorname{Im} \Phi_{|K_X|} = 1$, we have the following:
(0) $p_g(X) \leq (1/2)K_X{}^3 + 1$
or if not, we have the following two possible exceptional cases
(1) X is singular, the image is a rational curve, all the fibers are connected, $K_X{}^3 = 1$ and $p_g(X) = 2$
(2) The map $\Phi_{|K_X|}$ is a morphism and a general fiber S is a normal algebraic irreducible surface with only canonical singularities which have ample canonical divisor, $K_S{}^2 = 1$, $q(S) = 0$ and $p_g(S) = 1$ or 2,
where $q(S)$ and $p_g(S)$ are the irregularity and the genus of S respectively.

For detail matters, see M. Kobayashi [3]. But the existence of each possible exceptional case — the case (1) or the case (2) — he described is not known yet. However, in the case (1), we have the additional information about the genus p_g and $K_X{}^3$. In the case (2), we don't have any such information. Thus, we need an addi-

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tional information about the invariants of X to describe in detail. In our theorem, we have induced the inequalities between ${K_X}^3$ and invariants. From these inequalities, we can describe invariants like the irregularity, the genus and the Euler characteristic of a possible exceptional threefold. The main result is the following theorem.

Theorem. *Let X be a minimal threefold of general type which is factorial. Let* $\dim \operatorname{Im} \Phi_{|K_X|} = 1$. *Suppose that Noether inequality does not hold, i.e.,* $p_g > (1/2){K_X}^3 + 1$. *Suppose that we have the case* (2) *in the above. Then we have the following*:
(1) $q_1 \leq (1/2){K_X}^3 - 1$.
(2) $q_2 \leq \chi(\mathcal{O}_X) + {K_X}^3$.
(3) *If* $\chi(\mathcal{O}_X) > 0$, *then* $1 \leq \chi(\mathcal{O}_X) \leq 3$, $q_1 = 0$, *and* $p_g \leq q_2 \leq p_g + 2$.
(4) *If* $\chi(\mathcal{O}_X) \leq 0$, *then* $\chi(\mathcal{O}_X) \leq -2(q_1 - 1)$.

We are going to use the following notations to prove our main result. When X is a projective variety of general type with a canonical divisor K_X, we denote the genus of X by $p_g(X)$ and the irregularity $h^i(X, \mathcal{O}_X)$ ($i = 1, 2$) of X by $q_i(X)$ (or just simply p_g and q_i respectively unless there is some confusion). Let $\Phi_{|K_X|}$ be a rational map associated with a complete linear system $|K_X|$.

Proof of Theorem. By the works of M. Kobayashi, the map $\Phi_{|K_X|}$ is a morphism onto a curve C in $\mathbb{P}^{p_g - 1}$ and a general fiber S is a normal algebraic irreducible surface with ${K_S}^2 = 1$, $q(S) = 0$ and $p_g(S) = 1$ or 2.

There is a resolution of the singular locus of X, i.e., a birational morphism $g\colon X' \to X$ such that $f = \Phi_{|K_X|} \circ g$ is a morphism of a smooth threefold X' to C. The morphism $f\colon X' \to C$ has a connected general fiber S'. Then S' is a surface of a general type with $q(S') = 0$, $p_g(S') = 1$ or 2 since g is birational. Moreover, we have $a \leq {K_X}^3$, where a is the degree of C in $\mathbb{P}^{p_g - 1}$. We have the fiber space $f\colon X' \to C$ with a connected fiber S'. From the spectral sequence, we have

$$\begin{aligned}
h^0(X', \mathcal{O}_{X'}(K_{X'})) &= h^0(C, f_*\mathcal{O}_{X'}(K_{X'})) \\
h^1(X', \mathcal{O}_{X'}(K_{X'})) &= h^1(C, f_*\mathcal{O}_{X'}(K_{X'})) + h^0(C, R^1 f_*\mathcal{O}_{X'}(K_{X'})) \\
h^2(X', \mathcal{O}_{X'}(K_{X'})) &= h^1(C, R^1 f_*\mathcal{O}_{X'}(K_{X'})) + h^0(C, R^2 f_*\mathcal{O}_{X'}(K_{X'}))
\end{aligned}$$

We have $R^1 f_*\mathcal{O}_{X'}(K_{X'}) = 0$ since $q(S') = q(S) = 0$. Thus, we have $h^i(C, R^1 f_*\mathcal{O}_{X'}(K_{X'})) = 0$ for $i = 0, 1$. By the work of J. Kollár (see Kollár [4]), $R^2 f_*\mathcal{O}_{X'}(K_{X'})$ is isomorphic to $\mathcal{O}_C(K_C)$. Hence we have

$$\begin{aligned}
h^1(X', \mathcal{O}_{X'}(K_{X'})) &= h^1(C, f_*\mathcal{O}_{X'}(K_{X'})) \\
h^2(X', \mathcal{O}_{X'}(K_{X'})) &= h^0(C, R^2 f_*\mathcal{O}_{X'}(K_{X'})) = h^0(C, \mathcal{O}_C(K_C))
\end{aligned}$$

Thus $q_1 = p_g(C)$ since $q_1 = h^2(X', \mathcal{O}_{X'}(K_{X'}))$ by the duality.

If $2(p_g - 1) \leq a$, then $p_g \leq (1/2){K_X}^3 + 1$ since $a \leq {K_X}^3$. It contradicts our assumption. Thus $a < 2(p_g - 1)$. Then by the space curve genus formula (see P. Griffiths and J. Harris [1] p. 253), we have

$$q_1 = p_g(C) \leq a - p_g + 1 \leq {K_X}^3 - p_g + 1.$$

Thus, $q_1 + p_g \leq {K_X}^3 + 1$. Since $p_g > (1/2){K_X}^3 + 1$ by our assumption, we have

$$q_1 \leq \frac{1}{2}{K_X}^3 - 1.$$

For a proof of (2), we have

$$\chi(\mathcal{O}_X) = 1 - q_1 + q_2 - p_g \geq 1 + q_2 - {K_X}^3 - 1 = q_2 - {K_X}^3,$$

since $q_1 + p_g \leq {K_X}^3 + 1$.

For (3), recall that $f_* K_{X'/C} \stackrel{\text{def}}{:=} f_*(\mathcal{O}_{X'}(K_{X'}) \otimes f^*\mathcal{O}_C(K_C)^{-1})$ is semipositive and locally free of rank $p_g(S')$ (see Kawamata [2] or Ueno [5]). By Hirzebruch-Riemann-Roch Theorem, we have

$$\begin{aligned} p_g - q_2 &= h^0(X', \mathcal{O}_{X'}(K_{X'})) - h^1(X', \mathcal{O}_{X'}(K_{X'})) \\ &= h^0(C, f_*\mathcal{O}_{X'}(K_{X'})) - h^1(C, f_*\mathcal{O}_{X'}(K_{X'})) \\ &= \deg f_*\mathcal{O}_{X'}(K_{X'}) + p_g(S')(1 - p_g(C)) \\ &= \deg f_* K_{X'/C} + p_g(S')(p_g(C) - 1) \\ &\geq p_g(S')(p_g(C) - 1) \\ &= p_g(S')(q_1 - 1). \end{aligned}$$

Therefore, we have

$$(*) \qquad \chi(\mathcal{O}_X) \leq (p_g(S) + 1)(1 - q_1)$$

since $p_g(S') = p_g(S)$. If $\chi(\mathcal{O}_X) > 0$, then the inequality $(*)$ implies that $q_1 = 0$ and $1 \leq \chi(\mathcal{O}_X) \leq 3$ because $p_g(S) \leq 2$. Hence we have

$$p_g \leq q_2 \leq p_g + 2$$

since $1 \leq \chi(\mathcal{O}_X) \leq 3$ and $q_1 = 0$.

The inequality in (4) comes from the inequality $(*)$ since $\chi(\mathcal{O}_X) < 0$ and $1 \leq p_g(S) \leq 2$. $\square$

Corollary. *Suppose that a smooth threefold X with K_X nef and big has a canonical pencil. Then one of the following holds*:

$$p_g \leq \frac{1}{2}{K_X}^3 + 1 \ \textit{or}\ q_1 \leq \frac{1}{2}{K_X}^3 - 1.$$

Proof. This comes directly from Theorem. □

REMARK. Using inequalities in Theorem, we can describe the invariants of a possible exceptional threefold.

For an example, suppose that a smooth minimal threefold X of genral type with ${K_X}^3 = 2$ has a canonical pencil and suppose that Noether inequality does not hold on X. Since X is smooth, ${K_X}^3 \geq 2$ and $\chi(\mathcal{O}_X) \leq -1$. Thus, ${K_X}^3 = 2$. We have the following from inequalities in Theorem:

$$
\begin{aligned}
&q_1 \leq \frac{1}{2}{K_X}^3 - 1 = 0\\
&q_2 \leq \chi(\mathcal{O}_X) + {K_X}^2 = \chi(\mathcal{O}_X) + 2\\
&\frac{1}{2}{K_X}^3 + 2 \leq p_g \leq {K_X}^3 + 1 - q_1
\end{aligned}
$$

From above inequalities, we have $q_1 = 0$, $p_g = 3$ and $q_2 \leq 1$. Thus, if a canonical pencil with ${K_X}^3 = 2$ does not satisfy Noether inequality, then X must have $q_1 = 0$, $p_g = 3$, $q_2 \leq 1$ and $\chi(\mathcal{O}_X) = -1$ or -2.

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