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CONJECTURES ON CHARACTER DEGREES FOR THE SIMPLE THOMPSON GROUP

Dedicated to Professor Yukio Tsushima on his sixtieth birthday

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1. Introduction

Let Th be the simple Thompson group. It has order $2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$, and its Schur multiplier and the outer automorphism group are both trivial.

In this paper, we prove that Dade's conjecture for Th is true. Here we mean by saying the conjecture the ordinary form of the conjecture. (see [2].) According to [4], the ordinary form is equivalent to the inductive form if the Schur multiplier and the outer automorphism group are trivial. Moreover, if a defect group of a block is cyclic, then the ordinary form is proved to be true for this block in [2]. Thus, it suffices to treat the primes 2, 3, 5 and 7. The character tables of several subgroups of Th are available. We use those found in ATLAS [1] and GAP library. Also, maximal local subgroups of Th are determined in [12] and [18]. These information are quite useful. In the proof of the conjecture, we use a reduction theorem, which is a very special case of the one proved in §16 of [3].

Let G be a finite group and p a prime. Quite recently, a new type of a conjecture, which concerns the p'-part of character degrees, is proposed by Isaacs and Navarro [10]. Though the original version considers only height zero characters of G and the normalizer of a defect group, here we prove a version of Dade's type. Namely, we prove that the alternating sum of the numbers of relevant characters of the normalizers of p-chains with any fixed defect is zero. Since we count the number of characters satisfying a certain congruent relation such as $\chi(1)_{p'} \equiv \pm\kappa \mod p$, where $\chi(1)_{p'}$ is the p'-part of $\chi(1)$, it suffices to consider the case of $p \geq 5$. However, for $p = 13$, 19 or 31, a Sylow p-subgroup of Th is cyclic of prime order. Thus the normalizers of chains are just Th and the normalizer of a Sylow p-subgroup, and moreover, it is known that those p-blocks of the normalizers have only characters of height zero. Hence the assertion of our version of the conjecture is equivalent to that of Isaacs and Navarro, and the latter is proved for cyclic defect group cases in Theorem 2.1 of [10]. Thus when proving the alternating sum version of the conjecture, we treat only the cases of $p = 5$ and 7 in this paper. Our version of the conjecture is proposed in §3 below.

We denote by $A : B$ and $A \cdot B$ a split and a non-split extension of A by B, respectively. We use n to denote a cyclic group of order n. Moreover, for a prime p, an elementary abelian group of order p^n is denoted by p^n. For an odd prime p, we denote by p_+^{1+2n} an extra special group of order p^{2n+1} and exponent p. The plus type extra special group of order 2^{2n+1} is denoted by 2_+^{1+2n}. Finally, S_n and A_n denote the symmetric group and the alternating group of degree n, respectively, and D_8 denotes the dihedral group of order 8.

Throughout the paper, a character means an irreducible complex character unless otherwise noted. For a fixed prime p, we use the following notation. Let G be a finite group and H a subgroup of G. The principal block of H is denoted by $B_0(H)$, and the set of characters φ of H is denoted by $\mathrm{Irr}(H)$. If the p-part of $|H|/\varphi(1)$ is p^d, then we say that φ has defect $d = d(\varphi)$. For an integer d, we denote by $\mathrm{Irr}(H, d)$ the set of those φ in $\mathrm{Irr}(H)$ with $d(\varphi) = d$. Also for an integer κ, we denote by $\mathrm{Irr}(H, [\kappa])$ the set of those φ in $\mathrm{Irr}(H)$ such that

$$(|H|/\varphi(1))_{p'} \equiv \pm\kappa \mod p.$$

We also put $\mathrm{Irr}(H, d, [\kappa])$ by

$$\mathrm{Irr}(H, d, [\kappa]) = \mathrm{Irr}(H, d) \cap \mathrm{Irr}(H, [\kappa]).$$

For a normal subgroup K of H and a character θ of K, we use $\mathrm{Irr}(H|\theta)$ to denote the set of those φ in $\mathrm{Irr}(H)$ such that φ is lying over θ, that is, the restriction of φ to K has θ as its irreducible constituent. We denote the intersections of $\mathrm{Irr}(H|\theta)$ with $\mathrm{Irr}(H, d)$, $\mathrm{Irr}(H, [\kappa])$, and $\mathrm{Irr}(H, d, [\kappa])$ respectively by $\mathrm{Irr}(H, d|\theta)$, $\mathrm{Irr}(H, [\kappa]|\theta)$ and $\mathrm{Irr}(H, d, [\kappa]|\theta)$. Moreover, for a p-block B of G, we use $\mathrm{Irr}(H, B)$ to denote the set of characters of H belonging to some p-block b of H inducing B. The sets $\mathrm{Irr}(H, B|\theta)$, $\mathrm{Irr}(H, B, d)$, $\mathrm{Irr}(H, B, d|\theta)$ and so on denote the intersections of $\mathrm{Irr}(H, B)$ with the relevant sets. Sometimes θ is replaced by a subset Θ of $\mathrm{Irr}(K)$. If this is the case, for example, $\mathrm{Irr}(H|\Theta)$ means the set of those φ in $\mathrm{Irr}(H)$ such that φ is lying over some θ in Θ. The cardinalities of $\mathrm{Irr}(*)$ is in general denoted by $k(*)$.

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2. Dade's conjecture

We describe the ordinary form of the conjecture. Let G be a finite group and p a prime. A radical p-chain of G is a chain

$$\mathcal{C}\colon P_0 < P_1 < P_2 < \cdots < P_n$$

of p-subgroups P_i of G such that

$$\begin{aligned} &\text{(i)}\quad P_0 = O_p(G) \quad \text{and} \\ &\text{(ii)}\quad P_i = O_p\left(\textstyle\bigcap_{j=0}^{i} N_G(P_j)\right) \quad \text{for all } i = 0, 1, \ldots, n. \end{aligned}$$

For those $\mathcal{C}$, we write by $N_G(\mathcal{C})$ the normalizer $\bigcap_{i=0}^{n} N_G(P_i)$ of $\mathcal{C}$ in G, and by $|\mathcal{C}|$, the length n of $\mathcal{C}$. Let $\mathcal{R}_p(G)$ be the set of all radical p-chains of G. The group G acts on $\mathcal{R}_p(G)$ by conjugation. We denote a set of representatives of G-conjugacy classes of $\mathcal{R}_p(G)$ by $\mathcal{R}_p(G)/G$.

If a p-subgroup P of G satisfies $P = O_p(N_G(P))$, then we say that P is a radical p-subgroup of G. (Note that in this paper, $O_p(G)$ is also a radical subgroup of G.) The set of radical p-subgroups of G is denoted by $\mathcal{B}_p(G)$. Thus,

$$\mathcal{C}\colon P_0 < P_1 < P_2 < \cdots < P_n$$

is a radical p-chain if and only if $P_0 = O_p(G)$, $P_i \leq N_G(P_j)$ for i, j with $j \leq i$ and P_{i+1}/P_i lies in $\mathcal{B}_p(\bigcap_{j=0}^{i} N_G(P_j)/P_i)$ for all $i = 0, 1, \ldots, n-1$.

The ordinary form of Dade's conjecture is stated as follows.

Conjecture 2.1. *Let p be a prime and G a finite group with $O_p(G) = 1$. Consider the above situation. If the defect $d(B)$ of B is greater than 0, then*

$$\sum_{\mathcal{C}\in\mathcal{R}_p(G)/G} (-1)^{|\mathcal{C}|} k(N_G(\mathcal{C}), B, d) = 0$$

for all d.

3. Conjecture of Isaacs and Navarro

In this section, we describe the Isaacs and Navarro conjecture and an alternating sum version of it. Let G be a finite group and p a prime. Let B be a p-block of G with defect group D and b the p-block of $N_G(D)$ which corresponds to B by the first main theorem of Brauer. For an integer κ with $0 < \kappa < p$, let $M_\kappa(B)$ be the number of irreducible characters in B with height zero and the p'-part of the degrees are congruent to $\pm\kappa$ modulo p. Then Isaacs and Navarro propose the following conjecture in [10].

Conjecture 3.1 (Isaacs-Navarro). *In the notation above, let c be the p'-part of $|G : N_G(D)|$. Then*

$$M_{c\kappa}(B) = M_\kappa(b)$$

for all κ with $0 < \kappa < p$.

However, in the above, there seems to be no reason why we take only height zero characters if we deal with also the normalizers of chains. By considering resemblance to the relationship between McKay conjecture and the ordinary form of Dade's conjecture, we propose the following.

Conjecture 3.2. *Let p be a prime and G a finite group with $O_p(G) = 1$. Consider the above situation. If the defect $d(B)$ of B is greater than 0, then*

$$\sum_{\mathcal{C} \in \mathcal{R}_p(G)/G} (-1)^{|\mathcal{C}|} k\big(N_G(\mathcal{C}), B, d, [\kappa]\big) = 0$$

for all d and κ.

Of course the other forms such as invariant, projective forms of the conjecture can be set. However, detail of them and the relationship between Conjecture 3.2 and the original version of the conjecture by Isaacs and Navarro will be considered elsewhere. Here remark that Conjecture 3.2 implies Conjecture 2.1, since the alternating sum in the latter is the sum of those in the former taken over all κ's with $0 < \kappa \leq (p-1)/2$.

Now we give the following easy remarks.

Lemma 3.3. *Let G be a finite group and fix a radical p-subgroup P of G with $P \neq O_p(G)$. Suppose that a radical p-subgroup $\tilde{P}$ of G with $\tilde{P} > P$ satisfies $N_G(\tilde{P}) \subseteq N_G(P)$. Then $\tilde{P}$ is a radical p-subgroup of $N_G(P)$, and putting*

$$\begin{aligned} \mathcal{R}_1 &= \{\ \mathcal{C} \colon O_p(G) < P < \tilde{P} < \cdots < P_n \in \mathcal{R}_p(G) \mid |\mathcal{C}| \geq 2\ \} \text{ and} \\ \mathcal{R}_2 &= \{\ \mathcal{C} \colon O_p(G) < \tilde{P} < \cdots < P_n \in \mathcal{R}_p(G) \mid |\mathcal{C}| \geq 1\ \}, \end{aligned}$$

there is a bijection $f \colon \mathcal{R}_1 \to \mathcal{R}_2$ sending any $O_p(G) < P < \tilde{P} < \cdots$ in $\mathcal{R}_1$ into $O_p(G) < \tilde{P} < \cdots$. Moreover, for any $\mathcal{C}$ in $\mathcal{R}_1$, we have $N_G(\mathcal{C}) = N_G(f(\mathcal{C}))$ and $|\mathcal{C}| = |f(\mathcal{C})| + 1$.

By the above, in the alternating sum appearing in the conjectures, we can take radical p-chains which do not lie in $\mathcal{R}_1$ or $\mathcal{R}_2$. Moreover, concerning Clifford theory, the following is important.

Lemma 3.4. *Let G be a finite group and N a normal subgroup of G. Let θ be a character of N, and T the stabilizer of θ in G. Then there exists a bijection from $\mathrm{Irr}(T, d, [\kappa]|\theta)$ to $\mathrm{Irr}(G, d, [\kappa]|\theta)$ sending any φ in $\mathrm{Irr}(T, d, [\kappa]|\theta)$ to φ^G.*

4. A reduction theorem

In this section, we prove a theorem which reduces the problem to subsections. It is a very special case of Theorem 16.4 of [3], though here it is slightly modified and takes κ into account. We can give a proof here, because it is not very complicated in our case. Consider the following situation.

Hypothesis 4.1. *Let p be a prime and G a finite group of the form*

$$G = (Q \times H) : \langle \sigma \rangle,$$

for a p-subgroup Q, a subgroup H and a p-regular element σ of G such that σ normalizes Q and H.

For any $\mathcal{C} \in \mathcal{R}_p(G)$, each subgroup P appearing in $\mathcal{C}$ has the form $P = Q \times \overline{P}$ for a p-subgroup $\overline{P}$ of H. Furthermore, there is a bijection

$$f : \mathcal{R}_p(G)/G \to \mathcal{R}_p(H : \langle \sigma \rangle)/(H : \langle \sigma \rangle)$$

sending any class of

$$\mathcal{C}\colon\ O_p(G) = Q \times O_p(H) < Q \times \overline{P}_1 < \cdots$$

to the class of

$$\overline{\mathcal{C}}\colon\ O_p(H : \langle \sigma \rangle) = O_p(H) < \overline{P}_1 < \cdots ,$$

and we have $|\mathcal{C}| = |\overline{\mathcal{C}}|$ and $N_G(\mathcal{C}) = Q : N_{H:\langle \sigma \rangle}(\overline{\mathcal{C}})$.

Fix a character θ of Q. Let $\langle \tau \rangle$ be the stabilizer of θ in $\langle \sigma \rangle$, where $\tau = \sigma^m$ for some divisor m of $|\langle \sigma \rangle|$. The stabilizer of θ in G is $(Q \times H) : \langle \tau \rangle$, and the stabilizer $I(\mathcal{C}, \theta)$ of θ in $N_G(\mathcal{C})$ can be written as

$$I = I(\mathcal{C}, \theta) = Q : N_{H:\langle \tau \rangle}(\overline{\mathcal{C}}).$$

Note that θ can be extended to $(Q \times H) : \langle \tau \rangle$, since it has an extension $\theta \times 1_H$ to $Q \times H$ which is $\langle \tau \rangle$-invariant. In particular, θ can be extended to $I(\mathcal{C}, \theta)$ for any $\mathcal{C}$.

Let b be a p-block of $N_G(\mathcal{C})$ such that b contains a some character in $\mathrm{Irr}(N_G(\mathcal{C}))|\theta)$. Then there exists a p-block b' of I inducing b. (see V.3.1 of [13].) Let $\overline{b}_1, b_2, \ldots, b_t$ be p-blocks of $N_{H:\langle \tau \rangle}(\overline{\mathcal{C}})$ dominated by b'. See Chapter V, §8 of [13]. Since θ can be extended to $I(\mathcal{C}, \theta)$ and since $|Q|/\theta(1)$ is a $d(\theta)$-th power of p, it follows from Clifford theory that

$$k\big(I, b', d, [\kappa]|\theta\big) = \sum_{j=1}^{t} k\big(N_{H:\langle \tau \rangle}(\overline{C}), \overline{b}_j, d - d(\theta), [\kappa]\big).$$

Since $k(N_G(\mathcal{C}), b, d, [\kappa]|\theta)$ is the sum of $k(I, b', d, [\kappa]|\theta)$, where b' ranges in the set of p-blocks of I inducing b, we have the following.

Theorem 4.2. *Under* Hypothesis 4.1, *let θ be a character of Q with defect $d(\theta)$ and Θ the G-orbit of θ, and let $\langle\tau\rangle$ be the stabilizer of θ in $\langle\sigma\rangle$. If*

$$\sum_{\overline{\mathcal{C}}\in\mathcal{R}_p(H:\langle\tau\rangle)/H:\langle\tau\rangle}(-1)^{|\overline{\mathcal{C}}|}k\big(N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}),\overline{B},d,[\kappa]\big)=0$$

for all p-blocks $\overline{B}$ of $H:\langle\tau\rangle$, all d with $d\neq 0$, and all κ, then we have

$$\sum_{\mathcal{C}\in\mathcal{R}_p(G)/G}(-1)^{|\mathcal{C}|}k\big(N_G(\mathcal{C}),B,d,[\kappa]|\Theta\big)=0$$

for all p-block B of G, all d with $d\neq d(\theta)$, and all κ.

Proof. Use the notation given preceding the theorem. Since the stabilizer of θ in G is $(Q\times H)\langle\tau\rangle$, its G-orbit Θ consists of m characters. Since $\langle\sigma\rangle$ is abelian, $I(\mathcal{C},\theta)=I(\mathcal{C},\theta^g)=Q:N_{H:\langle\tau\rangle}(\overline{\mathcal{C}})$ for all $g\in G$, and thus the $N_G(\mathcal{C})$-orbits of θ and θ^g consist of the same number of characters. Since the $N_G(\mathcal{C})$-orbit of θ consists of $|N_G(\mathcal{C}):(Q:N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}))|$ characters, the number of $N_G(\mathcal{C})$-orbits in Θ is

$$\begin{aligned}m(\mathcal{C},\theta)&=m/|N_G(\mathcal{C}):(Q:N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}))|=m/|N_{H:\langle\sigma\rangle}(\overline{\mathcal{C}}):N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}))|\\&=|(H:\langle\sigma\rangle)|/|(H:\langle\tau\rangle)|\times(|N_{H:\langle\sigma\rangle}(\overline{\mathcal{C}})|/|N_{H:\langle\sigma\rangle}(\overline{\mathcal{C}})\cap(H:\langle\tau\rangle|)^{-1}\\&=|(H:\langle\sigma\rangle)|/|(H:\langle\tau\rangle)|\times(|N_{H:\langle\sigma\rangle}(\overline{\mathcal{C}})H:\langle\tau\rangle|/|H:\langle\tau\rangle|)^{-1}\\&=|(H:\langle\sigma\rangle)|\times|N_{H:\langle\sigma\rangle}(\overline{\mathcal{C}})(H:\langle\tau\rangle)|^{-1}.\end{aligned}$$

Let $\theta=\theta_1,\ \theta_2,\dots,\theta_{m(\mathcal{C},\theta)}$ form a set of representatives of $N_G(\mathcal{C})$-orbits in the G-orbit of θ.

Now, the $H:\langle\sigma\rangle$-orbit of the radical p-chain $\overline{C}$ of H consists of $|(H:\langle\sigma\rangle):N_{H:\langle\sigma\rangle}(\overline{C})|$ chains. But, it is a disjoint union of $H:\langle\tau\rangle$-orbits, each of which consists of $|H\langle\tau\rangle:N_{H:\langle\tau\rangle}(\overline{C})|$ chains and we have

$$|H\langle\tau\rangle:N_{H:\langle\tau\rangle}(\overline{C})|=|H\langle\tau\rangle:N_{H:\langle\sigma\rangle}(\overline{C})\cap H\langle\tau\rangle|=|N_{H:\langle\sigma\rangle}(\overline{C})(H:\langle\tau\rangle):N_{H:\langle\sigma\rangle}(\overline{C})|$$

Thus the number of $H\langle\tau\rangle$-orbits in the disjoint union is in total $m(\mathcal{C},\theta)$.

On the other hand, let B be a p-block of G and consider all p-blocks b of $N_G(\mathcal{C})$ inducing B and having some character in $\mathrm{Irr}(N_G(\mathcal{C}))|\theta)$. (If such a b exists, then B has a character lying over θ. see V.3.10 of [13].) Each of those is induced from some p-block b' of $I(\mathcal{C},\theta)$. Take all such b and b' and all p-blocks $\overline{b}_1,\ \overline{b}_2,\dots,\overline{b}_s$ of $N_{H:\langle\tau\rangle}(\overline{\mathcal{C}})$ dominated by one of these b''s. Note that each $\overline{b}_j$ can be induced to a p-block of $H:\langle\tau\rangle$. Of course, they depend on $\mathcal{C}$. Let $\mathcal{C}_0$ be the trivial chain of G consisting only one group $O_p(G)$. Then $I(\mathcal{C}_0,\theta)=(Q\times H):\langle\tau\rangle$, and we have p-blocks $\overline{B}_1,\ \overline{B}_2,\dots,\overline{B}_u$ of $N_{H:\langle\tau\rangle}(\overline{\mathcal{C}_0})=H:\langle\tau\rangle$ by the above process applied to $\mathcal{C}_0$. Then, it is easy to see that $\overline{B}_1,\ \overline{B}_2,\dots,\overline{B}_u$ are exactly the p-blocks of $H:\langle\tau\rangle$ induced from some $\overline{b}_j$. (This is proved in much more general situation in Theorem 14.3

of [3].) From the arguments above and preceding the theorem, we have

$$
\begin{aligned}
&\sum_{\mathcal{C}\in\mathcal{R}_p(G)/G} (-1)^{|\mathcal{C}|} k\big(N_G(\mathcal{C}), B, d, [\kappa]|\Theta\big)\\
&= \sum_{\mathcal{C}\in\mathcal{R}_p(G)/G} \sum_{i=1}^{m(\mathcal{C},\theta)} (-1)^{|\mathcal{C}|} k\big(N_G(\mathcal{C}), B, d, [\kappa]|\theta_i\big)\\
&= \sum_{\mathcal{C}\in\mathcal{R}_p(G)/G} \sum_{i=1}^{m(\mathcal{C},\theta)} (-1)^{|\mathcal{C}|} k\big(I(\mathcal{C}, \theta_i), B, d, [\kappa]|\theta_i\big)\\
&= \sum_{\mathcal{C}\in\mathcal{R}_p(G)/G} m(\mathcal{C}, \theta)(-1)^{|\mathcal{C}|} k\big(I(\mathcal{C}, \theta), B, d, [\kappa]|\theta\big)\\
&= \sum_{\overline{\mathcal{C}}\in\mathcal{R}_p(H:\langle\sigma\rangle)/(H:\langle\sigma\rangle)} m(\mathcal{C}, \theta)(-1)^{|\overline{\mathcal{C}}|} \sum_{j=1}^{u} k\big(N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}), \overline{B}_j, d - d(\theta), [\kappa]\big)\\
&= \sum_{\overline{\mathcal{C}}\in\mathcal{R}_p(H)/(H:\langle\sigma\rangle)} m(\mathcal{C}, \theta)(-1)^{|\overline{\mathcal{C}}|} \sum_{j=1}^{u} k\big(N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}), \overline{B}_j, d - d(\theta), [\kappa]\big)\\
&= \sum_{\overline{\mathcal{C}}\in\mathcal{R}_p(H)/(H:\langle\tau\rangle)} (-1)^{|\overline{\mathcal{C}}|} \sum_{j=1}^{u} k\big(N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}), \overline{B}_j, d - d(\theta), [\kappa]\big)\\
&= \sum_{\overline{\mathcal{C}}\in\mathcal{R}_p(H:\langle\tau\rangle)/(H:\langle\tau\rangle)} (-1)^{|\overline{\mathcal{C}}|} \sum_{j=1}^{u} k\big(N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}), \overline{B}_j, d - d(\theta), [\kappa]\big)\\
&= \sum_{j=1}^{u} \sum_{\overline{\mathcal{C}}\in\mathcal{R}_p(H:\langle\tau\rangle)/(H:\langle\tau\rangle)} (-1)^{|\overline{\mathcal{C}}|} k\big(N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}), \overline{B}_j, d - d(\theta), [\kappa]\big)
\end{aligned}
$$

for all d and κ. Now, by the assumption, the above alternating sum is zero for all d with $d \neq d(\theta)$. This completes the proof. □

REMARK. One might think that the condition $d \neq d(\theta)$ is not necessary in the conclusion of the above. However, if $O_p(H) = \{1\}$ and $\overline{B}$ is of defect zero, then the alternating sum in the assumption of the theorem is 1 for $d = 0$, since $k(N_{H:\langle\tau\rangle}(\overline{\mathcal{C}}), \overline{B}, 0) = 0$ for all non-trivial chain $\overline{\mathcal{C}}$ in $\mathcal{R}_p(H : \langle\tau\rangle)$. Thus it is reasonable to restrict the assumption either to p-blocks $\overline{B}$ of positive defect or to d with $d \neq 0$. However, if we take the former, it becomes slightly difficult to state the conclusion.

The above theorem will be applied in the following situation.

Hypothesis 4.3. *Let p be a prime and G a finite group such that G has a p-subgroup Q with*

$$N_G(Q) = (Q \times H) : \langle \sigma \rangle = N_G\big(Q \times O_p(H)\big)$$

for a subgroup H of G and a p-regular element σ of G which normalizes Q and H.

Then $Q \times O_p(H)$ is a radical p-subgroup of G, and Hypothesis 4.1 is satisfied for $N_G(Q)$ in place of G there. Assume further that $O_p(G) = \{1\}$ and let $\mathcal{R}'_p(G)$ be the set of radical p-chains of G starting with $1 < Q \times O_p(H)$. Then there is a bijection

$$\mathcal{F} \colon \mathcal{R}'_p(G)/N_G(Q) \to \mathcal{R}_p(N_G(Q))/N_G(Q)$$

sending any class of

$$\mathcal{C} \colon 1 < Q \times O_p(H) < P_1 < \cdots$$

to the class of

$$\mathcal{C}' \colon O_p(N_G(Q)) = Q \times O_p(H) < P_1 < \cdots ,$$

such that $|\mathcal{C}| = |\mathcal{F}(\mathcal{C})| + 1$ and $N_G(\mathcal{C}) = N_G(\mathcal{F}(\mathcal{C}))$. Thus Theorem 4.2 can be used to reduce partly the problem for G to that for $H : \langle \sigma \rangle$. In particular, if Q is abelian, we have the following.

Theorem 4.4. *Under* Hypothesis 4.3, *use the above notation. Suppose that $O_p(G) = \{1\}$ and that Q is abelian with $|Q| = p^s$. If*

$$\sum_{\overline{\mathcal{C}} \in \mathcal{R}_p(H_1)/H_1} (-1)^{|\overline{\mathcal{C}}|} k\big(N_{H_1}(\overline{\mathcal{C}}), \overline{B}, d, [\kappa]\big) = 0$$

for all p-blocks $\overline{B}$ of all H_1 with $H \le H_1 \le H : \langle \sigma \rangle$, all d with $d \ne 0$ and all κ, then we have

$$\sum_{\mathcal{C} \in \mathcal{R}'_p(G)/G} (-1)^{|\mathcal{C}|} k\big(N_G(\mathcal{C}), B, d, [\kappa]\big) = 0$$

for all p-blocks B of G, all d with $d \ne s$ and all κ.

Proof. Taking all $N_G(Q)$-orbits Θ in $\mathrm{Irr}(Q)$, and applying Theorem 4.2, we have the desired result, since all the characters of Q have defect s. $\square$

In the group Th, the above situation really occurs and to use Theorem 4.2 for it, we prove that the Conjecture 3.2 holds for certain groups.

Proposition 4.5. *For the following groups G and primes p, we have*

$$\sum_{\mathcal{C}\in\mathcal{R}_p(G)/G} (-1)^{|\mathcal{C}|} k\big(N_G(\mathcal{C}), B, d, [\kappa]\big) = 0$$

for any p-block B of G with positive defect and for all d and κ.
(1a) *Let $q = p^e$. The group G is $p^{2e} : GL_2(q) = (\mathbf{F}_q)^2 : GL_2(q)$ with the natural action of $GL_2(q)$ on $(\mathbf{F}_q)^2$.*
(1b) *Let $q = p^e$. The group G is $p^{2e} : SL_2(q) = (\mathbf{F}_q)^2 : SL_2(q)$ with the natural action of $SL_2(q)$ on $(\mathbf{F}_q)^2$.*
(1c) *Let $q = p^e$ with $p \neq 2$. Let σ be the Galois automorphism of $\mathbf{F}_{q^2}$ with $\sigma(x) = x^q$. The group G is $(p^{4e} : SL_2(q^2)) : \langle\sigma\rangle = ((\mathbf{F}_{q^2})^2 : SL_2(q^2)) : \langle\sigma\rangle$ with the natural action of $SL_2(q^2)$ and σ.*
(2) *Let $p = 3$. Denote by σ the graph automorphism of $G_2(3)$ of order two. The group G is $G_2(3)$ or $G_2(3) : \langle\sigma\rangle$.*

Proof. (1a), (1b). We omit the proof of (1a) and give a proof of (1b), since the argument for (1a) is similar to and simpler than that for (1b). Let $H(q)$ be the subgroup of upper triangular matrices in $SL_2(q)$.

$$H(q) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a, b \in \mathbf{F}_q,\ a \neq 0 \right\}.$$

Then the trivial chain $(\mathbf{F}_q)^2 = O_p(G)$ and $(\mathbf{F}_q)^2 < (\mathbf{F}_q)^2 : O_p(H(q))$ form a set of representatives of G-conjugacy classes of radical p-chains of G. The normalizer of the latter is $(\mathbf{F}_q)^2 : H(q)$. It follows from V.3.11 of [7] that G and $(\mathbf{F}_q)^2 : H(q)$ have only the principal p-blocks. We use the argument in §6 of [14]. Consider the action of $SL_2(q)$ on $\mathrm{Irr}((\mathbf{F}_q)^2)$. Define $\nu\colon (\mathbf{F}_q)^2 \to \mathbf{C}^*$ by $\nu((\alpha,\beta)) = (\sqrt[q]{1})^{\alpha}$ for $(\alpha,\beta) \in (\mathbf{F}_q)^2$. Then ν is a character of $(\mathbf{F}_q)^2$. The stabilizer of ν in $SL_2(q)$ is

$$H_1(q) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbf{F}_q \right\}.$$

Clearly, the trivial character $1_{(\mathbf{F}_q)^2}$ and ν form a set of representatives of G-conjugacy classes of $\mathrm{Irr}((\mathbf{F}_q)^2)$. However, there are three $H(q)$-orbits, one consisting of $1_{(\mathbf{F}_q)^2}$, the orbit of ν and the one consisting of all the others. Let ν' be the character of $(\mathbf{F}_q)^2$ defined by $\nu'((\alpha,\beta)) = (\sqrt[q]{1})^{\beta}$ for $(\alpha,\beta) \in (\mathbf{F}_q)^2$. Then ν and ν' are not $H(q)$-conjugate and the stabilizer of ν' in $H(q)$ is trivial. It follows from Clifford theory that

$$\begin{aligned} k\big((\mathbf{F}_q)^2 : SL_2(q), d, [\kappa]\big) &= k\big(SL_2(q), d-2e, [\kappa]) + k(H_1(q), d-2e, [\kappa]\big) \quad \text{and} \\ k\big((\mathbf{F}_q)^2 : H(q), d, [\kappa]\big) &= k\big(H(q), d-2e, [\kappa]) + k(H_1(q), d-2e, [\kappa]\big) \\ &\quad + k\big(\{1\}, d-2e, [\kappa]\big), \end{aligned}$$

for all d and κ, since $1_{\mathbf{F}_q}$ and ν can be extended to their stabilizers. (see also 5.20 of [11].) If $p \geq 5$, then we have

(d, κ)	$(e, 1)$	$(e, 2)$	$(0, 1)$
$k(SL_2(q), d, [\kappa])$	$q - 1$	4	1
$k(H(q), d, [\kappa])$	$q - 1$	4	0

For other d's and κ's which do not appear above, the numbers of relevant characters are zero. This convention is used throughout the paper. For $p = 2$ and 3, we can ignore κ and have the following.

$p = 3$

d	e	0
$k(SL_2(q), d)$	$q + 3$	1
$k(H(q), d)$	$q + 3$	0

$p = 2$

d	e	0
$k(SL_2(q), d)$	q	1
$k(H(q), d)$	q	0

Thus the result holds. See also [16].

(1c). Use the argument in (1b). Our G has two conjugacy classes of radical p-chains. However, we have $((\mathbf{F}_{q^2})^2 : H(q^2)) : \langle\sigma\rangle$ as the normalizer of a non-trivial radical p-chain. Define $\theta\colon (\mathbf{F}_{q^2})^2 \to \mathbf{C}^*$ by $\theta((\alpha, \beta)) = \nu(T(\alpha), T(\beta))$ for $(\alpha, \beta) \in (\mathbf{F}_{q^2})^2$, where $T\colon \mathbf{F}_{q^2} \to \mathbf{F}_q$ is the usual trace map. Define θ' similarly by using ν' instead of ν. Then θ and θ' are $\langle\sigma\rangle$-invariant. Moreover, $1_{(\mathbf{F}_q)^2}$ and θ form a set of representatives of $SL_2(q^2) : \langle\sigma\rangle$-orbits in $\mathrm{Irr}((\mathbf{F}_{q^2})^2)$, and $1_{(\mathbf{F}_q)^2}$, θ and θ' form that of $H(q^2) : \langle\sigma\rangle$-orbits. Since $1_{\mathbf{F}_{q^2}}$, θ and θ' can be extended to their stabilizers, we have

$$
\begin{aligned}
k\big((\mathbf{F}_{q^2} : SL_2(q^2)) : \langle\sigma\rangle, d, [\kappa]\big) &= k\big(SL_2(q^2) : \langle\sigma\rangle, d - 4e, [\kappa]\big) \\
&\quad + k\big(H_1(q^2) : \langle\sigma\rangle, d - 4e, [\kappa]\big) \quad \text{and} \\
k\big((\mathbf{F}_{q^2} : H(q^2)) : \langle\sigma\rangle, d, [\kappa]\big) &= k\big(H(q^2) : \langle\sigma\rangle, d - 4e, [\kappa]\big) \\
&\quad + k\big(H_1(q^2) : \langle\sigma\rangle, d - 4e, [\kappa]\big) + k\big(\langle\sigma\rangle, d - 4e, [\kappa]\big).
\end{aligned}
$$

Now look at the σ action on the characters of $SL_2(q^2)$ and $H(q^2)$. For characters of $SL_2(q^2)$, see for example, [5]. The group $SL_2(q^2)$ has $(q^2 - 3)/2$ characters of degree $q^2 + 1$. It is easily seen that among those, $q - 2$ are σ-invariant and $(q - 1)^2/2$ are not. It has also $(q^2 - 1)/2$ characters of degree $q^2 - 1$. None of them are σ-invariant. There exist characters of degree 1, q^2, one for each, and of degree $(q^2 - 1)/2$ and $(q^2 + 1)/2$, two for each. These six characters are all σ-invariant. On the other hand, $H(q^2)$ has $q^2 - 1$ linear characters, and among those, exactly $q - 1$ are σ-invariant. It also has four characters of degree $(q^2 - 1)/2$. They are all σ-invariant. Thus, for $p \geq 5$, we have

(d, κ)	$(2e, 2)$	$(2e, 1)$	$(2e, 4)$	$(0, 1)$
$k(SL_2(q^2) : \langle\sigma\rangle, d, [\kappa])$	$2(q - 1)$	$q(q - 1)/2$	8	2
$k(H(q^2) : \langle\sigma\rangle, d, [\kappa])$	$2(q - 1)$	$q(q - 1)/2$	8	0

and for $p = 3$, we have

d	$2e$	0
$k(SL_2(q^2) : \langle\sigma\rangle, d)$	$(q^2 + 3q + 12)/2$	2
$k(H(q^2) : \langle\sigma\rangle, d)$	$(q^2 + 3q + 12)/2$	0

Hence the result holds.

(2). Since $p = 3$, we can ignore κ in the argument. The group $G_2(3)$ has 3 non-conjugate non-trivial parabolic subgroups U, U^σ and U_0, where U is a maximal parabolic subgroup and U_0 is a Borel subgroup. Hence the following form a set of representatives of radical 3-chains of $G_2(3)$. Note that two maximal parabolic subgroups are σ-conjugate and U_0 is σ-invariant.

radical 3-chain	normalizer	parity
$1,$	$G_2(3),$	$+$
$1 < O_3(U),$	$U,$	$-$
$1 < O_3(U)^\sigma,$	$U^\sigma,$	$-$
$1 < O_3(U) < O_3(U_0),$	$U_0,$	$+$
$1 < O_3(U)^\sigma < O_3(U_0),$	$U_0,$	$+$
$1 < O_3(U_0),$	$U_0,$	$-$

The group $G_2(3)$ has one block of defect zero and the principal block $B_0(G_2(3))$, and U and U_0 have only the principal blocks. From [6] we have the following. Note that U_0 is denoted by B in [6].

d	6	5	4
$k(G_2(3), B_0(G_2(3)), d)$	15	6	1
$k(U, B_0(G_2(3)), d)$	15	15	1
$k(U_0, B_0(G_2(3)), d)$	15	24	1

Thus the result follows for $G_2(3)$. The group $G_2(3) : \langle\sigma\rangle$ has the following set of representatives of conjugacy classes of radical 3-chains.

radical 3-chain	normalizer	parity
$1,$	$G_2(3) : \langle\sigma\rangle,$	$+$
$1 < O_3(U),$	$U,$	$-$
$1 < O_3(U) < O_3(U_0),$	$U_0,$	$+$
$1 < O_3(U_0),$	$U_0 : \langle\sigma\rangle,$	$-$

The group $G_2(3) : \langle\sigma\rangle$ has two blocks of defect zero and the principal block $B_0(G_2(3) : \langle\sigma\rangle)$, and the subgroup $U_0 : \langle\sigma\rangle$ has only the principal block. In the notation of [6], eleven characters, θ_i for $i = 0, 1, 2, 5, 6, 7, 10, 11$, $\theta_{12}(\pm 1)$ and $\chi_{14}(1)$, of $G_2(3)$ are $\langle\sigma\rangle$-invariant, and so are the ten characters $\chi_1(0, 0)$, $\chi_1((q-1)/2, 0)$, $\theta_5(0)$,

$\theta_5(1)$, $\theta_6(0)$, $\theta_6(1)$, θ_7, θ_8, $\theta_9(\pm 1)$ of U_0. The character θ_5 of $G_2(3)$ has degree 3^6 and thus lies in a block of defect zero. All the other characters lie in the corresponding principal blocks. Thus we have the following.

d	6	5	4
$k(G_2(3):\langle\sigma\rangle, B_0(G_2(3):\langle\sigma\rangle), d)$	18	6	2
$k(U_0:\langle\sigma\rangle, B_0(G_2(3):\langle\sigma\rangle), d)$	18	15	2

Thus the result follows for $G_2(3):\langle\sigma\rangle$. □

5. The case of $p = 2$ for Th

In this section, we assume that $G = Th$ and $p = 2$. Thus we can ignore κ.

5.1. Radical 2-chains. By Theorem 20 of [20], there are 19 G-conjugacy classes of non-trivial radical 2-subgroups. Among them, there are radical 2-subgroups $E \cong 2_+^{1+8}$ and $A \cong 2^5$, and we have $N_G(E) \cong E \cdot A_9$ and $N_G(A) \cong A \cdot L_5(2)$ which are only maximal 2-local subgroups of G up to G-conjugacy. (Theorem 2.2 of [18].) Moreover, $C_G(z) = N_G(E) \cong E \cdot A_9$ for the central involution z of E. Furthermore, a radical 2-subgroup of $N_G(E)$ is $N_G(E)$-conjugate to the inverse image ER in $N_G(E)$ of a radical 2-subgroup R of A_9 under the surjective map $N_G(E) \to A_9$, and each ER is a radical 2-subgroup of G. Note that A_9 has eleven classes of radical 2-subgroups, and those ER give representatives of radical 2-subgroups of G having G-conjugates whose normalizers are contained in $N_G(E)$. Thus, it follows from Lemma 3.3 that radical 2-chains of G starting with $1 < ER$ or $1 < E < ER$ for each non-trivial radical 2-subgroup R of A_9 can be eliminated in the alternating sum. In particular, it suffices to consider only $1 < E$ among those starting with $1 < E$. To examine the situation for A, let us introduce the following notation. Fix a Borel subgroup of $L_5(2)$ and let an ordered set $\Pi = \{r_1, r_2, r_3, r_4\}$ be the set of fundamental positive roots of $L_5(2)$. Then $L_5(2)$-conjugacy classes of parabolic subgroups of $L_5(2)$ are parameterized by subsets of Π. For a subset J of Π, we denote the corresponding parabolic subgroup by P_J, and the unipotent radical of P_J by U_J. A radical 2-subgroup of $L_5(2)$ is $L_5(2)$-conjugate to U_J for some J. Thus a radical 2-subgroup of $N_G(A)$ is $N_G(A)$-conjugate to AU_J for some J. However, as is shown in the proof of Theorem 20 of [20], if $r_1 \notin J$, then $N_G(AU_J) \leq N_G(E)$ up to conjugacy. Thus AU_J is G-conjugate to a radical 2-subgroup of $N_G(E)$. Hence to examine radical 2-subgroups of G contained in $N_G(A)$, it suffices to look at AU_J such that $r_1 \in J$. Those eight AU_J's with $r_1 \in J$ are representatives of remaining radical 2-subgroups of G. (Note that $AU_\Pi = A$.) Thus, again by Lemma 3.3, radical 2-chains of G starting with $1 < AU_J$ or $1 < A < AU_J$ for a subset J of Π with $r_1 \in J$ and $J \neq \Pi$ can be eliminated in the alternating sum. In particular, among those starting with $1 < A$, it suffices to consider only $1 < A$ and

$$\mathcal{C}\colon 1 < A < AU_{J_1} < AU_{J_2} < \cdots < AU_{J_t} \quad \text{with} \quad \Pi \supset J_1 \supset J_2 \supset \cdots \supset J_t$$

such that $r_1 \notin J_1$. Note that $N_G(\mathcal{C}) = AP_{J_t}$. Let us fix a subset J of Π with $r_1 \notin J$, and let $\mathcal{R}'_2(G, A, AU_J)$ be the subset of $\mathcal{R}_2(G)$ consisting of those $\mathcal{C}$ above such that the last subgroup in $\mathcal{C}$ is AU_J. (So $J_t = J$ in the above.) If $J \neq \{r_2, r_3, r_4\}$, then we can define a map $f\colon \mathcal{R}'_2(G, A, AU_J) \to \mathcal{R}'_2(G, A, AU_J)$ by

$$f(\mathcal{C}) = \begin{cases} 1 < A < AU_{J_2} < \cdots < AU_{J_t}, & \text{if } J_1 = \{r_2, r_3, r_4\}, \\ 1 < A < AU_{\{r_2,r_3,r_4\}} < AU_{J_1} < AU_{J_2} < \cdots < AU_{J_t}, & \text{otherwise,} \end{cases}$$

in the above notation. Then f gives an involutive bijection on $\mathcal{R}'_2(G, A, AU_J)$ and we have $N_G(\mathcal{C}) = N_G(f(\mathcal{C}))$ and $|\mathcal{C}| = |f(\mathcal{C})| \pm 1$. Thus we can eliminate the chains in $\mathcal{R}'_2(G, A, AU_J)$ in the alternating sum. Now the remaining radical 2-chains are $1 < A$ and $1 < A < AU_{J_0}$, where $J_0 = \{r_2, r_3, r_4\}$. Note that $U_{J_0} \cong 2^4$ and $P_{J_0} \cong 2^4 : L_4(2)$. Thus the normalizer of $1 < A < AU_{J_0}$ has the structure $A \cdot (2^4 : L_4(2))$. Moreover, since we may assume $N_G(AU_{J_0}) \leq N_G(E)$, we have $AU_{J_0} = E$ and $A \cdot (2^4 : L_4(2)) = E \cdot L_4(2) = E \cdot A_8$. Hence, we have the following.

Proposition 5.1. *To prove that the conjecture holds for G and $p = 2$, it suffices to consider the alternating sum with respect to the following radical 2-chains.*

$$\begin{array}{ll} C_0\colon 1, & N_G(C_0) = G, \\ C_1\colon 1 < E, & N_G(C_1) = N_G(E) \cong E \cdot A_9, \\ C_2\colon 1 < A, & N_G(C_2) = N_G(A) \cong A \cdot L_5(2), \\ C_3\colon 1 < A < A \cdot 2^4, & N_G(C_3) = N_G(E) \cap N_G(A) \cong A \cdot (2^4 : L_4(2)) = E \cdot A_8. \end{array}$$

5.2. 2-blocks of G, $N_G(E)$ and $N_G(A)$. As is shown in ATLAS, G has 48 irreducible characters. From the character table of G, it follows that G has four 2-blocks,

$$B_0 = B_0(G), \{\chi_{22}\}, \{\chi_{23}\}, \{\chi_{38}\},$$

where $\chi_{22}(1) = \chi_{23}(1) = 2^{15} \cdot 5^3$ and $\chi_{38}(1) = 2^{15} \cdot 7^2 \cdot 19$.

Since $N_G(E) \cong 2_+^{1+8} \cdot A_9$ and $N_G(A) \cong A \cdot L_5(2)$ have normal selfcentralizing 2-subgroups, they have only the principal 2-blocks, which of course induce B_0. (see V.3.11 of [7].) They are maximal subgroups of G and appear in the GAP library.

From their character tables, we have the following.

d	15	14	13	12	11	10	9	8	7	6	5
$k(G, B_0, d)$	16	4	2	14	2	0	3	0	2	2	0
$k(N_G(E), B_0, d)$	16	4	2	14	1	0	3	7	3	2	0
$k(N_G(A), B_0, d)$	16	4	2	14	2	0	2	0	0	0	1

5.3. 2-blocks of $N_G(E) \cap N_G(A)$. By 5.1, $N_G(E) \cap N_G(A)$ has a normal subgroup $E \cong 2^5.2^4 \cong 2_+^{1+8}$, which is self-centralizing. Thus $N_G(E) \cap N_G(A)$ has only the principal 2-block, which induces B_0. Moreover, we have $(N_G(E) \cap N_G(A))/E \cong L_4(2) \cong A_8$.

The set $\mathrm{Irr}(E)$ consists of one character η of degree 2^4 and 2^8 characters of degree 1. From the character table of $N_G(E)$, we can compute the $N_G(E)$-orbits in $\mathrm{Irr}(E)$. Note first that in the notation of GAP library, $1a$, $2a$, $2b$ and $4a$ elements of $N_G(E)$ form E. Of course, the trivial character 1_E of E and η form their own orbits of length one. We use the same numbering of characters of $N_G(E)$ as appeared in GAP library. First of all, by looking at the values at $2b$ and $4a$ elements of $N_G(E)$, we can see that $(\chi_{21})_E$ and $(\chi_{32})_E$ lie over different orbits. Moreover, it is easy to see that a character of $N_G(E)$ lying over a non-trivial linear character of E has a restriction which is a multiple of either $(\chi_{21})_E$ or $(\chi_{32})_E$. Thus, Clifford theory implies that there are exactly two $N_G(E)$-orbits of non-trivial linear characters of E. Moreover, we have $[(\chi_{21})_E, (\chi_{21})_E] = 120$ and $[(\chi_{32})_E, (\chi_{32})_E] = 135$. Since E has $2^8 - 1 = 255$ non-trivial linear characters, we can conclude that the two orbits consist of 120 and 135 characters, respectively. Let θ_1 and θ_2 be representatives of these orbits of length 120 and 135, respectively. Then 1_E, η, θ_1 and θ_2 form a set of representatives of $N_G(E)$-orbits in $\mathrm{Irr}(E)$. For $i = 1, 2$, let T_i be the stabilizer of θ_i in $N_G(E)$. Note also that $(\chi_{21})_E$ and $(\chi_{32})_E$ are exactly the sum of characters in the orbits of θ_1 and θ_2, respectively. This implies that θ_i can be extended to T_i, for $i = 1, 2$.

Let us examine the individual orbits. From the character table of $L_4(2) \cong A_8$, we have

(1)

d	15	14	13	12	11	10	9	8	7	6	5
$k(N_G(E) \cap N_G(A), d \mid 1_E)$	8	2	2	1	0	0	1	0	0	0	0

The character η is $N_G(E)$-invariant. Recall that $N_G(E)/E \cong A_9$. If η has an extension to $N_G(E)$, then, since A_9 has eight characters of odd degree by ATLAS, $N_G(E)$ has at least eight characters of defect 11. However, as is seen in the previous subsection, this is not the case. Hence, η can not be extended to $N_G(E)$. Consider $\mathrm{Irr}(N_G(E) \mid \eta)$. Let ζ be the non-trivial character of $Z(2.A_9)$. Since η does not have an extension to $N_G(E)$ and since $(N_G(E) \cap N_G(A))/E$ corresponds to $L_4(2) \cong A_8$ in $N_G(E)/E \cong A_9$, the theory of character triple (Chap. 11 of [9]) tells us that there is a bijection σ from $\mathrm{Irr}(N_G(E) \cap N_G(A) \mid \eta)$ to $\mathrm{Irr}(2.A_8 \mid \zeta)$ such that $\varphi(1)/\eta(1) = \sigma(\varphi)(1)/\zeta(1)$ for all φ in $\mathrm{Irr}(N_G(E) \cap N_G(A) \mid \eta)$. (see (11.28), (11.23) and (11.24) of [9].) Thus the character table of $2.A_8$ gives the following.

(2)

d	15	14	13	12	11	10	9	8	7	6	5
$k(N_G(E) \cap N_G(A), d \mid \eta)$	0	0	0	0	0	0	0	7	1	0	1

The group T_1/E has index 120 in $N_G(E)/E \cong A_9$. Form p. 37 of ATLAS, it must follow that $T_1/E \cong L_2(8):3$. To obtain $N_G(E) \cap N_G(A)$-orbit of θ_1, we consider $A_8 \cap (L_2(8):3)$. Since $|A_8| = 2^6{\cdot}3^2{\cdot}5{\cdot}7$ and $|L_2(8):3| = 2^3{\cdot}3^3{\cdot}5{\cdot}7$, the group $A_8 \cap (L_2(8):3)$

is a proper subgroup of $L_2(8):3$. On the other hand, $|A_8 : (A_8 \cap T_1)| \leq 120$. Thus it follows from p. 6 of ATLAS that $A_8 \cap (L_2(8):3)$ is isomorphic to either $L_2(8)$ or $2^3:7:3$. However, if the former was true, then it would follow from p. 22 of ATLAS that $L_2(8) \leq A_7$, which would contradict p. 10 of ATLAS, since A_7 does not have a subgroup isomorphic to $L_2(8)$. Thus we can conclude that $A_8 \cap (L_2(8):3) \cong 2^3:7:3$. In particular, the $N_G(E) \cap N_G(A)$-orbit of θ_1 consists of 120 characters. Since θ_1 can be extended to $N_G(A) \cap T_1$ and since the 2-part of $|A_8 : A_8 \cap (L_2(8):3)|$ is 2^3, the character table of $2^3:7:3$ gives the following.

(3)

d	15	14	13	12	11	10	9	8	7	6	5
$k(N_G(E) \cap N_G(A), d \mid \theta_1)$	0	0	0	8	0	0	0	0	0	0	0

The group T_2/E has index 135 in A_9. From p. 37 of ATLAS, we may assume that $T_2/E \leq A_8$ and p. 22 of ATLAS implies that $T_2/E \cong 2^3 : L_3(2)$. Thus we may assume that $T_2 \leq N_G(E) \cap N_G(A)$, and then it follows that the $N_G(E) \cap N_G(A)$-orbit of θ_2 consists of 15 characters. Since θ_2 can be extended to T_2, the character table of $2^3 : L_3(2)$ gives the following.

(4)

d	15	14	13	12	11	10	9	8	7	6	5
$k(N_G(E) \cap N_G(A), d \mid \theta_2)$	8	2	0	1	0	0	0	0	0	0	0

Finally, let θ_3 be a character in the $N_G(E)$-orbit of θ_2 such that θ_2 and θ_3 are not $N_G(E) \cap N_G(A)$-conjugate. Let T_3 be the stabilizer of θ_3 in $N_G(E)$. Since we assume that $T_2/E \leq A_8$, it follows that T_3/E is contained in an A_9-conjugate of A_8 which is different from the original A_8. This means that $(N_G(E) \cap N_G(A) \cap T_3)/E$ is isomorphic to a subgroup of A_7. Recall that $|T_3/E| = |T_2/E| = 2^6 \cdot 3 \cdot 7$, $|A_7| = 2^3 \cdot 3^2 \cdot 5 \cdot 7$ and $|N_G(E) \cap N_G(A) : (N_G(E) \cap N_G(A) \cap T_3)| \leq 135 - 15 = 120$. From these, it must follow that $|N_G(E) \cap N_G(A) : (N_G(E) \cap N_G(A) \cap T_3)| = 120$. In particular, the $N_G(E) \cap N_G(A)$-orbit of θ_3 consists of 120 characters. Moreover, since $|(N_G(E) \cap N_G(A) \cap T_3)/E| = 2^3 \cdot 3 \cdot 7$, p. 10 of ATLAS implies that $(N_G(E) \cap N_G(A) \cap T_3)/E \cong L_2(7)$. Since θ_3 can be extended to T_3, the character table of $L_2(7)$ gives the following.

(5)

d	15	14	13	12	11	10	9	8	7	6	5
$k(N_G(E) \cap N_G(A), d \mid \theta_3)$	0	0	0	4	1	0	1	0	0	0	0

Since 1_E, η, θ_i, $i = 1, 2, 3$ form a set of representatives of $N_G(E) \cap N_G(A)$-orbits in $\mathrm{Irr}(E)$, the above tables (1)–(5) give the following.

d	15	14	13	12	11	10	9	8	7	6	5
$k(N_G(E) \cap N_G(A), d)$	16	4	2	14	1	0	2	7	1	0	1

5.4. Conjecture. Summarizing the above, we have the following.

d	15	14	13	12	11	10	9	8	7	6	5	parity
$k(G, B_0, d)$	16	4	2	14	2	0	3	0	2	2	0	+
$k(N_G(E), B_0, d)$	16	4	2	14	1	0	3	7	3	2	0	−
$k(N_G(A), B_0, d)$	16	4	2	14	2	0	2	0	0	0	1	−
$k(N_G(E) \cap N_G(A), B_0, d)$	16	4	2	14	1	0	2	7	1	0	1	+

Hence, by Proposition 5.1, Conjecture 2.1 holds for G in the case of $p = 2$.

6. The case of $p = 3$ for Th

In this section, we assume that $G = Th$ and $p = 3$. Thus we can ignore κ.

6.1. Radical 3-subgroups and radical 3-chains. Maximal 3-local subgroups are classified in §4 of [18]. There are three conjugacy classes $3A$, $3B$ and $3C$ of elements of order three, and we have

$$\begin{aligned} L_1 &= N_G(3A) \cong (3 \times G_2(3)) : 2, \\ L_2 &= N_G(3B) \cong (3^3 \times 3_+^{1+2}) \cdot 3_+^{1+2} : 2S_4, \\ L_3 &= N_G(3C) \cong (3 \times 3^4 : 2A_6) : 2 \cong 3^5 : 2S_6. \end{aligned}$$

Let T denote the normal subgroup $3^3 \times 3_+^{1+2}$ of $N_G(3B)$ and let $C = O_3(N_G(3A))$. There is a central $3B^2$ group in T, and we have

$$L_4 = N_G(3B^2) \cong (3^2 \cdot [3^7]) \cdot 2S_4.$$

Any 3-local subgroups of G is contained in one of L_1, L_2, L_3 and L_4. (see Theorem 4.1 of [18].) Thus a radical 3-subgroup of G is that of L_i for some i.

Let us examine radical 3-subgroups of L_i's. First of all, $O_3(L_i)$ is a radical 3-subgroups of L_i and thus of G for each i. Moreover, a radical 3-subgroup R of L_i must contain $O_3(L_i)$. (Prop. 1.4 of [2]) In particular, $R/O_3(L_i)$ is a radical 3-subgroup of $L_i/O_3(L_i)$. Let us treat L_i individually. The group $G_2(3) : 2$ is the finite Chevalley group $G_2(3)$ with the non-trivial graph automorphism of order 2. Thus, up to $G_2(3) : 2$-conjugate, there are two non-trivial radical 3-subgroups of $G_2(3) : 2$. Namely, $U \cong 3^2 \times 3_+^{1+2}$ with normalizer $U : GL_2(3)$ and $U_0 \cong 3^2 \cdot (3 \times 3_+^{1+2})$ with normalizer $U_0 : D_8$ in $G_2(3) : 2$, the unipotent radical of a maximal parabolic subgroup and the unipotent radical of a Borel subgroup, respectively. On the other hand, the groups $2S_4$ and $2A_6 : 2$ have only one non-trivial radical 3-subgroup up to conjugate. Namely, their Sylow 3-subgroups. Let S and P be Sylow 3-subgroups of L_2 and L_3, respectively. Note that S is a Sylow 3-subgroup of L_2, L_4 and of G.

Let R be a 3-radical subgroup of G. Suppose that $N_G(R) \leq L_1$. Then $R \cap (3 \times G_2(3))$ is a radical subgroup of $3 \times G_2(3)$ and thus R is conjugate to $C = O_3(L_1)$,

$C \times U$ or $C \times U_0$. From the information in ATLAS [1] (pp. 60–61), it follows that $[U, U]$ is of order 3 generated by $3A$ or $3B$ element of $G_2(3)$ and the second commutator subgroup of U_0 is also of order 3 generated by $3A$ or $3B$ element of $G_2(3)$. Hence, if R is conjugate to $C \times U$ or $C \times U_0$, then by p. 20 of [18], $N_G(R)$ is contained in $L_2 = N_G(3B)$, and thus, R is a radical 3-subgroup of L_2. But, by comparing order of R and radical subgroups of L_2, we can see that it is not the case. Hence we can conclude that $R = O_3(L_1)$. Now suppose that $N_G(R) \leq L_3$. Then R is conjugate to $O_3(L_3)$ or P. Let z be a $3C$ element of G. Then $C_G(z) = 3 \times 3^4 : 2A_6 \cong 3 \times 3^4 : SL_2(9)$, in which the group $SL_2(9)$ acts naturally on $3^4 \cong (\mathbf{F}_9)^2$ (p. 21 of [18]). Thus P is a Sylow 3-subgroup of $C_G(z)$, and $[P, P]$ is a one-dimensional subgroup of $(\mathbf{F}_9)^2$, in which all the nonzero vectors are transitive under the action of an element of order 8 of $SL_2(9)$. Consider conjugacy classes of non-trivial elements of $[P, P]$ in G. Since there is no elementary abelian subgroup of order 9 whose non-trivial elements are all $3C$-elements (the 5th paragraph of p. 21 of [18]), every non-trivial element of $[P, P]$ is either $3A$ or $3B$-element. Moreover, since $O_3(L_3) \cong 3^5$ has type $3C_1B_{40}C_{80}$ (2nd paragraph of p. 21 of [18]), we can conclude that all the non-trivial elements of $[P, P]$ are $3B$-elements. On the other hand, by the argument in the 2nd paragraph of p. 22 of [18], $O_3(L_3) \cong 3^5$ is conjugate to a subgroup of $T = 3^3 \times 3^{1+2} \trianglelefteq N_G(3B)$. Hence $[P, P]$ is a $3B^2$-group in T. As is stated in the 3rd paragraph of p. 23 of [18], $N_G([P, P])$ is either L_4 or is contained in L_2. Thus we have $N_G(P) \leq N_G([P, P]) \leq L_i$ for $i = 2$ or $i = 4$, and P is a radical 3-subgroup of L_2 or L_4. But, by comparing orders of P and radical subgroups of L_2 and L_4, we can see that it is not the case. Hence we can conclude that $R = O_3(L_3)$. Now, suppose that $N_G(R) \leq L_2$ or $N_G(R) \leq L_4$. Then R is conjugate to $O_3(L_2)$, $O_3(L_4)$ or S.

The above arguments imply that a non-trivial radical 3-subgroup of G is conjugate to one of the following.

$$O_3(L_1), O_3(L_2), O_3(L_3), O_3(L_4), S$$

Now look at radical 3-chains of G. By the argument above, radical 3-chains starting with $1 < C$ are conjugate to

$$\begin{aligned} &1 < C, \\ &1 < C < C \times U, \\ &1 < C < C \times U_0 \quad \text{or} \\ &1 < C < C \times U < C \times U_0. \end{aligned}$$

Radical 3-chains starting with $1 < O_3(L_2)$ or $1 < O_3(L_3)$ are conjugate to $1 < O_3(L_2)$, $1 < O_3(L_2) < S$, $1 < O_3(L_3)$ or $1 < O_3(L_3) < P$. Finally, radical 3-chains starting with $1 < O_3(L_4)$ are conjugate to $1 < O_3(L_4)$ or $1 < O_3(L_4) < S$, and $1 < S$ is the unique radical 3-chain starting with $1 < S$. Since $1 < O_3(L_4) < S$ and $1 < S$ have the same normalizers and their length have opposite parity, we will not take them into account in the alternating sum. Summarizing the above, we have the following.

Proposition 6.1. *To prove that the conjecture holds for G and $p = 3$, it suffices to consider the alternating sum with respect to the following radical 3-chains.*

radical 3-chain	normalizer
$C_0: 1,$	$G,$
$C_1: 1 < C = O_3(N_G(3A)),$	$(3 \times G_2(3)) : 2,$
$C_2: 1 < C < C \times U,$	$3 \times U : GL_2(3)(< 3 \times G_2(3)),$
$C_3: 1 < C < C \times U_0$	$(3 \times U_0) : D_8$
$C_4: 1 < C < C \times U < C \times U_0$	$3 \times U_0 : 2^2(< 3 \times G_2(3))$
$C_5: 1 < O_3(N_G(3B)),$	$(3^3 \times 3_+^{1+2}) \cdot 3_+^{1+2} : 2S_4,$
$C_6: 1 < O_3(N_G(3B)) < S,$	$(3^3 \times 3_+^{1+2}) \cdot 3_+^{1+2} : (3 : 2^2),$
$C_7: 1 < O_3(N_G(3C)),$	$(3 \times 3^4 : 2A_6) : 2 \cong 3^5 : 2S_6,$
$C_8: 1 < O_3(N_G(3C)) < P,$	$(3 \times (3^4 : (3^2 : 8))) : 2,$
$C_9: 1 < O_3(N_G(3B^2)),$	$(3^2 \cdot [3^7]) \cdot 2S_4.$

The notation used in this subsection will be used throughout this section, though sometimes other letters have different meanings when appearing in different subsections.

6.2. 3-blocks of G and a reduction. The character table tells us that G has the following 3-blocks. (see also [8].)

$$B_0 = B_0(G), \ \ B' = \{\chi_{25}, \chi_{43}, \chi_{45}\}, \ \ \{\chi_{12}\}, \ \ \{\chi_{13}\}, \ \ \{\chi_{42}\}, \ \ \{\chi_{48}\}.$$

Here $\chi_{25}(1) = 2^3 \cdot 3^9 \cdot 31$, $\chi_{43}(1) = 3^9 \cdot 5^3 \cdot 31$, $\chi_{45}(1) = 3^9 \cdot 7 \cdot 19 \cdot 31$, $\chi_{12}(1) = \chi_{13}(1) = 3^{10} \cdot 13$, $\chi_{42}(1) = 3^{10} \cdot 5 \cdot 13 \cdot 19$, $\chi_{48}(1) = 2^3 \cdot 3^{10} \cdot 13 \cdot 31$. The block B_0 and B' are of defect 10 and 1, respectively. Indeed, B' has C as a defect group. Thus, it suffices to consider B_0. The character table of G tells us that we have the following.

d	10	9	8	7	6	5
$k(G, B_0, d)$	15	6	1	18	0	1

Note that C_1, C_2, C_3 and C_4 are representatives of radical 3-chains of G starting with $1 < O_3(N_G(3A))$ and C_7 and C_8 are representatives of radical 3-chains of G starting with $1 < O_3(N_G(3C))$. Furthermore, we have

$$\begin{aligned} &C_G(3A) \cong 3 \times G_2(3), && N_G(3A) = N_G(C) \cong (3 \times G_2(3)) : 2, \\ &C_G(3C) \cong 3 \times 3^4 : 2A_6 \cong 3 \times 3^4 : SL_2(9), && N_G(3C) \cong (3^5 : SL_2(9)) : \langle \sigma \rangle, \end{aligned}$$

where in $C_G(3C)$ the group $SL_2(9)$ acts naturally on $3^4 \cong (\mathbf{F}_9)^2$, and in $N_G(3C)$ the element σ inverts the central $3C$-element and acts as the Galois automorphism of order two on $(\mathbf{F}_9)^2 : SL_2(9)$. (For the structure of $N_G(3A)$, see the previous subsection.)

Hence Theorem 4.4 and Proposition 4.5 imply that

$$\sum_{i=1}^{4}(-1)^{|C_i|}k\big(N_G(C_i), B_0, d\big) = \sum_{i=7}^{8}(-1)^{|C_i|}k\big(N_G(C_i), B_0, d\big) = 0$$

for all d with $d \neq 1$. (Here a cyclic group generated by a $3A$- or a $3C$-element is the group Q in Theorem 4.4.) Consider the case of $d = 1$. Note that the factor groups $N_G(C_i)/O_3(N_G(3A))$, for $i = 2, 3, 4$, and $N_G(C_i)/O_3(N_G(3C))$ for $i = 7, 8$ do not have characters of defect zero. Thus $k(N_G(C_i), B_0, 1) = 0$ for $i = 2, 3, 4, 7$ and 8. (see also the proof of Theorems 4.2 and 4.4.) The group $N_G(3A)$ appears as the 5th maximal subgroup of G in the GAP library. It has the principal block and one non-principal 3-block consisting of two characters of degree 3^6 and one character of degree $3^6 \cdot 2$. (In the notation given in GAP library they are χ_{23}, χ_{24} and χ_{49}. This block is of defect 1 and induces B'.) The principal block of $N_G(3A)$ does not have a character of defect 1. Thus $k(N_G(C_1), B_0, 1) = 0$. Hence we have the following.

Proposition 6.2. *To prove that the conjecture holds for G and $p = 3$, it suffices to consider the alternating sum with respect to the radical 3-chains C_0, C_5, C_6 and C_9.*

6.3. 3-blocks of $N_G(3B)$ and $N_G(3B) \cap N_G(S)$. The group $N_G(3B) \cong (3^3 \times 3_+^{1+2}) \cdot 3_+^{1+2} : 2S_4$ appears as the 6th maximal subgroup of G in the GAP library. It has 62 characters and only the principal block. Since the local principal block induces B_0, we have the following.

d	10	9	8	7	6	5
$k(N_G(3B), B_0, d)$	15	15	7	18	6	1

Let us turn to $N_G(3B) \cap N_G(S)$. Let N be $O_3(N_G(3B)) \cong (3^3 \times 3_+^{1+2}) \cdot 3_+^{1+2}$. From the character table of $N_G(3B)$, we can compute the character degrees of N. Our strategy is as follows. Let χ be a character of $N_G(3B)$. By Clifford theory, the restriction χ_N of χ to N has a form

$$\chi_N = e\sum_{i=1}^{m}\theta_i,$$

where e is an integer and θ_i's are $N_G(3B)$-conjugate characters of N. The inner product $[\chi_N, \chi_N]$ is then e^2m. Moreover, $\theta_1(1) = \chi(1)/em$ is a power of 3 and $(1/e)\chi_N$ is a (not necessarily irreducible) character of N. Using the above, we can determine $N_G(3B)$-orbits of $\mathrm{Irr}(N)$ only by computing $[\chi_N, \chi_N]$. The result is as follows. We use the same numbering of characters of $N_G(3B)$ as appeared in GAP library. Notice also that, if $e = 1$ for some χ, then characters appearing in χ_N can be extended to its stabilizer, and that the order of the stabilizer is $|N_G(3B)|/m = 48|N|/m$.

First of all, $\chi_1, \chi_2, \dots, \chi_8$ have N in their kernels, and thus they lie over 1_N. Characters $\chi_9, \chi_{10}, \dots, \chi_{18}$ have $T = 3^3 \times 3_+^{1+2}$ in their kernels, and thus they can be considered as characters lying over those of $O_3(N_G(3B))/T \cong 3_+^{1+2}$. In fact, we have the following.

$$
\begin{array}{llll}
i = 9, 10, & \chi_i(1) = 8, & (\chi_i)_N = \sum_{j=1}^{8} \lambda'_j, & \lambda'_j(1) = 1 \\
i = 11, & \chi_{11}(1) = 16, & (\chi_{11})_N = 2\sum_{j=1}^{8} \lambda'_j, & \\
i = 12, 13, 14, & \chi_i(1) = 6, & (\chi_i)_N = \sum_{j=1}^{2} \mu'_j, & \mu'_j(1) = 3 \\
i = 15, 16, 17, & \chi_i(1) = 12, & (\chi_i)_N = 2\sum_{j=1}^{2} \mu'_j, & \\
i = 18, & \chi_{18}(1) = 18, & (\chi_{18})_N = 3\sum_{j=1}^{2} \mu'_j. &
\end{array}
$$

For the other characters, we have the following. Here for characters of N, different letters signify different characters.

$\chi \in \mathrm{Irr}(N_G(3B))$	Character degree	$[\chi_N, \chi_N]$	χ_N	Character degree
χ_{19}, χ_{20}	$\chi_{19}(1) = 8$	8	$\sum_{i=1}^{8} \lambda_i$	$\lambda_1(1) = 1$
χ_{21}	$\chi_{21}(1) = 16$	32	$2\sum_{i=1}^{8} \lambda_i$	$\lambda_1(1) = 1$
$\chi_{22}, \chi_{23}, \chi_{24}$	$\chi_{22}(1) = 16$	16	$\sum_{i=1}^{16} \nu_i$	$\nu_1(1) = 1$
χ_{25}, χ_{28}	$\chi_{25}(1) = 24$	24	$\sum_{i=1}^{24} \tau_i$	$\tau_1(1) = 1$
χ_{26}, χ_{27}	$\chi_{26}(1) = 24$	24	$\sum_{i=1}^{24} \tau'_i$	$\tau'_1(1) = 1$
$\chi_{29}, \chi_{30}, \chi_{31}$	$\chi_{29}(1) = 48$	16	$\sum_{i=1}^{16} \xi_i$	$\xi_1(1) = 3$
$\chi_{32}, \chi_{33}, \chi_{34}, \chi_{35}$	$\chi_{32}(1) = 27$	3	$\sum_{i=1}^{3} \theta_i$	$\theta_1(1) = 9$
$\chi_{36}, \chi_{37}, \chi_{38}, \chi_{39}$	$\chi_{36}(1) = 27$	3	$\sum_{i=1}^{3} \theta'_i$	$\theta'_1(1) = 9$
$\chi_{40}, \chi_{41}, \chi_{44}$	$\chi_{40}(1) = 54$	12	$2\sum_{i=1}^{3} \theta_i$	$\theta_1(1) = 9$
$\chi_{42}, \chi_{43}, \chi_{45}$	$\chi_{42}(1) = 54$	12	$2\sum_{i=1}^{3} \theta'_i$	$\theta'_1(1) = 9$
χ_{46}, χ_{47}	$\chi_{46}(1) = 72$	8	$\sum_{i=1}^{8} \phi_i$	$\phi_1(1) = 9$
χ_{49}	$\chi_{49}(1) = 144$	32	$2\sum_{i=1}^{8} \phi_i$	$\phi_1(1) = 9$
$\chi_{48}, \chi_{50}, \chi_{51}$	$\chi_{48}(1) = 144$	16	$\sum_{i=1}^{16} \eta_i$	$\eta_1(1) = 9$
χ_{52}, χ_{53}	$\chi_{52}(1) = 216$	24	$\sum_{i=1}^{24} \zeta_i$	$\zeta_1(1) = 9$
χ_{54}, χ_{55}	$\chi_{54}(1) = 216$	24	$\sum_{i=1}^{24} \zeta'_i$	$\zeta'_1(1) = 9$
$\chi_{56}, \chi_{57}, \chi_{58}$	$\chi_{56}(1) = 162$	2	$\sum_{i=1}^{2} \psi_i$	$\psi_1(1) = 81$
$\chi_{59}, \chi_{60}, \chi_{61}$	$\chi_{59}(1) = 324$	8	$2\sum_{i=1}^{2} \psi_i$	$\psi_1(1) = 81$
χ_{62}	$\chi_{62}(1) = 486$	18	$3\sum_{i=1}^{2} \psi_i$	$\psi_1(1) = 81$

Notice that all characters of N have extensions to their stabilizers. Let I and I' be the stabilizers in $N_G(3B)$ and in $N_G(3B) \cap N_G(S)$, respectively, of representatives of $N_G(3B)$-orbits in $\mathrm{Irr}(N)$. The structure of I/N can easily be obtained from the length of orbits. There are two isomorphism classes of subgroups of $N_G(3B)/N \cong 2S_4$ of order 6. However, from the degrees of characters of $N_G(3B)$ lying over the character of N, we can see $I/N \cong S_3$ when the orbit length is 8. The structures of I/N are as follows. Also I' for typical characters of N are given. Here D and D' are Sylow 2-subgroups of $N_G(3B)$ and $N_G(3B) \cap N_G(S)$, respectively.

	1_N	λ'_1	μ'_1	λ_1	ν_1	τ_1	τ'_1	ξ_1	θ_1	θ'_1	ϕ_1	η_1	ζ_1	ζ'_1	ψ_1
I/N	$2.S_4$	S_3	$SL_2(3)$	S_3	3	2	2	3	D	D	S_3	3	2	2	$SL_2(3)$
I'/N	$2S_3$	S_3	6	S_3	3	2	2	3	D'	D'	S_3	3	2	2	6

If $|I : I'| = 4$, then the $N_G(3B)$-orbit is also an $N_G(3B) \cap N_G(S)$-orbit. There are two conjugacy classes of involutions in $N_G(3B)$. Let z and z' be representatives of them, and we assume that z corresponds to the central involution of $2S_4$ (indicated as $2a$ in GAP library) and z' does to a non-central one. Note that z lies in $N_G(3B) \cap N_G(S)$ and we may suppose that z' lies in $N_G(3B) \cap N_G(S)$. If $I/N \cong S_3$, then z does not lie in I. We may suppose that z' lies in the stabilizer of λ'_1. Then the $N_G(3B) \cap N_G(S)$-orbit of λ'_1 consists of λ'_1 and $(\lambda'_1)^z$, since $I = I'$. Say $(\lambda'_1)^z = \lambda'_2$. Note that the stabilizers of λ'_1 and λ'_2 in $N_G(3B)$ are the same. Since the image of z' in $2S_4$ is contained in exactly two $N_G(3B)$-conjugates of $I/N \cong S_3$, we may assume that z' lies in the stabilizer of λ'_3 in $N_G(3B) \cap N_G(S)$, which is $\langle z' \rangle N$, since $I/N(\cong S_3)$ is a unique $N_G(3B)$-conjugate of it contained in $N_G(3B) \cap N_G(S)/N$. Then the $N_G(3B) \cap N_G(S)$-orbit of λ'_3 is $\{\lambda'_3, \lambda'_4, \dots, \lambda'_8\}$. This argument applies also for λ_i's and ϕ_i's. On the other hand, if z stabilizes some character of N, then it stabilizes all the characters in its $N_G(3B)$-orbit. Thus, it is easy to show that, if $\chi(z) = 0$, then z does not stabilize any constituents of χ_N. Hence z is not in the stabilizer of τ_i, τ'_i, ζ_i and ζ'_i. The $N_G(3B)/N$-conjugacy class of $z'N$ consists of 12 elements and among them 6 elements lie in $N_G(3B) \cap N_G(S)/N$, while the $N_G(3B) \cap N_G(S)/N$-conjugacy class of $z'N$ consists of 3 elements. Thus, we may assume that each of the stabilizers of τ_{2i-1} in $N_G(3B) \cap N_G(S)$ for $1 \le i \le 3$ consists of some $N_G(3B) \cap N_G(S)$-conjugate of z' and each of the stabilizers of τ_{2i-1} in $N_G(3B) \cap N_G(S)$ for $4 \le i \le 6$ consists of some $N_G(3B)$-conjugate but not an $N_G(3B) \cap N_G(S)$-conjugate of z'. Then we can conclude that $N_G(3B) \cap N_G(S)$-orbits of $\{\tau_i\}$ are $\{\tau_1, \tau_2, \dots, \tau_6\}$, $\{\tau_7, \tau_8, \dots, \tau_{12}\}$ and $\{\tau_{13}, \tau_{14}, \dots, \tau_{24}\}$, where $\tau_{2i} = (\tau_{2i-1})^z$ for $1 \le i \le 6$. In particular, the stabilizer of τ_{13} in $N_G(3B) \cap N_G(S)$ is N. This argument applies also for τ'_i's, ζ_i's and ζ'_i's. Finally, suppose that the stabilizer of ν_1 in $N_G(3B) \cap N_G(S)$ has order 3 and its $N_G(3B) \cap N_G(S)$-orbit is $\{\nu_1, \nu_2, \nu_3, \nu_4\}$. Then the stabilizer of ν_5 in $N_G(3B) \cap N_G(S)$ must be N and its $N_G(3B) \cap N_G(S)$-orbit is $\{\nu_5, \nu_6, \dots, \nu_{16}\}$. This argument applies also for ξ_i's and η_i's.

From the above observation we can compute $k(N_G(3B) \cap N_G(S), d|\gamma)$ for a character γ of N and d, $5 \leq d \leq 10$.

$\gamma\backslash d$	10	9	8	7	6	5
1_N	6	0	0	0	0	0
λ_1, λ_1'	3	0	0	0	0	0
λ_3, λ_3'	0	2	0	0	0	0
μ_1'	0	6	0	0	0	0
ν_1	3	0	0	0	0	0
ν_5	0	1	0	0	0	0
$\tau_1, \tau_7, \tau_1', \tau_7'$	0	2	0	0	0	0
τ_{13}, τ_{13}'	0	1	0	0	0	0
ξ_1	0	3	0	0	0	0
ξ_5	0	0	1	0	0	0
θ_1, θ_1'	0	0	0	4	0	0
ϕ_1	0	0	3	0	0	0
ϕ_3	0	0	0	2	0	0
η_1	0	0	3	0	0	0
η_5	0	0	0	1	0	0
$\zeta_1, \zeta_7, \zeta_1', \zeta_7'$	0	0	0	2	0	0
ζ_{13}, ζ_{13}'	0	0	0	1	0	0
ψ_1	0	0	0	0	6	0

Consequently, we obtain the following.

d	10	9	8	7	6	5
$k(N_G(3B) \cap N_G(P), B_0, d)$	15	24	7	21	6	0

6.4. 3-blocks of $N_G(3B^2)$. The group $N_G(3B^2) \cong (3^2 \cdot [3^7]) \cdot 2S_4$ appears as the 7th maximal subgroup of G in the GAP library. It has 52 characters and only the principal block. Since the local principal block induces B_0, we have the following.

d	10	9	8	7	6	5
$k(N_G(3B^2), B_0, d)$	15	15	1	21	0	0

6.5. Conjecture. From the results in 6.2, 6.3 and 6.4, we have the following.

d	10	9	8	7	6	5	parity
$k(G, B_0, d)$	15	6	1	18	0	1	+
$k(N_G(3B), B_0, d)$	15	15	7	18	6	1	–
$k(N_G(3B) \cap N_G(S), B_0, d)$	15	24	7	21	6	0	+
$k(N_G(3B^2), B_0, d)$	15	15	1	21	0	0	–

Hence, by Proposition 6.2, Conjecture 2.1 holds for G in the case of $p = 3$.

7. The case of $p = 5$ for Th

In this section, we assume that $G = Th$ and $p = 5$.

7.1. Radical 5-chains. ATLAS shows that there is the unique conjugacy class of elements of order 5, and a cyclic subgroup C of order 5 satisfies $|N_G(C)| = 2^5 \cdot 3 \cdot 5^3$. On the other hand, a Sylow 5-subgroup of G is an extra special group $S \cong 5_+^{1+2}$ and we have $|N_G(S)| = 2^5 \cdot 3 \cdot 5^3$. (see p. 299 of [19].) Since $N_G(S) \leq N_G(Z(S))$, we must have $N_G(S) = N_G(Z(S))$. Thus, a subgroup of order 5 is not a radical 5-subgroup of G. There exists a subgroup $E \cong 5^2$ of G with $N_G(E) \cong E : GL_2(5)$. (see [1].) This E is a radical 5-subgroup of G and any subgroup of order 5^2 is conjugate to E. Hence the trivial subgroup, E and S form a set of representatives of G-conjugacy classes of radical 5-subgroups of G. Thus representatives of G-conjugacy classes of radical 5-chains of G are as follows. Here P is a Borel subgroup of $GL_2(5)$.

$$\begin{array}{ll} C_0 : 1, & N_G(C_0) = G, \\ C_1 : 1 < E, & N_G(C_1) = N_G(E) \cong E : GL_2(5), \\ C_2 : 1 < E < S, & N_G(C_2) \cong E : P, \\ C_3 : 1 < S, & N_G(C_3) = N_G(S) \cong 5_+^{1+2} : 4S_4. \end{array}$$

Now Proposition 4.5 implies that $N_G(E)$ and $E : P$ have only the principal 5-blocks and $k(N_G(E), B_0(G), d, [\kappa]) = k(E : P, B_0(G), d, [\kappa])$ for all d and κ. (see also Proposition 2.3 of [17] or Theorem 6.2 of [14].) Hence we have the following.

Proposition 7.1. *To prove that the conjecture holds for G and $p = 5$, it suffices to consider the alternating sum with respect to the following radical 5-chains.*

$$\begin{array}{ll} C_0 : 1, & N_G(C_0) = G, \\ C_3 : 1 < S, & N_G(C_3) = N_G(S) \cong 5_+^{1+2} : 4S_4. \end{array}$$

7.2. 5-blocks of G and $N_G(S)$. The group G has no block of defect one ([8]), and by V.3.11 of [7], $N_G(E)$ has only the principal block. Thus G does not have a block of defect two by Brauer's first main theorem. The character table shows that the characters whose degrees are not divisible by 5^3 form $B_0(G)$ which has 27 characters. Each of the other 21 forms of course a block of defect zero. Thus G has the following twenty two 5-blocks. We write characters in $B_0(G)$ by their degrees and the multiplicity is written exponentially.

$$\begin{aligned} B_0 &= B_0(G) \\ &= \{1, 248, 4123, 30628, 61256, (5 \cdot 17199)^2, 767637^2, 779247^2, 1707264^2, \\ &\quad 5 \cdot 490048, 2572752, 3376737, 4881384, 11577384, 5 \cdot 3307824, (5 \cdot 4265352)^2, \\ &\quad 30507008, 44330496, 5 \cdot 14585103, 81153009, 91171899, 190373976\}, \end{aligned}$$

$\{\chi_4\}, \{\chi_5\}, \{\chi_7\}, \{\chi_{11}\}, \{\chi_{16}\}, \{\chi_{22}\}, \{\chi_{23}\},$
$\{\chi_{24}\}, \{\chi_{26}\}, \{\chi_{27}\}, \{\chi_{28}\}, \{\chi_{29}\}, \{\chi_{30}\}, \{\chi_{31}\},$
$\{\chi_{34}\}, \{\chi_{37}\}, \{\chi_{39}\}, \{\chi_{41}\}, \{\chi_{43}\}, \{\chi_{44}\}, \{\chi_{47}\}.$

The group $N_G(S) \cong 5_+^{1+2} : 4S_4$ appears as the 9th maximal subgroup of G in the GAP library. Since it has only the principal 5-block, we have the following.

$$B_0(N_G(S)) = \{1^4, 2^6, 3^4, 4^2, (5\cdot 4)^3, 24^4, (5\cdot 8)^3, 5\cdot 12\}.$$

Since $|G|_{5'} \equiv |N_G(S)|_{5'} \equiv 1 \mod 5$, we obtain the following, and Conjecture 3.2 holds for G in the case of $p = 5$ by Proposition 7.1.

(d, κ)	$(3, 1)$	$(3, 2)$	$(2, 1)$	$(2, 2)$
$k(G, B_0, d, [\kappa])$	10	10	3	4
$k(N_G(S), B_0, d, [\kappa])$	10	10	3	4

8. The case of $p = 7$ for Th

In this section, we assume that $G = Th$ and $p = 7$.

8.1. Radical 7-chains. ATLAS shows that there is the unique conjugacy class of elements of order 7, and a cyclic subgroup C of order 7 has the normalizer $N_G(C) \cong (7 : 3 \times L_2(7)) : 2$ (p. 20 of [18]). Thus C is a radical 7-subgroup of G. Hence, the trivial subgroup, C and a Sylow 7-subgroup $E \cong 7^2$ of G form a set of representatives of G-conjugacy classes of 7-radical subgroups of G. Hence, representatives of G-conjugacy classes of radical 7-chains of G are as follows.

$$\begin{array}{ll} C_0: 1, & N_G(C_0) = G, \\ C_1: 1 < C, & N_G(C_1) \cong (7 : 3 \times L_2(7)) : 2 \\ C_2: 1 < E, & N_G(C_2) \cong 7^2 : (3 \times 2S_4) \\ C_3: 1 < C < E, & N_G(C_3) \cong (7 : 3 \times P) : 2. \end{array}$$

8.2. 7-blocks of G and a reduction. The character table tells us that G has the principal 7-block $B_0 = B_0(G)$, one 7-block B' of defect 1, and fourteen 7-blocks of defect 0. We have $\mathrm{Irr}(G, B_0) = \mathrm{Irr}(G, B_0, 2)$ and $\mathrm{Irr}(G, B') = \mathrm{Irr}(G, B', 1)$, and since $|G|_{7'} \equiv 4 \mod 7$, the characters are distributed as follows.

$\mathrm{Irr}(G, B_0, 2, [1]) = \{248, 30628, 767637^2, 4881384, 6696000^2, 16539120, 76271625\},$
$\mathrm{Irr}(G, B_0, 2, [2]) = \{30875, 147250, 2450240, 6669000^2, 51684750, 72925515,$
$77376000, 190373976\},$
$\mathrm{Irr}(G, B_0, 2, [3]) = \{1, 27000^2, 61256, 957125, 1707264^2, 4096000^2\},$
$\mathrm{Irr}(G, B', 1) = \mathrm{Irr}(G, B', 1, [3]) = \{\chi_3, \chi_{20}, \chi_{24}, \chi_{32}, \chi_{39}, \chi_{45}, \chi_{47}\},$
$\{\chi_9\}, \{\chi_{10}\}, \{\chi_{14}\}, \{\chi_{15}\}, \{\chi_{21}\}, \{\chi_{26}\}, \{\chi_{31}\},$
$\{\chi_{34}\}, \{\chi_{35}\}, \{\chi_{36}\}, \{\chi_{37}\}, \{\chi_{38}\}, \{\chi_{40}\}, \{\chi_{46}\}.$

(see also 3.5.2 of [15] or [19] for B_0 and [8] for B'.) We have

(d, κ)	$(2, 1)$	$(2, 2)$	$(2, 3)$
$k(G, B_0, d, [\kappa])$	9	9	9

Note that C_1 and C_3 are representatives of radical 7-chains of G starting with $1 < C$, and

$$C_G(C) \cong 7 \times L_2(7),\ \ N_G(C) \cong (7 : 3 \times L_2(7)) : 2 = (7 \times L_2(7)) : 6.$$

Since Conjecture 3.2 holds for the principal blocks of $3 \times L_2(7)$ and $3 \times L_2(7) : 2$, Theorem 4.4 implies that

$$k\big(N_G(C_1), B_0, d, [\kappa]\big) = k\big(N_G(C_3), B_0, d, [\kappa]\big)$$

for all d with $d \neq 1$ and all κ. Moreover, it is easy to see that $k(N_G(C_1), 1) = 7$ and $k(N_G(C_3), 1) = 0$. However, by Brauer's first main theorem, the seven characters of $N_G(C)$ with defect 1 must form a unique 7-block inducing B', and this implies that $k(N_G(C_1), B_0, 1) = 0$. Moreover, since $|N_G(C)|_{7'} \equiv 4 \mod 7$, we have $\mathrm{Irr}(N_G(C), B', 1) = \mathrm{Irr}(N_G(C), B', 1, [3])$. Consequently, we have the following.

Proposition 8.1. *To prove that the conjecture holds for G and $p = 7$, it suffices to consider B_0 and the alternating sum with respect to the radical 7-chains C_0 and C_2.*

8.3. 7-blocks of $N_G(E)$. By V.3.11 of [7], $N_G(E) \cong 7^2 : (3 \times 2S_4)$ has only the principal 7-block $B_0(N_G(E))$. It appears as the 11th maximal subgroup of G in the GAP library. Its character table gives (see also p. 298 of [19].)

$$B_0(N_G(E)) = \{1^6, 2^9, 3^6, 4^3, 48^3\},$$

which implies the following, since $|N_G(E)|_{7'} \equiv 4 \mod 7$.

(d, κ)	$(2, 1)$	$(2, 2)$	$(2, 3)$
$k(N_G(E), B_0, d, [\kappa])$	9	9	9

From the results in 8.2, 8.3 and Proposition 8.1 imply that Conjecture 3.2 holds for G in the case of $p = 7$.

Note that Rouquier [15] proved that there exists a perfect isometry between B_0 and its Brauer correspondent, the principal 7-block of a Sylow normalizer $N_G(E)$.

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