

## NORMAL AFFINE SURFACES WITH $\mathbb{C}^*$ -ACTIONS

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### Introduction

A classification of (normal) affine surfaces admitting a  $\mathbb{C}^*$ -action was given e.g., in [5, 6, 21, 22, 1, 25] and [12]–[14]. Here we obtain a simple alternative description of normal affine surfaces  $V$  with a  $\mathbb{C}^*$ -action in terms of their graded coordinate rings as well as by defining equations. Our approach is based on a generalization of the Dolgachev-Pinkham-Demazure construction [11, 22, 10]. Recall (see [12]–[14]) that a  $\mathbb{C}^*$ -action on a normal affine surface  $V$  is called *elliptic* if it has a unique fixed point which belongs to the closure of every 1-dimensional orbit, *parabolic* if the set of its fixed points is 1-dimensional, and *hyperbolic* if  $V$  has only a finite number of fixed points, and these fixed points are of hyperbolic type, that is each one of them belongs to the closure of exactly two 1-dimensional orbits.

In the elliptic case, the complement  $V^*$  of the unique fixed point in  $V$  is fibered by the 1-dimensional orbits over a projective curve  $C$ . In the other two cases  $V$  is fibered over an affine curve  $C$ , and this fibration is invariant under the  $\mathbb{C}^*$ -action.

Vice versa, given a smooth curve  $C$  and a  $\mathbb{Q}$ -divisor  $D$  on  $C$ , the Dolgachev-Pinkham-Demazure construction provides a normal affine surface  $V = V_{C,D}$  with a  $\mathbb{C}^*$ -action such that  $C$  is just the algebraic quotient of  $V^*$  or of  $V$ , respectively. This surface  $V$  is of elliptic type if  $C$  is projective and of parabolic type if  $C$  is affine.

We remind this construction in Sections 1 and 2 below. In Section 3 we use it to present any normal affine surface  $V$  with a parabolic  $\mathbb{C}^*$ -action as a normalization of the surface  $x^d - P(z)y = 0$  in  $\mathbb{A}_{\mathbb{C}}^3$  for a certain  $d \in \mathbb{N}$  and a certain polynomial  $P \in \mathbb{C}[t]$  (see Theorem 3.11).

In Section 4 we deal with the hyperbolic case. We generalize the Dolgachev-Pinkham-Demazure construction in order to make it work for any hyperbolic  $\mathbb{C}^*$ -surface. Instead of one  $\mathbb{Q}$ -divisor  $D$  on a smooth affine curve  $C$  as before, it involves now two  $\mathbb{Q}$ -divisors  $D_+$  and  $D_-$  on  $C$ . By our result *isomorphism classes of normal affine hyperbolic  $\mathbb{C}^*$ -surfaces are in 1-1-correspondence to equivalence classes*

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of triples  $(C, D_+, D_-)$ , where  $C$  is a smooth affine curve and  $D_+, D_-$  is a pair of  $\mathbb{Q}$ -divisors on  $C$  with  $D_+ + D_- \leq 0$ ; two such triples  $(C, D_+, D_-)$  and  $(C', D'_+, D'_-)$  are considered to be equivalent if and only if  $C \cong C'$  and  $D_\pm = D'_\pm \pm D_0$  with a principal divisor  $D_0$ ; cf. Theorem 4.3. We also determine the structure of the singularities, the orbits, the divisor class group and the canonical divisor in terms of the divisors  $D_\pm$ , see Theorems 4.15, 4.18, 4.22 and Corollary 4.24.

Using our description it is possible to represent any normal hyperbolic  $\mathbb{C}^*$ -surface fibered over  $C = \mathbb{A}^1_{\mathbb{C}}$  as the normalization of a surface in  $\mathbb{A}^4_{\mathbb{C}}$  given by

$$x^{dk} - P(t)y = 0, \quad x^{ek}z - Q(t) = 0 \quad \text{and} \quad y^e z^d - R(t) = 0,$$

for certain polynomials  $P, Q, R \in \mathbb{C}[t]$  satisfying the relation  $P^e R = Q^d$ , where  $e, d$  are coprime. These polynomials can be easily computed in terms of the data  $(D_+, D_-)$  (see Proposition 4.8). For instance, if the divisor  $D_-$  is integral then this system reduces to one equation  $x^e z - Q(t) = 0$  in  $\mathbb{A}^3_{\mathbb{C}}$ , and vice versa. When  $k = 1$  then it again reduces to one equation  $y^e z^d - R(t) = 0$  in  $\mathbb{A}^3_{\mathbb{C}}$ .

In Proposition 4.12 we show how the pair  $(D_+, D_-)$  is transformed when passing to an equivariant cyclic cover of  $V$ . We deduce, in particular, a characterization of normal hyperbolic  $\mathbb{C}^*$ -surfaces over  $C = \mathbb{A}^1_{\mathbb{C}}$  with the fractional part of  $D_-$  supported at one point, as normalized cyclic quotients of the surfaces  $x^e z - Q(t) = 0$  in  $\mathbb{A}^3_{\mathbb{C}}$ .

In the forthcoming paper [15], which is actually Part II of the present one, we will apply these results to give a simple description of all normal affine  $\mathbb{C}^*$ -surfaces equipped in addition by a  $\mathbb{C}^+$ -action. In fact, this class consists of all normal affine surfaces which admit an algebraic group action with an open orbit.

We note that the results of this paper hold *m.m.* for graded 2-dimensional normal algebras of finite type over a Dedekind domain.

### 1. Generalities on graded rings

A  $\mathbb{Z}$ -graded ring  $A = \bigoplus_{i \in \mathbb{Z}} A_i$  contains  $A_{\geq 0} = \bigoplus_{i \geq 0} A_i$  and  $A_{\leq 0} = \bigoplus_{i \leq 0} A_i$  as subrings. The following lemma is “well known”; in lack of a reference we provide a short argument.

**Lemma 1.1.** *If  $A = \bigoplus_{i \in \mathbb{Z}} A_i$  is a finitely generated  $A_0$ -algebra, then so are  $A_{\geq 0}$  and  $A_{\leq 0}$ . Moreover,  $A$  is normal if and only if so are both  $A_{\geq 0}$  and  $A_{\leq 0}$ .*

*Proof.* Reversing the grading interchanges the subrings  $A_{\geq 0}$  and  $A_{\leq 0}$ . Thus it is sufficient to prove the first part for  $A_{\geq 0}$ . If  $a_{ij} \in A_i$  with  $-n \leq i \leq n, j = 1, \dots, n_i$ , is a system of homogeneous generators of  $A$ , then  $A_{\geq 0}$  is generated (as a module over  $A_0$ ) by the multiplicatively closed system of monomials

$$a^k := \prod_{i,j} a_{ij}^{k_{ij}},$$

where  $k := (k_{ij}) \in \mathbb{Z}^N$  satisfies the inequalities

$$(1) \quad k_{ij} \geq 0, \quad -n \leq i \leq n, \quad j = 1, \dots, n_i, \quad \sum_{i,j} ik_{ij} \geq 0.$$

By Gordan’s Lemma (see [20]) the rational polyhedral lattice cone  $K \subseteq \mathbb{Z}^N$  defined by (1) is a finitely generated semigroup. Hence the algebra  $A_{\geq 0}$  is generated by a finite system of monomials  $a^k \in A_{\geq 0}$ .

Next we show that the subalgebra  $A_{\geq 0}$  (and then also  $A_{\leq 0}$ ) is normal if so is  $A$ . Indeed, the integral closure  $(A_{\geq 0})_{\text{norm}} \subseteq A = A_{\text{norm}}$  is graded. Take a homogeneous element  $x \in (A_{\geq 0})_{\text{norm}}$  of degree  $d := \deg x$ , and let

$$(2) \quad x^n + \sum_{i=1}^n b_i x^{n-i} = 0, \quad \text{where } b_i \in A_{\geq 0},$$

be an equation of integral dependence. We may assume that  $b_i$  are also homogeneous, of degree  $\deg b_i = di \geq 0$ . Since  $\deg b_i \geq 0$  we have  $d \geq 0$ , and so  $x \in A_{\geq 0}$ .

Conversely, suppose that both  $A_{\geq 0}$  and  $A_{\leq 0}$  are normal. The ring  $A \otimes_{A_0} \text{Frac}(A_0)$  is normal and so is equal to  $\text{Frac}(A_0)[u, u^{-1}]$  for a homogeneous element  $u$  of minimal degree  $> 0$  in  $A \otimes_{A_0} \text{Frac}(A_0)$ . Hence  $A_{\text{norm}}$  is contained in this subring of  $\text{Frac} A$ . If  $f \in A \otimes_{A_0} \text{Frac}(A_0)$  belongs to the normalization  $A_{\text{norm}}$  of  $A$  then so does its top homogeneous component. Thus it is enough to deal with homogeneous elements. Let  $a$  be such an element satisfying an equation of integral dependence (2) over  $A$ . We may suppose as above that  $b_i \in A_{di}$  ( $i = 1, \dots, n$ ). Since  $di$  has the same sign as  $d := \deg a$ , we have  $a \in (A_{\geq 0})_{\text{norm}} = A_{\geq 0}$  if  $d \geq 0$  and  $a \in (A_{\leq 0})_{\text{norm}} = A_{\leq 0}$  if  $d \leq 0$ , respectively. Anyhow,  $a \in A$ , whence  $A$  is normal, as stated.  $\square$

NOTATION 1.2. Let  $V = \text{Spec} A$  be a normal affine surface over  $\mathbb{C}$  with an effective  $\mathbb{C}^*$ -action. The coordinate ring  $A = \bigoplus_{i \in \mathbb{Z}} A_i$  is then naturally graded so that  $A_i$  is the set of elements of  $A$  on which  $t \in \mathbb{C}^*$  acts via  $t.f = t^i f$ . Thus,  $A_0 = A^{\mathbb{C}^*}$  is the subalgebra of invariants, and  $A_i$  ( $i \neq 0$ ) consists of the quasi-invariants of weight  $i$ . Up to reversing the grading we may assume that  $A_+ := \bigoplus_{i>0} A_i \neq 0$ . The subsets  $A_+$  and  $A_- := \bigoplus_{i<0} A_i$  of  $A$  are ideals in  $A_{\geq 0}$  and  $A_{\leq 0}$ , respectively.

The following lemma is well known (see e.g., [10], [12, Lemma 1.5]).

**Lemma 1.3.** (a) *If  $A_0 \neq \mathbb{C}$  then the set  $M := \{i \in \mathbb{Z} \mid A_i \neq 0\}$  coincides either with  $\mathbb{N}$  or with  $\mathbb{Z}$ , and  $A_i$  is a locally free  $A_0$ -module of rank 1 for all  $i \in M$ . Moreover, if  $u \in \text{Frac}(A_0) \cdot A_1$  is a non-zero element then*

$$A \subseteq \text{Frac}(A_0)[u, u^{-1}], \quad \text{and even} \quad A \subseteq \text{Frac}(A_0)[u] \quad \text{if } M = \mathbb{N}.$$

(b) *In particular, if  $A_0 \cong \mathbb{C}[t]$  then  $A_i$  is a free  $A_0$ -module of rank 1 for all  $i \in M$ .*

Proof. (a) The  $K_0 := \text{Frac}(A_0)$ -algebra  $A \otimes_{A_0} K_0$  is a 1-dimensional normal graded domain over the field  $K_0$ . Hence it is isomorphic to the free polynomial ring  $K_0[u]$  or the ring of Laurent polynomials  $K_0[u, u^{-1}]$ , where  $u \in K_0 A_d$  and  $d > 0$ . As the  $\mathbb{C}^*$ -action is effective  $d = 1$ , and (a) follows.

(b) follows from [7, Ch. VII, §4, Corollary 2]. □

Lemma 1.3(a) does not hold in general without the assumption that  $A_0 \neq \mathbb{C}$  as is seen by the Pham-Brieskorn surfaces  $V_{p,q,r} := \{x^p + y^q + z^r = 0\} \subseteq \mathbb{C}^3$ .

**1.4.** Usually (cf. [12]) one distinguishes between the following three cases.

- (i) *The elliptic case:*  $A_- = 0, A_0 = \mathbb{C}$ .
- (ii) *The parabolic case:*  $A_- = 0, A_0 \neq \mathbb{C}$ .
- (iii) *The hyperbolic case:*  $A_- \neq 0$ .

Below we provide more information in each of these cases.

## 2. The elliptic case

In the elliptic case the  $\mathbb{C}^*$ -action on  $V$  is good. In particular, its fixed point set  $F := V^{\mathbb{C}^*}$  (which is the zero set of the augmentation ideal  $A_+$  of  $A$ ) consists of a unique point called *the vertex* of  $V$ , and the surface  $V$  is smooth outside the vertex. One considers the smooth projective curve  $C := \text{Proj } A \cong V^*/\mathbb{C}^*$ , where  $V^* := V \setminus F$ , together with the orbit morphism  $\pi: V^* \rightarrow C$  (the fibers of  $\pi$  are the orbits of the  $\mathbb{C}^*$ -action on  $V^*$ ).

A useful class of examples of normal affine surfaces with a good  $\mathbb{C}^*$ -action is provided by the affine cones over projective curves. For an ample divisor  $D$  on a smooth projective curve  $C$  the ring

$$A_{C,D} := \bigoplus_{k \geq 0} H^0(C, \mathcal{O}_C(kD)) \cdot u^k \subseteq \text{Frac}(C)[u],$$

where  $u$  is an indeterminate, is the coordinate ring of a normal affine surface  $V := \text{Spec } A_{C,D}$  with a good  $\mathbb{C}^*$ -action. Alternatively this surface  $V$  is obtained by blowing down the zero section of the line bundle associated to  $\mathcal{O}_C(-D)$ . We will refer to such surfaces as affine cones over  $C$  (although  $A_{C,D}$  is not generated by elements of degree one, in general).

Let furthermore a finite group  $G$  act on  $V$  freely off the vertex, and assume that this action commutes with the given good  $\mathbb{C}^*$ -action on  $V$ . Then the quotient  $V/G$  is again a normal affine surface with a good  $\mathbb{C}^*$ -action. Conversely, the following result is true.

**Theorem 2.1** ([11, 22, 10, 24]). *Every normal affine surface with a good  $\mathbb{C}^*$ -action appears as the quotient of an affine cone over a smooth projective curve by a finite group acting freely off the vertex of the cone.*

Generalizing the construction above, for a smooth projective curve  $C$  and a  $\mathbb{Q}$ -divisor  $D$  on  $C$  one considers the graded ring

$$A_{C,D} := \bigoplus_{k \geq 0} H^0(C, \mathcal{O}([kD])) \cdot u^k,$$

where  $\lfloor E \rfloor$  denotes the integral part of a  $\mathbb{Q}$ -divisor  $E$ . We have the following result.

**Theorem 2.2** ([22], [10, Theorem 3.5]). *Given a normal affine surface  $V = \text{Spec } A$  with a good  $\mathbb{C}^*$ -action there exists a  $\mathbb{Q}$ -divisor  $D$  on the curve  $C = \text{Proj } A$  such that  $A \cong A_{C,D}$ .*

The affine toric surfaces provide an interesting family of elliptic  $\mathbb{C}^*$ -surfaces.

EXAMPLE 2.3 ([20, 9]). We remind that a normal affine toric surface  $V = V_\sigma$  is associated to a strictly convex rational polyhedral cone  $\sigma \subseteq \mathbb{R}^2$ . If  $\dim \sigma = 0$  or  $= 1$  then  $V_\sigma \cong \mathbb{C}^* \times \mathbb{C}^*$  or  $V_\sigma \cong \mathbb{A}_{\mathbb{C}}^1 \times \mathbb{C}^*$ , respectively, and so  $A^\times \neq \mathbb{C}^*$ . Consequently, these two cannot be elliptic  $\mathbb{C}^*$ -surfaces. Otherwise, if  $\dim \sigma = 2$  then choosing an appropriate base  $e_1, e_2$  of the lattice one may suppose that  $\sigma$  is the cone  $C(e_2, de_1 - ee_2)$ , where  $d \geq 1, 0 \leq e < d$  and  $\text{gcd}(e, d) = 1$ . We denote  $V_{d,e} := V_\sigma$ ; then  $V_{d,e} = \text{Spec } A_{d,e}$ , where

$$A_{d,e} := \bigoplus_{b \geq 0, ad - be \geq 0} \mathbb{C} \cdot x^a y^b \subseteq \mathbb{C}[x, y]$$

is the semigroup algebra of the dual cone  $\sigma^\vee = C(e_1, ee_1 + de_2)$ .

The 2-torus  $\mathbb{T} = (\mathbb{C}^*)^2$  acts on  $V_{d,e}$  with an open orbit  $V_{d,e}^* := V_{d,e} \setminus \{0\}$ . Thus one can introduce on  $V_{d,e}$  a number of elliptic, parabolic as well as hyperbolic  $\mathbb{C}^*$ -actions by choosing appropriate 1-parameter algebraic subgroups of the torus  $\mathbb{T}$ .

In [23, 2, 3, 9] one can find a description of minimal sets of generators of the algebras  $A_{d,e}$  as above, as well as defining equations for the affine varieties  $V_{d,e} = \text{Spec } A_{d,e} \hookrightarrow \mathbb{C}^N$ . An explicit presentation of these algebras as in Theorem 2.2 is given in [10, 5.1].

We would like to emphasize the well known relation between affine toric surfaces and cyclic quotient singularities (see [10, 5.2] or [20, Proposition 1.24]).

**Lemma 2.4.** *If  $B$  is the normalization of  $A := A_{d,e}$  in the field  $L := \text{Frac}(A)[u]$  with  $u := \sqrt[d]{x}$ , then  $B$  is the polynomial ring  $B = \mathbb{C}[u, v]$  with  $v := ue^y$ . The Galois group  $\langle \zeta \rangle \cong \mathbb{Z}_d$  of  $L : \text{Frac}(A)$  acts on  $B$  via the representation, say  $G_{d,e}$*

$$\zeta \cdot u = \zeta u, \quad \zeta \cdot v = \zeta^e v,$$

and  $A = B^{\mathbb{Z}_d}$ . Consequently, there is an isomorphism

$$V_{d,e} \cong \mathbb{A}_{\mathbb{C}}^2/G_{d,e} = \mathbb{A}_{\mathbb{C}}^2/\mathbb{Z}_d.$$

Proof. For the convenience of the reader we give a short argument. By definition,  $A$  is generated over  $\mathbb{C}$  by the monomials

$$x^a y^b \quad \text{with} \quad b \geq 0, \quad ad - be \geq 0.$$

As  $x^a y^b = u^{ad-be} v^b$ , this shows that  $A$  embeds naturally into  $\mathbb{C}[u, v]$  and that even  $A = \mathbb{C}[x, y] \cap \mathbb{C}[u, v]$ . In particular  $A$  is a normal domain. Because of  $u^d = x \in A$  and  $v^d = x^e y^d \in A$  the ring  $B$  is integral over  $A$ , whence it is the normalization of  $A$ .

The second part follows from the first one, since  $L$  is a cyclic extension of  $\text{Frac}(A)$  with Galois group  $\mathbb{Z}_d$  acting via  $\zeta.u = \zeta u$  and  $\zeta.z = z$  for all  $z \in A$ . □

REMARK 2.5. Assuming that  $e > 0$  and letting  $\xi := \zeta^e$  one obtains

$$(\zeta u, \zeta^e v) = (\xi^{e'} u, \xi v),$$

where  $0 \leq e' < d$  and  $ee' \equiv 1 \pmod d$  (note that for  $d = 1$  this means  $e' = 0$ ). Hence, with  $\tau(u, v) := (v, u)$  the conjugate  $\mathbb{Z}_d$ -action  $G'_{d,e'} := \tau^{-1} G_{d,e} \tau$  on  $\mathbb{A}_{\mathbb{C}}^2$

$$\xi.(u, v) = (\xi^{e'} u, \xi v)$$

has the same orbits as  $G_{d,e}$  thus providing an isomorphism of affine surfaces

$$V_{d,e} \cong \mathbb{A}_{\mathbb{C}}^2/G_{d,e} \cong \mathbb{A}_{\mathbb{C}}^2/G'_{d,e'} \cong \mathbb{A}_{\mathbb{C}}^2/G_{d,e'} \cong V_{d,e'}.$$

Moreover,  $V_{d,e} \cong V_{d',e'}$  if and only if  $d = d'$  and either  $e = e'$  or  $ee' \equiv 1 \pmod d$ .

### 3. The parabolic case

In the parabolic case one considers a normal affine surface  $V$  with a  $\mathbb{C}^*$ -action such that the coordinate ring  $A = \bigoplus_{i \geq 0} A_i$  is positively graded and  $A_0$  is a 1-dimensional domain. Thus  $A_0$  corresponds to a smooth affine curve  $C = \text{Spec } A_0$ , which can be identified with the algebraic quotient  $V//\mathbb{C}^*$  (indeed,  $A_0 = A^{\mathbb{C}^*}$  is the ring of invariants of the  $\mathbb{C}^*$ -action on  $A$ ). The embedding  $A_0 \hookrightarrow A$  corresponds to the quotient morphism  $\pi: V \rightarrow C$ , and the projection  $A \rightarrow A_0$  gives an embedding  $\iota: C \hookrightarrow V$  which provides a retraction of  $\pi$  and whose image is the fixed point set. Every fiber of  $\pi: V \rightarrow C$  is the closure of a non-trivial orbit; it contains a unique fixed point (a *source* of this orbit) [12, Lemma 1.7].

A simple example of a parabolic  $\mathbb{C}^*$ -surface is the cylinder  $C \times \mathbb{A}_{\mathbb{C}}^1$  over a smooth affine curve  $C$ , where  $\mathbb{C}^*$  acts on the second factor. More examples can be produced

by applying equivariant affine modifications to  $C \times \mathbb{A}_{\mathbb{C}}^1$  (see [16, Theorem 1.1]). Actually, one obtains in this way all normal affine surfaces with a parabolic  $\mathbb{C}^*$ -action.

**3.1.** The Dolgachev-Pinkham-Demazure construction (see Theorem 2.2) is available also in the parabolic case. Let  $C = \text{Spec } A_0$  be an affine curve over  $\mathbb{C}$  with function field  $K_0 := \text{Frac}(A_0)$ , and let  $D$  be a  $\mathbb{Q}$ -Cartier divisor on  $C$ . Similarly as in the elliptic case we can introduce the algebra

$$A_0[D] := A_{C,D} = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C([nD])) \cdot u^n \subseteq K_0[u].$$

More explicitly, if  $f \in K_0$  then

$$(3) \quad fu^n \in A := A_0[D] \Leftrightarrow \text{div } f + nD \geq 0.$$

By [10, 2.2] the algebra  $A$  is finitely generated over  $A_0$  and normal (see also Corollary 3.8(b) below). Notice also that  $u \in A_1$  if and only if  $D \geq 0$ .

The following theorem is well known (cf. [10, Theorem 3.5]); for the convenience of the reader we include a short proof.

**Theorem 3.2.** *Let  $C = \text{Spec } A_0$  be a normal affine algebraic curve with function field  $K_0 := \text{Frac}(A_0)$ . If  $A = \bigoplus_{i \geq 0} A_i$  is a normal finitely generated  $A_0$ -algebra of dimension 2 with  $A_1 \neq 0$  then the following hold.*

(a)  *$A$  is isomorphic to  $A_0[D]$  for some  $\mathbb{Q}$ -divisor  $D$  on  $C$ . More precisely, if  $u \in K_0 \cdot A_1$  is a non-zero element and if the divisor  $D$  is defined by the equality*

$$\pi^* D = \text{div } u - \iota(C),$$

*then  $A$  and  $A_0[D]$  are equal when considered as subrings of  $K_0[u]$ .*

(b) *For two  $\mathbb{Q}$ -divisors  $D$  and  $D'$  on  $C$ , the rings  $A = A_0[D]$  and  $A' = A_0[D']$  are isomorphic as graded  $A_0$ -algebras if and only if  $D$  and  $D'$  are linearly equivalent.*

**Proof.** (a) Since  $u \in K_0 \cdot A_1$  is homogeneous, the divisor  $\text{div } u$  on the normal surface  $V = \text{Spec } A$  is invariant under the induced  $\mathbb{C}^*$ -action on  $V$ , and so we have

$$\text{div } u = \sum_{i=1}^m p_i F_i + \iota(C)$$

with  $p_i \in \mathbb{Z}$ , where  $F_i = \pi^{-1}(x_i)_{\text{red}}$  are the fibers of  $\pi$  over distinct points  $x_i \in C$ ,  $i = 1, \dots, m$ . Letting  $\pi^* x_i = q_i F_i$  with  $q_i \in \mathbb{N}$  ( $i = 1, \dots, m$ ), the  $\mathbb{Q}$ -divisor  $D := \sum_{i=1}^m p_i/q_i x_i$  on  $V$  satisfies

$$\text{div } u = \pi^*(D) + \iota(C).$$

Since  $V$  is normal, for a rational function  $\varphi \in K_0$  on  $C$  the following equivalences hold:

$$\begin{aligned} \varphi u^n \in A_n &\Leftrightarrow \operatorname{div}(\varphi u^n) \geq 0 \Leftrightarrow \pi^* \operatorname{div} \varphi + n \operatorname{div} u \geq 0 \Leftrightarrow \\ \pi^* \operatorname{div} \varphi + n\pi^*(D) + n\nu(C) \geq 0 &\Leftrightarrow \operatorname{div} \varphi + nD \geq 0 \Leftrightarrow \varphi \in H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)). \end{aligned}$$

Hence  $A_n = H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) \cdot u^n$  for all  $n \geq 0$ , as desired.

(b) Any isomorphism of graded  $A_0$ -algebras

$$\varphi: A_0[D] = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) \cdot u^n \longrightarrow A_0[D'] = \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C(\lfloor nD' \rfloor)) \cdot u'^n,$$

extends to an isomorphism of graded  $K_0$ -algebras

$$\varphi_{K_0}: K_0[u] \rightarrow K_0[u']$$

and so has the form  $u^n \mapsto f^n u'^n$ ,  $n \geq 0$ , for some non-zero  $f \in K_0$ . Conversely, such a morphism  $\varphi_{K_0}$  maps  $A_0[D]$  isomorphically onto  $A_0[D']$  if and only if

$$H^0(C, \mathcal{O}_C(\lfloor nD' \rfloor)) = f^n \cdot H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) \quad \forall n.$$

As

$$f^n \cdot H^0(C, \mathcal{O}_C(\lfloor nD \rfloor)) = H^0(C, \mathcal{O}_C(\lfloor nD - n \operatorname{div} f \rfloor)),$$

the existence of an isomorphism  $\varphi$  as above is equivalent to the existence of an element  $f \in K_0$  with  $D' = D - \operatorname{div} f$ . □

**3.3.** We denote  $\{D\} = D - \lfloor D \rfloor$  the fractional part of a  $\mathbb{Q}$ -divisor  $D$ . Since principal divisors are  $\mathbb{Z}$ -divisors, we have  $\{D\} = \{D'\}$  as soon as  $D \sim D'$ .

If  $C = \operatorname{Spec} \mathbb{C}[t] = \mathbb{A}_{\mathbb{C}}^1$  then the converse is also true. Indeed, any  $\mathbb{Z}$ -divisor on  $\mathbb{A}_{\mathbb{C}}^1$  is principal, and so the linear equivalence class of a  $\mathbb{Q}$ -divisor  $D$  on  $\mathbb{A}_{\mathbb{C}}^1$  is uniquely determined by the fractional part  $\{D\}$  of  $D$ . Thus we obtain the following corollary.

**Corollary 3.4.** *For every normal parabolic  $\mathbb{C}^*$ -surface  $V = \operatorname{Spec} A$  with  $A = \bigoplus_{n \geq 0} A_n$  and  $A_0 = \mathbb{C}[t]$ , there is a unique isomorphism  $A \cong A_0[D]$  of graded  $A_0$ -algebras, where  $D = 0$  or  $D = \sum_{i=1}^n (p_i/q_i)x_i$  with  $0 < p_i < q_i$ ,  $\operatorname{gcd}(p_i, q_i) = 1 \forall i = 1, \dots, n$  and  $x_i \in \mathbb{A}_{\mathbb{C}}^1$ ,  $x_i \neq x_j$  for  $i \neq j$ .*

The next lemma is also well known; in lack of a reference we provide a short argument.

**Lemma 3.5.** *Let  $D$  be a  $\mathbb{Q}$ -divisor on a normal affine variety  $S$  and consider the graded ring  $A := \bigoplus_{i \geq 0} A_i$ , where  $A_i := H^0(S, \mathcal{O}_S(\lfloor iD \rfloor)) \cdot u^i$ . For  $d \in \mathbb{N}$  the following conditions are equivalent.*



- (i)  $dD$  is integral.
- (ii)  $A_{d+m} = A_d A_m$  for all  $m \geq 0$ .
- (iii) The  $d$ -th Veronese subring  $A^{(d)} := \bigoplus_{m \geq 0} A_{md}$  is isomorphic to the symmetric algebra  $S_{A_0}(A_d)$  i.e.,  $A_{md} = S_{A_0}^m A_d$ .

Proof. Condition (ii) is equivalent to

$$\mathcal{O}_S(\lfloor (m+d)D \rfloor) \cong \mathcal{O}_S(\lfloor mD \rfloor) \otimes \mathcal{O}_S(\lfloor dD \rfloor) \quad \forall m \geq 0,$$

and the latter condition is equivalent to

$$(ii') \quad \lfloor (m+d)D \rfloor = \lfloor mD \rfloor + \lfloor dD \rfloor \quad \forall m \geq 0.$$

Similarly, (iii) is equivalent to

$$(iii') \quad \lfloor mdD \rfloor = m \lfloor dD \rfloor \quad \forall m \geq 0.$$

The equivalence of (i), (ii') and (iii') now follows from the elementary fact that for a rational number  $r = p/q$  and  $d \in \mathbb{N}$  the following conditions are equivalent:

$$(1) \ dr \in \mathbb{Z} \quad (2) \ \lfloor (m+d)r \rfloor = \lfloor mr \rfloor + \lfloor dr \rfloor \ \forall m \geq 0 \quad (3) \ \lfloor mdr \rfloor = m \lfloor dr \rfloor \ \forall m \geq 0.$$

□

NOTATION 3.6. We denote  $d(A)$  the smallest positive integer  $d$  satisfying the equivalent conditions of Lemma 3.5.

REMARK 3.7. In the situation of Theorem 3.2, one can recover  $D$  from the graded ring  $A = A_0[D]$  more algebraically as follows. Consider  $d \in \mathbb{N}$  with  $A_d A_i = A_{d+i}$  for all  $i \geq 0$  (or, equivalently,  $A_{id} = S^i(A_d)$ , see Lemma 3.5) and let  $v$  be a generator of  $A_d$  as  $A_0$ -module; this exists after a suitable localization of  $A_0$ . If  $u^d = fv$  with  $f \in \text{Frac } A_0$ , then  $D = \text{div}(f)/d$ . In fact, the ideal  $vA$  is equal to  $A_{\geq d}$  and so its zero set has no irreducible components in the fibers of  $\pi$ . Thus  $\text{div } v = d \cdot \iota(C)$  on  $V$ . Since

$$\pi^*(D) = \text{div } u - \iota(C) \quad \text{and} \quad d \cdot \text{div } u = \text{div } v + \text{div } f$$

as divisors on  $V$ , we obtain  $D = \text{div}(f)/d$ .

A parabolic  $\mathbb{C}^*$ -surface  $V = \text{Spec } A_0[D]$  has at most cyclic quotient singularities, as follows from Miyanishi's Theorem (see [17, Lemma 1.4.4(1)]). In the next result (see [10, Section 5]) we describe their structure in terms of the divisor  $D$ .

**Proposition 3.8.** (a) If  $A_0 = \mathbb{C}[t]$  and if  $D$  is supported on the origin in  $\text{Spec } A_0 = \mathbb{A}_{\mathbb{C}}^1$  so that  $D = -(e/d)[0]$  with  $\text{gcd}(e, d) = 1$ , then  $A := A_0[-(e/d)[0]]$

is naturally isomorphic to the semigroup algebra

$$A_{d,e} = \bigoplus_{b \geq 0, ad-be \geq 0} \mathbb{C} \cdot t^a u^b$$

graded via  $\deg t = 0, \deg u = 1$  (cf. Example 2.3). Consequently,  $V := \text{Spec } A$  is isomorphic to the toric surface  $V_{d,e'} = \text{Spec } A_{d,e'} \cong \mathbb{A}_{\mathbb{C}}^2 / G_{d,e'}$ , where  $e' \equiv e \pmod d$  and  $0 \leq e' < d$ .

(b) If  $C = \text{Spec } A_0$  is any normal affine curve over  $\mathbb{C}$  and  $D$  is a  $\mathbb{Q}$ -divisor on  $C$ , then the surface  $V = \text{Spec } A_0[D]$  is normal with at most cyclic quotient singularities. More precisely, if  $D(a) = -e/d$  with  $\gcd(e, d) = 1$  then  $V$  has a quotient singularity of type  $(d, e')$  at  $\iota(a)$ , where  $e'$  is as in (a).

Proof. The first part of (a) follows immediately from (3) in 3.1, whereas the second one is a consequence of Lemma 2.4.

Tensoring the isomorphism in (a) with  $- \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$  we obtain that (b) holds if  $A_0 \cong \mathbb{C}[[t]]$ . The general case follows from this by taking completions at the maximal ideals of  $A_0$ . □

The algebra  $A_0[D]$  is finitely generated over  $A_0$ , so there exist  $f_1, \dots, f_n \in K_0$  and  $m_1, \dots, m_n \in \mathbb{N}$  such that

$$A = A_0[f_1 u^{m_1}, \dots, f_n u^{m_n}] \subseteq K_0[u].$$

In the next result we show how to compute  $D$  from such a representation.

**Proposition 3.9.** *Let  $C = \text{Spec } A_0$  be a smooth affine curve and  $K_0 := \text{Frac } A_0$ . If a 2-dimensional subring  $B$  of the polynomial ring  $K_0[u]$  is represented as*

$$B = A_0[f_1 u^{m_1}, \dots, f_n u^{m_n}] \subseteq K_0[u], \quad m_i > 0 \ \forall i$$

with  $f_1, \dots, f_n \in K_0$  and  $\gcd(m_1, \dots, m_n) = 1$ , then its normalization  $A = B_{\text{norm}}$  coincides as an  $A_0$ -subalgebra of  $K_0[u]$  with  $A_0[D]$ , where

$$D := - \min_{1 \leq i \leq n} \frac{\text{div } f_i}{m_i}.$$

Proof. By definition of  $D$  we have  $\text{div } f_i + m_i D \geq 0$  so by (3)  $f_i u^{m_i} \in A_0[D]$  and  $B$  is a subring of  $A_0[D]$ . As  $A_0[D]$  is normal (see Proposition 3.8(b)),  $A$  is also contained in  $A_0[D]$ . Let us show that these subrings coincide.

According to Theorem 3.2, we can represent  $A$  as  $A = A_0[D']$  with  $\pi^*(D') = \text{div } u - \iota(C)$ . In particular  $f_i u^{m_i} \in A = A_0[D']$ , so again by (3)  $\text{div } f_i + m_i D' \geq 0$  or, equivalently,  $D' \geq -(1/m_i) \text{div } f_i$ . Thus  $D' \geq D$  and  $A_0[D] \subseteq A_0[D'] = A$ . As

we have already shown the converse inclusion we obtain that  $A = A_0[D]$ , as desired.  $\square$

The following examples of parabolic  $\mathbb{C}^*$ -surfaces ruled over  $\mathbb{A}_{\mathbb{C}}^1$  are basic (see Theorem 3.11 below).

EXAMPLE 3.10. For a unitary polynomial  $P \in \mathbb{C}[t]$  and for an integer  $d \geq 1$  we let

$$B_{d,P}^+ := \mathbb{C}[t, u, v]/(u^d - P(t)v) \cong \mathbb{C} \left[ t, u, \frac{u^d}{P(t)} \right]$$

graded via

$$\deg t = 0, \quad \deg u = 1, \quad \deg v = d.$$

The normalization

$$A_{d,P}^+ := (B_{d,P}^+)_{\text{norm}}$$

is a positively graded finitely generated  $\mathbb{C}$ -algebra of dimension 2 with  $A_0 = \mathbb{C}[t]$ . By Proposition 3.9 and Corollary 3.4 we have

$$A_{d,P}^+ \cong A_0[D] \cong A_0[\{D\}], \quad \text{where} \quad D = D(d, P) := \frac{\text{div}(P)}{d}.$$

For  $P(t) = \prod_{i=1}^n (t - x_i)^{r_i}$  (where  $x_i \neq x_j$  if  $i \neq j$ ) we obtain

$$D = \sum_{i=1}^n \frac{r_i}{d} x_i, \quad \text{and} \quad \{D\} = \sum_{i=1}^n \left\{ \frac{r_i}{d} \right\} x_i,$$

whereas  $D = 0$  if  $P = 1$ . Replacing  $D$  by  $\{D\}$  we may suppose that

$$(*) \quad \gcd(d, r_1, \dots, r_n) = 1, \quad 0 < r_i < d \quad \forall i = 1, \dots, n, \quad \text{if } d \geq 2, \quad \text{and } P = 1 \text{ if } d = 1.$$

If two pairs  $(d, P)$  and  $(\tilde{d}, \tilde{P})$  satisfy  $(*)$  and if  $A_{d,P}^+ \cong A_{\tilde{d},\tilde{P}}^+$  as graded  $A_0$ -algebras then by Corollary 3.4 we have  $\text{div}(P)/d = \text{div}(\tilde{P})/\tilde{d}$ , and so  $d = \tilde{d}$  and  $P = \tilde{P}$ .

Thus we obtain the following classification result.

**Theorem 3.11.** *For every normal affine surface  $V = \text{Spec } A$ , where  $A = \bigoplus_{i \geq 0} A_i$  with  $A_0 = \mathbb{C}[t]$ , there is a unique pair  $(d, P)$  satisfying condition  $(*)$  and an equivariant isomorphism of  $A_0$ -schemes*

$$\varphi: V \longrightarrow V_{d,P}^+ := \text{Spec } A_{d,P}^+.$$

REMARK 3.12. 1. In the situation of Theorem 3.11 above, the Veronese subring  $A^{(d)}$  is equal to  $A_0[v] = \mathbb{C}[t, v]$ . The cyclic group  $\mathbb{Z}_d$  acts on  $A$  via the  $\mathbb{C}^*$ -action and  $A^{(d)}$  coincides with the ring of invariants  $A^{\mathbb{Z}_d}$ , whereas  $A$  is the normalization of  $A^{(d)}$  in the fraction field  $\text{Frac}(A)$ . Thus the morphism  $V \rightarrow \mathbb{A}_{\mathbb{C}}^2 = \text{Spec } \mathbb{C}[t, v]$  induced by the inclusion  $\mathbb{C}[t, v] \subseteq A$  represents  $V$  as a cyclic covering of the plane branched along the curve  $u = 0$ , and  $V$  is the normalization of a surface  $\{u^d - P(t)v = 0\}$  in  $\mathbb{C}^3$ . 2. More generally, let  $C = \text{Spec } A_0$  be any smooth affine curve and let  $A = \bigoplus_{i \geq 0} A_i$  be a normal 2-dimensional  $A_0$ -algebra of finite type. If  $A_1 = u \cdot A_0$  and  $A_d = v \cdot A_0$ ,  $d := d(A)$ , for suitable elements  $u \in A_1$  and  $v \in A_d$  then  $A$  is the normalization of an algebra  $A_0[u, v]/(u^d - P_+v)$  graded via  $\deg u = 1, \deg v = d$ , for a certain  $d \in \mathbb{N}$  and a certain element  $P_+ \in A_0$ .

4. The hyperbolic case

Let  $A = \bigoplus_{i \in \mathbb{Z}} A_i$  be the coordinate ring of a normal affine surface  $V = \text{Spec } A$  with  $\mathbb{C}^*$ -action such that  $A_+, A_-$  are both non-zero. Here again there is a quotient morphism  $\pi: V \rightarrow C = \text{Spec } A_0$  induced by the inclusion  $A_0 \hookrightarrow A$ . Every fiber of  $\pi$  is either a non-trivial orbit or a union of two 1-dimensional orbits and a hyperbolic fixed point, which is a source for one of them and a sink for the other one [12, Lemma 1.7]. Thus the fixed point set  $F$  is finite and contains  $\text{Sing } V$ .

By Lemma 1.1 the proper subalgebras  $A_{\geq 0}$  and  $A_{\leq 0}$  of  $A$  are normal and finitely generated, and so  $V_+ := \text{Spec } A_{\geq 0}$  and  $V_- := \text{Spec } A_{\leq 0}$  are normal affine surfaces with a parabolic  $\mathbb{C}^*$ -action birationally dominated by  $V$ . The natural embeddings  $A_0 \hookrightarrow A_{\geq 0} \hookrightarrow A$  and  $A_0 \hookrightarrow A_{\leq 0} \hookrightarrow A$  yield the commutative diagram

$$(4) \quad \begin{array}{ccccc} & & V_+ & \xleftarrow{\sigma_+} & V & \xrightarrow{\sigma_-} & V_- & & \\ & & \searrow & & \downarrow \pi & & \swarrow & & \\ & & \pi_+ & & C & & \pi_- & & \end{array}$$

where  $\sigma_{\pm}$  are equivariant birational morphisms. Hence  $\sigma_{\pm}$  are equivariant affine modifications [16, Theorem 1.1]. More precisely the following result holds.

**Proposition 4.1.**  *$V$  can be obtained from  $V_{\pm}$  by blowing up a  $\mathbb{C}^*$ -invariant subscheme and deleting the proper transform of a  $\mathbb{C}^*$ -invariant divisor  $D^{\pm}$  on  $V_{\pm}$ , which contains the fixed point curve  $\iota_{\pm}(C) \subseteq V_{\pm}$ .*

Proof. Let us show this for  $V_+$ , the proof for  $V_-$  being similar. Choose a system of homogeneous generators  $a_1, \dots, a_n$  of the finitely generated  $A_0$ -subalgebra  $A_{\leq 0}$  and let  $f_0 \in A_+$  be a non-zero element of degree  $m = -\min_i \deg a_i$ . Letting  $f_i := a_i f_0$  for

$i = 1, \dots, n$  we obtain

$$A = A_{\geq 0} \left[ \frac{f_1}{f_0}, \dots, \frac{f_n}{f_0} \right] = A_{\geq 0} \left[ \frac{I}{f_0} \right] := \left\{ \frac{x_k}{f_0^k} \mid x_k \in I^k, k \geq 0 \right\},$$

where  $I$  is the graded ideal of  $A_{\geq 0}$  generated by  $f_0, \dots, f_n$ . Thus  $V = \text{Spec } A$  is obtained by blowing up  $V_+ = \text{Spec } A_{\geq 0}$  with center  $I$  and deleting the proper transform of the  $\mathbb{C}^*$ -invariant divisor  $\text{div } f_0$  on  $V_+$ . As this divisor contains  $\iota_+(C)$ , the result follows. □

For a more precise description of the affine modifications  $\sigma_{\pm}$  see Remark 4.20.

**4.2.** The Dolgachev-Pinkham-Demazure construction is still available in the hyperbolic case. In [10, Theorem 3.5] it is done under the additional assumption that  $A_{-n} \otimes A_n \rightarrow A_0$  is an isomorphism for all  $n$ . Here we generalize the construction in order to make it work for any hyperbolic  $\mathbb{C}^*$ -surface.

Let  $D_+, D_-$  be  $\mathbb{Q}$ -divisors on the smooth affine curve  $C := \text{Spec } A_0$ . For  $n \geq 0$  we consider the  $A_0$ -submodules

$$A_{-n} := H^0(C, \mathcal{O}_C(\lfloor nD_- \rfloor)) \cdot u^{-n} \quad \text{and} \quad A_n := H^0(C, \mathcal{O}_C(\lfloor nD_+ \rfloor)) \cdot u^n$$

of  $\text{Frac}(A_0)[u, u^{-1}]$ , where  $u$  is an indeterminate of degree 1. If  $D_+ + D_- \leq 0$  then for  $n \geq m \geq 0$  we have

$$\lfloor nD_+ \rfloor + \lfloor mD_- \rfloor \leq \lfloor (n - m)D_+ \rfloor,$$

whence  $A_n \cdot A_{-m} \subseteq A_{n-m}$ . Similarly, for  $0 \leq n \leq m$  we have  $A_n \cdot A_{-m} \subseteq A_{n-m}$ . Thus

$$A := A_0[D_+, D_-] := \bigoplus_{n \in \mathbb{Z}} A_n$$

is a finitely generated  $A_0$ -subalgebra of  $\text{Frac}(A_0)[u, u^{-1}]$  with  $A_{\geq 0} = A_0[D_+]$  and  $A_{\leq 0} \cong A_0[D_-]$ . The grading on  $A$  defines a natural hyperbolic  $\mathbb{C}^*$ -action on the surface  $V := \text{Spec } A$ . The latter surface is normal as so are the algebras  $A_0[D_+]$  and  $A_0[D_-]$  (see Lemma 1.1 and Corollary 3.8(b)). Conversely, we have the following theorem.

**Theorem 4.3.** *If  $C = \text{Spec } A_0$  is a smooth affine curve and  $A = \bigoplus_{i \in \mathbb{Z}} A_i$  is a normal graded finitely generated domain of dimension 2 with  $A_{\pm} \neq 0$ , then the following hold.*

(a)  *$A$  is isomorphic to  $A_0[D_+, D_-]$ , where  $D_+, D_-$  are  $\mathbb{Q}$ -divisors on  $C$  satisfying  $D_+ + D_- \leq 0$ . More precisely, if  $u \in \text{Frac}(A_0) \cdot A_1$  and if the divisors  $D_+, D_-$  on  $C$  are defined by*

$$(5) \quad \pi_+^*(D_+) = \text{div}(u) - \iota_+(C) \quad \text{and} \quad \pi_-^*(D_-) = \text{div}(u^{-1}) - \iota_-(C),$$

where  $\pi_{\pm}$  are as in diagram (4) above and  $\iota_{\pm}: C \hookrightarrow V_{\pm}$  are the natural embeddings, then  $D_+ + D_- \leq 0$  and  $A \cong A[D_+, D_-]$ .

(b)  $A_0[D_+, D_-] \cong A_0[D'_+, D'_-]$  as graded  $A_0$ -algebras if and only if, for a rational function  $\varphi \in \text{Frac}(A_0)$ , one has

$$D'_+ = D_+ + \text{div } \varphi \quad \text{and} \quad D'_- = D_- - \text{div } \varphi.$$

Proof. (a) By Theorem 3.2 and its proof we have equalities

$$A_{\geq 0} = A_0[D_+] \quad \text{and} \quad A_{\leq 0} = A_0[D_-]$$

as subalgebras of  $\text{Frac}(A_0)[u, u^{-1}]$ , whence  $A = A_0[D_+, D_-]$ . It remains to show that  $D_+ + D_- \leq 0$ . Applying in (5) the functors  $\sigma_+^*$  and  $\sigma_-^*$  respectively, we obtain

$$\pi^*(D_+) = \text{div}(u) - \sigma_+^* \iota_+^*(C) \quad \text{and} \quad \pi^*(D_-) = \text{div}(u^{-1}) - \sigma_-^* \iota_-^*(C).$$

Taking the sum of these equalities yields  $\pi^*(D_+ + D_-) = -(\sigma_+^* \iota_+^*(C) + \sigma_-^* \iota_-^*(C))$ , whence  $D_+ + D_- \leq 0$ , as required. Finally (b) follows from Theorem 3.2(b) and its proof. □

Consequently, if  $A_0 = \mathbb{C}[t]$  then  $A$  admits a unique presentation  $A = A_0[D_+, D_-]$  with  $D_+ = \{D_+\}$  and  $D_+ + D_- \leq 0$ .

It follows from Theorem 4.3 that outside  $|D_+| \cup |D_-|$ , the map  $\pi: V \rightarrow C$  is a locally trivial principal  $\mathbb{C}^*$ -bundle. More generally, the Dolgachev-Pinkham-Demazure construction shows the following result (cf. [1], [12, Proposition 1.11]).

**Corollary 4.4.** *In all three cases, outside of a finite subset of the curve  $C$  the projection  $\pi: V^* \rightarrow C$  and  $\pi: V \rightarrow C$ , respectively, defines a locally trivial fiber bundle. This is a principal  $\mathbb{C}^*$ -bundle in the elliptic and hyperbolic cases, and a line bundle in the parabolic case.*

Note that if  $u \in A_1 \cup A_{-1}$  is a non-zero element then its restriction to a general fiber of  $\pi$  gives a fiber coordinate and so a trivialization over a Zariski open subset of  $C$ .

REMARK 4.5. *The algebra  $A = A_0[D_+, D_-]$  contains an invertible element of degree  $d > 0$  if and only if  $D_- = -D_+$  and  $dD_+$  is a principal divisor on  $C = \text{Spec } A_0$ . In fact, if  $v \in A$  is an invertible element of degree  $d > 0$  then we can write*

$$v = fu^d \in A_d \quad \text{and} \quad v^{-1} = f^{-1}u^{-d} \in A_{-d},$$

where  $f \in \text{Frac}(A_0)$  satisfies

$$\text{div}(f) + dD_+ \geq 0 \quad \text{and} \quad -\text{div}(f) + dD_- \geq 0.$$

Thus  $0 \geq D_+ + D_- \geq 0$ , whence  $D_- = -D_+$ . Since  $A_d = vA_0$  it also follows that  $dD_+$  is principal. Conversely, if  $D_+ = -D_-$  and if  $dD_+$  is principal, then  $vA_0 = A_d$  is free over  $A_0$  and  $v = fu^d$  with  $\operatorname{div} f + dD_+ = 0$  by Remark 3.7. Hence also  $\operatorname{div} f^{-1} + dD_- = 0$ , so  $f^{-1}u^{-d} \in A$  and  $v = fu^d$  is a unit in  $A$ .

The following analogue of Proposition 3.9 holds with a similar proof.

**Lemma 4.6.** *Let  $C = \operatorname{Spec} A_0$  be a smooth affine curve with function field  $K_0 = \operatorname{Frac}(A_0)$ . If a graded 2-dimensional domain  $B \subseteq K_0[u, u^{-1}]$  is represented as*

$$B = A_0[h_1u^{-n_1}, \dots, h_ku^{-n_k}, f_1u^{m_1}, \dots, f_nu^{m_n}] \quad (\text{where } n_i, m_j > 0 \ \forall i, j)$$

with  $h_1, \dots, h_k, f_1, \dots, f_n \in K_0$  and  $B_0 = A_0$ , then its normalization  $A = B_{\text{norm}}$  coincides (as a graded  $A_0$ -subalgebra of  $K_0[u, u^{-1}]$ ) with  $A_0[D_+, D_-]$ , where

$$D_- = - \min_{1 \leq i \leq k} \frac{\operatorname{div} h_i}{n_i} \quad \text{and} \quad D_+ = - \min_{1 \leq j \leq n} \frac{\operatorname{div} f_j}{m_j}.$$

We notice that the assumption  $A_0 = B_0$  amounts to the inequalities

$$\frac{\operatorname{div} h_i}{n_i} + \frac{\operatorname{div} f_j}{m_j} \geq 0 \quad \forall i, j,$$

which in turn are equivalent to  $D_+ + D_- \leq 0$ .

The following lemma provides additional information in the case that  $[D_{\pm}]$  and  $d_{\pm}(A)D_{\pm}$  are principal divisors<sup>1</sup>.

**Lemma 4.7.** *Let  $A = \bigoplus_{i \in \mathbb{Z}} A_i = A_0[D_+, D_-] \subseteq \operatorname{Frac}(A_0)[u, u^{-1}]$ , and let  $d_{\pm} = d_{\pm}(A)$  be the minimal positive integer such that the divisor  $d_{\pm}D_{\pm}$  is integral. If  $A_{\pm 1} = u_{\pm} \cdot A_0$ ,  $A_{\pm d_{\pm}} = v_{\pm} \cdot A_0$  and*

$$u_+u_- = Q, \quad u_{\pm}^{d_{\pm}} = P_{\pm}v_{\pm}$$

for some elements  $Q, P_{\pm} \in A_0$ , then

$$(6) \quad D_+ = \frac{\operatorname{div} P_+}{d_+} + D_0 \quad \text{and} \quad D_- = \frac{\operatorname{div} P_-}{d_-} - D_0 - \operatorname{div} Q,$$

where  $D_0$  is the integral divisor  $D_0 = \operatorname{div}(u/u_+)$  on  $C = \operatorname{Spec} A_0$ . Consequently,

$$(7) \quad \frac{\operatorname{div} P_+}{d_+} + \frac{\operatorname{div} P_-}{d_-} \leq \operatorname{div} Q.$$

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<sup>1</sup>or, equivalently, that  $A_{\pm 1}$  and  $A_{\pm d_{\pm}}$  are free  $A_0$ -modules of rank 1.

Furthermore,  $P_+$  and  $P_-$  are uniquely determined by  $D_+$  and  $D_-$  through

$$(8) \quad \{D_+\} = \frac{\operatorname{div} P_+}{d_+} \quad \text{and} \quad \{D_-\} = \frac{\operatorname{div} P_-}{d_-}.$$

Proof. We have  $u^{d_+} = P_+ \cdot (u/u_+)^{d_+} v_+$  and  $u^{-d_-} = P_- \cdot (u/u_+)^{-d_-} Q^{-d_-} v_-$  and so by Remark 3.7

$$D_+ = \frac{\operatorname{div}(P_+ \cdot (u/u_+)^{d_+})}{d_+} = \frac{\operatorname{div} P_+}{d_+} + D_0, \quad \text{and}$$

$$D_- = \frac{\operatorname{div}(P_- \cdot (u/u_+)^{-d_-} Q^{-d_-})}{d_-} = \frac{\operatorname{div} P_-}{d_-} - D_0 - \operatorname{div} Q.$$

Now (7) follows from the inequality  $D_+ + D_- \leq 0$ . To show (8), after localizing  $A_0$  we can assume that  $P_\pm = S_\pm^{d_\pm} T_\pm$ , where  $S_\pm, T_\pm \in A_0$  are elements with

$$\operatorname{div} S_\pm = \left\lfloor \frac{\operatorname{div} P_\pm}{d_\pm} \right\rfloor \quad \text{and} \quad \operatorname{div} T_\pm = \left\{ \frac{\operatorname{div} P_\pm}{d_\pm} \right\},$$

respectively. The relation  $(u_\pm/S_\pm)^{d_\pm} = T_\pm v_\pm$  then shows that  $u_\pm/S_\pm$  is integral over  $A$  and so by the normality of  $A$  is contained in  $A_{\pm 1}$ . As  $u_\pm$  is a generator of  $A_{\pm 1}$  this forces that  $S_\pm \in A_0^\times$  are units, proving (8).  $\square$

In many cases the surfaces  $V = \operatorname{Spec} A_0[D_+, D_-]$  can be represented by explicit equations as follows.

**Proposition 4.8.** *With the assumptions as in Lemma 4.7 the following hold.*

(a)  $A = A_0[D_+, D_-]$  is the normalization of the  $A_0$ -algebra

$$(9) \quad B := A_0[u_-, v_+, v_-] / \left( u_-^{d_-} - v_- P_-, v_+^{d'_+} v_-^{d'_-} - P, v_+ u_-^{d_+} - Q_+ \right)$$

graded via  $\deg u_- = -1, \deg v_\pm = \pm d_\pm$ , where  $k := \operatorname{gcd}(d_+, d_-), d'_\pm := d_\pm/k$  and

$$(10) \quad P := \frac{Q^{kd'_+ d'_-}}{P_+^{d'_-} P_-^{d'_+}} \in A_0, \quad Q_+ := \frac{Q^{d_+}}{P_+} \in A_0.$$

(b)  $V = \operatorname{Spec} A$  is a cyclic branched covering of degree  $k$  of the normalization of the hypersurface  $\{v_+^{d'_-} v_-^{d'_+} - P = 0\}$  in  $C \times \mathbb{A}_C^2$ .

(c) If  $k = 1$  i.e., if  $d_+$  and  $d_-$  are coprime and if  $v_+$  is not invertible, then  $V = \operatorname{Spec} A$  can be represented as the normalization of a hypersurface  $X$  in  $A_C^3 = \operatorname{Spec} \mathbb{C}[s, v_+, v_-]$  with equation

$$q(s, v_+^{d_-} \cdot v_-^{d_+}) = 0,$$



where  $q \in \mathbb{C}[s, t]$  is a suitable irreducible polynomial.

Proof. (a) First we note that  $A$  is integral over the subring  $A_0[v_{\pm}]$ . Indeed, if  $w \in A_k$  with  $k \neq 0$  then  $w^{d_+} = av_+^k$  if  $k > 0$  and  $w^{d_-} = av_-^k$  if  $k < 0$ , where  $a \in A_0$  (see Lemma 3.5). Since  $A$  and its subring  $A_0[u_-, v_{\pm}]$  have the same field of fractions, it follows that  $A$  is the normalization of  $A_0[u_-, v_{\pm}]$ .

To find the relations between the generators of  $A_0[u_-, v_{\pm}]$ , note that  $v_{\pm} = u_{\pm}^{d_{\pm}}/P_{\pm}$  and so

$$v_+^{d'_-} v_-^{d'_+} = \frac{u_+^{d_+ d'_-} u_-^{d'_+ d_-}}{P_+^{d'_-} P_-^{d'_+}} = \frac{Q^{kd'_+ d'_-}}{P_+^{d'_-} P_-^{d'_+}} = P \in A_0.$$

Similarly

$$v_+ u_-^{d_+} = \frac{u_+^{d_+} u_-^{d_+}}{P_+} = \frac{Q^{d_+}}{P_+} = Q_+ \in A_0.$$

The general fibers of the natural map  $\text{Spec } B \rightarrow C = \text{Spec } A_0$  are irreducible, and every fiber is 1-dimensional and in the closure of the generic fiber. Thus the surface  $\text{Spec } B$  is irreducible, and (a) follows.

(b) Since  $k = \text{gcd}(d_+, d_-)$ , the ring  $A_0[v_{\pm}]$  contains nonzero elements of degree  $k$  and is contained in the Veronese subring  $A^{(k)}$  of  $A$ . Hence the fraction fields of both rings coincide. As  $A$  and then also  $A^{(k)}$  is integral over  $A_0[v_{\pm}]$  the normalization of  $A_0[v_{\pm}]$  is just  $A^{(k)}$ . The cyclic group  $\mathbb{Z}_k$  acts on  $A$  via the  $\mathbb{C}^*$ -action with invariant ring  $A^{(k)}$ . Thus  $V \rightarrow \text{Spec } A^{(k)}$  is a cyclic branched covering of degree  $k$ , and (b) follows.

(c) In case  $k = 1$  the algebra  $A = A^{(k)}$  is itself the normalization of the hypersurface  $A_0[v_+, v_-]/(v_+^{d_+} v_-^{d_-} - P)$ . Notice that  $P$  is non-constant as  $A$  is a domain and, by our assumption, the elements  $v_{\pm}$  are not invertible. For a general element  $s$  of  $A_0$  the map  $\varphi = (s, t)$  is a finite morphism of  $C = \text{Spec } A_0$  onto a plane curve  $\tilde{C} \subseteq \mathbb{A}_{\mathbb{C}}^2$  with an irreducible equation  $q(s, t) = 0$ , where  $t := P = v_+^{d_+} v_-^{d_-} \in A_0$ . This implies (c).  $\square$

**Remarks 4.9.** 1. It is worthwhile mentioning how to get, under the assumptions as in (c), a representation  $A \cong A_0[D_+, D_-]$  in terms of  $P$  in (10). Choose  $p, q \in \mathbb{Z}$  with  $|\frac{d_+ p}{d_- q}| = 1$  so that  $u' := v_+^q v_-^p$  has degree 1. By an easy calculation  $u'^{d_+} = v_+ P^p$  and  $u'^{-d_-} = v_- / P^q$ , whence by Remark 3.7  $A \cong A_0[D_+, D_-]$  with

$$D_+ = \frac{p}{d_+} \text{div } P, \quad D_- = -\frac{q}{d_-} \text{div } P, \quad \text{and} \quad D_+ + D_- = -\frac{\text{div } P}{d_+ d_-}.$$

2. In analogy with (c), any parabolic  $\mathbb{C}^*$ -surface  $V = \text{Spec } A$  with  $A = A_0[D]$ , where  $[D]$  and  $d(A)D$  are principal divisors on  $C = \text{Spec } A_0$ , can be obtained as the normalization of a surface  $u^d - tv = 0 = q(s, t)$  in  $\mathbb{A}_{\mathbb{C}}^4 = \text{Spec } \mathbb{C}[s, t, u, v]$  graded via  $\text{deg } s = \text{deg } t = 0, \text{deg } u = 1, \text{deg } v = d$ , where  $q \in \mathbb{C}[s, t]$  is a suitable irreducible polynomial (see also Remark 3.12(2)).

The special case  $d_+ = 1$  leads to the following example.

EXAMPLE 4.10 (cf. [4, Example 4.11]). For a unitary polynomial  $P \in \mathbb{C}[t]$ , we let  $A = A_{d,P} = B_{\text{norm}}$  be the normalization of the  $\mathbb{C}$ -algebra

$$B = B_{d,P} := \mathbb{C}[t, u, v]/(u^d v - P(t))$$

graded via  $\deg t = 0$ ,  $\deg u = 1$ ,  $\deg v = -d$  so that the normal affine surface  $V := \text{Spec } A$  is equipped with a hyperbolic  $\mathbb{C}^*$ -action. As  $B \cong A_0[u, Pu^{-d}]$  we can write

$$A \cong A_0[D_+, D_-], \quad \text{where} \quad D_+ = 0 \quad \text{and} \quad D_- = -\frac{\text{div } P}{d}$$

(see Lemma 4.6). We can recover  $P_{\pm}$  and  $Q$  in Lemma 4.7 as follows. By the construction given there  $P_+ = 1$  and by (8)  $\{D_-\} = \text{div}(P_-)/d_-$ . This gives

$$(11) \quad \text{div } P_- = d_- \left\{ -\frac{\text{div } P}{d} \right\} \quad \text{and} \quad \text{div } Q = \frac{\text{div } P}{d} + \frac{\text{div } P_-}{d_-}$$

(see (6)). In particular,

$$A_{\geq 0} \cong A_0[u] \cong \mathbb{C}[t, u] \quad \text{and} \quad A_{\leq 0} \cong A_{d_-, P_-}^+$$

(cf. Example 3.10) as graded  $A_0$ -algebras, where for the second isomorphism we have to reverse the grading of one of the rings.

This discussion provides the following characterization of the algebras  $A_{d,P}$ .

**Proposition 4.11.** *If  $A = A_0[D_+, D_-]$ , where  $A_0 \cong \mathbb{C}[t]$  and  $D_+, D_-$  are  $\mathbb{Q}$ -divisors on  $\mathbb{A}_{\mathbb{C}}^1$  with  $D_+ + D_- \leq 0$ , then the following conditions are equivalent.*

- (i)  $D_+$  is integral i.e.,  $\{D_+\} = 0$ .
- (ii)  $A_{\geq 0} \cong A_0[u]$  as graded  $A_0$ -algebras, where  $\deg u = 1$ .
- (iii)  $A \cong A_{d,P}$  as graded  $A_0$ -algebras, where  $D_+ + D_- = -\text{div}(P)/d$ .

Next we study the effect of base change to the Dolgachev-Pinkham-Demazure representation.

**Proposition 4.12.** *Let  $C = \text{Spec } A_0$  be an affine curve with function field  $K_0 = \text{Frac}(A_0)$  and let*

$$A := A_0[D_+, D_-] \subseteq K_0[u, u^{-1}],$$

where  $D_{\pm}$  are  $\mathbb{Q}$ -divisors on  $C$  satisfying  $D_+ + D_- \leq 0$ . Let  $L$  be the field  $L := \text{Frac}(A)[\sqrt[d]{tu^b}]$ , where  $t \in K_0$  and  $b \geq 0$ ,  $d > 0$ . If  $A'$  is the normalization of  $A$  in  $L$  then the following hold.

1.  $A'_0$  is the normalization of  $A_0$  in  $K_0[s]$  with  $s := \sqrt[k]{t}$ , where  $k := \gcd(b, d)$ .
2.  $A' \cong A'_0[D'_+, D'_-]$  with

$$D'_\pm := \frac{k}{d} (p^*(D_\pm) \pm \beta \operatorname{div} s) ,$$

where  $p: C' := \operatorname{Spec} A'_0 \rightarrow C$  is the projection and  $\beta$  is defined by  $\beta b \equiv k \pmod{d}$ .

Proof. We let  $b = b'k$  and  $d = d'k$ . The normalization  $A'$  admits a natural  $(1/d)$ -grading, and the element  $u^* := \sqrt[d]{tu^b}$  is of degree  $b/d = b'/d'$ . If we write  $k = \beta b + \delta d$ , then the element  $u' := u^{*\beta} u^\delta \in \operatorname{Frac}(A')$  has minimal possible positive degree  $1/d'$ . Thus

$$A' \subseteq \operatorname{Frac}(A'_0)[u', u'^{-1}].$$

To compute  $A'_0$ , we note that  $u^{*n} u^{-m}$  with  $n, m \in \mathbb{N}$  has degree 0 if and only if  $nb'/d' = m$ . In particular,  $n = n'd'$  is an integer multiple of  $d'$ . Thus  $K'_0 := \operatorname{Frac} A'_0$  is generated over  $K_0$  by  $u^{*d'} u^{-b'} = t^{1/k}$  (i.e.,  $n' = 1$ ). As  $d'$  and  $k$  are coprime, it follows that  $s = \sqrt[k]{t}$  also belongs to  $K'_0$  and that this field is actually generated by  $s$  over  $K_0$ , proving (1).

After localizing  $A_0$  we may assume that there is an element  $v_+ \in A$  of degree  $d_+ = d(A_{\geq 0})$  with  $A_{d_+} = v_+ A_0$  (see 3.6). We claim that then  $A'_{sd_+} = v_+^s A'_0$  for all  $s \geq 0$ . If not, then for some  $s > 0$  and some non-unit  $x \in A'_0$  the element  $v_+^s/x$  belongs to  $A'$ , so it is integral over  $A$  and there is an equation

$$\frac{v_+^{sm}}{x^m} + a_1 \frac{v_+^{s(m-1)}}{x^{m-1}} + \dots + a_m = 0 ,$$

where  $m \geq 0$  and  $a_i \in A_{i s d_+}$ . Thus  $a_i = v_+^{s i} q_i$  for some elements  $q_i \in A_0$ , whence dividing the equation above by  $v_+^{sm}$  we obtain that

$$\frac{1}{x^m} + q_1 \frac{1}{x^{m-1}} + \dots + q_m = 0 .$$

As  $A'_0$  is integrally closed this is only possible if  $x \in A'_0$  contradicting the choice of  $x$ .

Thus  $v = v_+$  is an element satisfying the assumptions of Remark 3.7, and we compute with it the divisor  $D'_+$  as follows (the calculation for  $D'_-$  is analogous). If we consider the new grading of  $A'$  by assigning to  $u'$  the degree 1, then  $v_+^k$  becomes an element of degree  $dd_+$ . Moreover, if  $u^{d_+} = P_+ v_+$  with  $P_+ \in K_0$  then by Remark 3.7  $D_+ = (\operatorname{div}(P_+))/d_+$ . Since

$$\begin{aligned} u'^{d d_+} &= (u^{*\beta} u^\delta)^{d d_+} = (t u^b)^{\beta d_+} u^{\delta d d_+} \\ &= t^{\beta d_+} u^{d_+ (\beta b + \delta d)} = t^{\beta d_+} u^{d_+ k} \\ &= t^{\beta d_+} P_+^k v_+^k \end{aligned}$$

we obtain again by Remark 3.7 that on  $C'$

$$D'_+ = \frac{\operatorname{div}(t^{\beta d_+} P_+^k)}{dd_+} = \frac{\beta}{d} \operatorname{div}(t) + \frac{k}{d} p^*(D_+),$$

and (2) follows. □

Let us consider the following important example.

EXAMPLE 4.13. With  $A_0 := \mathbb{C}[t]$ , suppose that  $D_+ = -(e/d)[0]$  and that  $D_-$  is any  $\mathbb{Q}$ -divisor on  $\mathbb{A}_{\mathbb{C}}^1 = \operatorname{Spec} A_0$  satisfying  $D_+ + D_- \leq 0$ . Applying Proposition 4.12 to  $s := \sqrt[d]{t}$  (i.e.  $b = 0$ ) we get that the normalization of  $A := A_0[D_+, D_-]$  in the field  $L := \operatorname{Frac}(A)[s]$  is given by

$$A' = A'_0[-e[0], D'_-] \subseteq \mathbb{C}(s)[u, u^{-1}],$$

where  $A'_0 = \mathbb{C}[s]$  and  $D'_- = p^*(D_-)$  (as before,  $p: \operatorname{Spec} \mathbb{C}[s] \rightarrow \operatorname{Spec} \mathbb{C}[t]$  denotes the projection  $s \mapsto s^d$ ). The divisor  $D'_+ = -e[0]$  being integral we have

$$A' \cong A'_0[0, D'_+ + D'_-] \subseteq \mathbb{C}(s)[\tilde{u}, \tilde{u}^{-1}],$$

where  $\tilde{u} := s^e u$ .

More concretely, if  $k := d_-(A)$ ,  $e := k \cdot D_-(0)$  and if we choose a unitary polynomial  $Q \in \mathbb{C}[t]$  with  $D_- = -(\operatorname{div}(Qt^e))/k$  then  $D'_+ + D'_- = -\{\operatorname{div}(Q(s^d)s^{ke+de})\}/k$ . By Example 4.10  $A' \cong A_{k,P}$  is the normalization of

$$(12) \quad B_{k,P} = \mathbb{C}[s, \tilde{u}, v]/(\tilde{u}^k v - P(s)), \quad \text{where } P(s) := Q(s^d)s^{ke+de}.$$

The field extension  $\operatorname{Frac}(A) \subseteq \operatorname{Frac}(A)[s]$  is Galois with Galois group  $\mathbb{Z}_d = \langle \zeta \rangle$ , where  $\zeta \cdot s = \zeta s$ . Thus

$$A \cong (A_{k,P})^{\mathbb{Z}_d},$$

and the action of  $\zeta$  on  $\tilde{u} = s^e u$  is given by  $\zeta \cdot \tilde{u} = \zeta^e \tilde{u}$ . Therefore, the group  $\mathbb{Z}_d$  acts on  $A_{k,P}$  via

$$(13) \quad \zeta \cdot s = \zeta s, \quad \zeta \cdot \tilde{u} = \zeta^e \tilde{u} \quad \text{and} \quad \zeta \cdot v = v.$$

Thus we obtain the following characterization.

**Proposition 4.14.** *For an algebra  $A = A_0[D_+, D_-]$  with  $A_0 = \mathbb{C}[s]$  the following conditions are equivalent.*

- (i)  $\{-D_+\} = (e/d)[0]$ , where  $0 \leq e < d$  and  $\operatorname{gcd}(e, d) = 1$ .

(ii)  $A \cong (A_{k,P})^{\mathbb{Z}_d}$ , where  $A_{k,P}$  is the normalization of  $B_{k,P}$  in (12) and where  $\mathbb{Z}_d = \langle \zeta \rangle$  acts via the formulas in (13).

Like in the parabolic case  $V$  may possess at most cyclic quotient singularities. The type of quotient singularities is determined from the divisors  $D_+, D_-$  by the following result. As before,  $C = \text{Spec } A_0$  is a smooth affine curve with function field  $K_0 = \text{Frac } A_0$  and  $A := A_0[D_+, D_-]$  with  $\mathbb{Q}$ -divisors  $D_+$  and  $D_-$  on  $C$ . Denote  $\pi: V = \text{Spec } A \rightarrow C$  the canonical projection.

**Theorem 4.15.** (a) *The set of singular points  $\text{Sing } V$  is contained in the fixed point set  $F$  which is the zero locus  $F = V(I)$  of the ideal  $I := A_+A + A_-A$  of  $A$ .*  
 (b) *The map  $\pi|_F: F \rightarrow C$  is injective, and  $\pi(F) = \{a \in C \mid D_+(a) + D_-(a) < 0\}$ .*  
 (c) *For a point  $a' \in F$  with image  $a := \pi(a') \in C$  we write*

$$D_+(a) = -\frac{e_+}{m_+} \quad \text{and} \quad D_-(a) = \frac{e_-}{m_-}$$

with the convention that

$$m_+ > 0, \quad m_- < 0, \quad \gcd(e_+, m_+) = \gcd(e_-, m_-) = 1 \quad \text{and} \\ m_+ = 1 \quad \text{if} \quad D_+(a) = 0, \quad m_- = -1 \quad \text{if} \quad D_-(a) = 0.$$

Let  $p, q \in \mathbb{Z}$  with  $\left| \frac{p}{q} \frac{e_+}{m_+} \right| = 1$ . Then  $a' \in F$  is a quotient singularity of type

$$(\Delta(a), e), \quad \text{where} \quad \Delta(a) := -\left| \frac{e_+}{m_+} \frac{e_-}{m_-} \right| \quad \text{and} \quad e \equiv \left| \frac{p}{q} \frac{e_-}{m_-} \right| \pmod{\Delta(a)}.$$

In particular,  $a' \in \text{Sing } V$  if and only if  $\Delta(a) \neq 1$ .

*Proof.* As in the proof of Proposition 3.8(b) we can reduce the statement to the case that  $A_0 = \mathbb{C}[t]$  and  $|D_+| \cup |D_-|$  is contained in the origin, so that  $D_{\pm} = \mp e_{\pm}/m_{\pm}[0]$ .

(a) The set  $\text{Sing } V$  is finite and invariant under the  $\mathbb{C}^*$ -action. Hence it is contained in the fixed point set  $F$ .  
 (b) The map  $A_0 \rightarrow A/I$  is obviously surjective. Thus

$$\pi|_F: F = \text{Spec} \left( \frac{A}{I} \right) \rightarrow C$$

is a closed embedding. Moreover,  $F = \emptyset$  if and only if  $1 \in I$  if and only if  $1 = a_+a_-$  for some homogeneous elements of  $A$  of opposite degrees, and the latter happens if and only if  $D_+ + D_- = 0$  by Remark 4.5.

(c) Notice first that the elements

$$v_+ := t^{e_+} u^{m_+}, \quad v_- := t^{e_-} u^{m_-} \in K_0[u, u^{-1}]$$

belong to  $A$ . Indeed, by definition, the ideal  $I+tA$  of  $A$  (this is just the maximal ideal of the point  $a' \in F$ ) is generated by the monomials  $t^e u^m$  with  $(e, m) \in \mathbb{Z} \times \mathbb{Z}$ , where  $(e, m) \neq (0, 0)$  and

$$e + mD_+(0) \geq 0 \quad \text{if } m \geq 0, \quad e - mD_-(0) \geq 0 \quad \text{if } m \leq 0.$$

In other words,  $(e, m)$  is an element of the cone  $\Gamma := C((e_+, m_+), (e_-, m_-))$  generated by the vectors  $(e_\pm, m_\pm)$  in the plane. Hence  $A$  is a toric algebra generated by the semigroup  $\Gamma \cap \mathbb{Z}^2$ , and so is a quotient  $A_{d,e}$  for some  $d, e \geq 0$  (see Lemma 2.4). To determine  $d, e$ , we must find a basis of  $\mathbb{Z}^2$  such that  $(e_+, m_+)$  is one of the basis vectors. This is done as follows.

If we choose  $p, q \in \mathbb{Z}$  with  $\begin{vmatrix} p & e_+ \\ q & m_+ \end{vmatrix} = 1$ , then the vectors  $\tilde{e}_1 := (e_+, m_+)$  and  $\tilde{e}_2 := (p, q)$  form a basis of  $\mathbb{Z}^2$ , and

$$(e_-, m_-) = \Delta' \tilde{e}_1 + \Delta \tilde{e}_2, \quad \text{where } \Delta' := \begin{vmatrix} p & e_- \\ q & m_- \end{vmatrix} \text{ and } \Delta := \Delta(0).$$

As  $\tilde{e}_1$  and  $(e_-, m_-)$  form a basis of the cone  $\Gamma$ , it follows from Lemma 2.4 that  $A$  has a quotient singularity of type  $(\Delta, e)$ , where  $0 \leq e < \Delta$  and  $e \equiv \begin{vmatrix} p & e_- \\ q & m_- \end{vmatrix} \pmod{\Delta}$ . Note that  $\Delta$  and  $\Delta'$  are coprime since so are  $e_-$  and  $m_-$ .

The determinant  $\Delta$  has always positive sign as

$$(14) \quad D_+(0) + D_-(0) = \frac{\Delta}{m_+ m_-} \leq 0 \quad \text{and} \quad m_+ > 0, \quad m_- < 0,$$

and so (c) follows. □

**Corollary 4.16.** *If  $A_{d,P}$  is the normalization of the algebra*

$$B_{d,P} = \mathbb{C}[t, u, v] / (u^d v - P(t)),$$

where  $P(t) = \prod_{i=1}^k (t - a_i)^{r_i}$  with  $a_i \neq a_j$  for  $i \neq j$  (see Example 4.10), then the singular points of the surface  $V_{d,P} = \text{Spec } A_{d,P}$  are the points  $a'_i \in V_{d,P}$  ( $1 \leq i \leq k$ ), where  $t = a_i, u = v = 0$  and  $r_i \nmid d$ .

*Proof.* It was shown in Example 4.10 that  $D_+ = 0$  and  $D_-(a_i) = -r_i/d$ . Therefore,  $\Delta(a_i) = e_+ > 1$  if and only if  $r_i \nmid d$ , which implies our assertion. □

In the sequel we use the following notation.

**DEFINITION 4.17.** Let  $O = \mathbb{C}^*_z$  be the orbit through a point  $z \in V \setminus F$ . Following [12] we say that  $O$  is of type  $(d, q)$  if  $d$  is the order of the stabilizer

$$\text{Stab}_z = \ker(\mathbb{C}^* \rightarrow \text{Aut } O) \subseteq \mathbb{C}^*, \quad \text{so that } \text{Stab}_z = \langle \zeta \rangle \cong \mathbb{Z}_d,$$

and  $q$  ( $0 \leq q < d$ ) is determined from the tangent representation of  $\text{Stab}_z$  on the tangent plane  $T_z V$  via pseudo-reflections

$$\text{Stab}_z \ni \zeta \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & \zeta^q \end{pmatrix}.$$

The orbit  $O$  is called *principal* if  $d = 1$  and *exceptional* otherwise (see [12]–[14] for a detailed description of the structure of  $V$  near the exceptional orbits).

In the next result we will characterize the orbit types of the surface  $V = \text{Spec } A$  with  $A := A_0[D_+, D_-]$ , where  $D_+$  and  $D_-$  are  $\mathbb{Q}$ -divisors on the smooth affine curve  $C = \text{Spec } A_0$ . Let  $\pi: V \rightarrow C$  denote the projection. To examine the orbits over a point  $a \in C$ , we write

$$D_+(a) = -\frac{e_+}{m_+} \quad \text{and} \quad D_-(a) = \frac{e_-}{m_-}$$

with the conventions as in Theorem 4.15(c). Let  $q_+$  be defined by  $0 \leq q_+ < m_+$  and  $q_+ e_+ \equiv -1 \pmod{m_+}$ , and similarly  $q_-$  by  $0 \leq q_- < -m_-$  and  $q_- e_- \equiv 1 \pmod{m_-}$ . With this notation the following result holds.

**Theorem 4.18.** *The exceptional orbits of  $V$  are located over  $|D_+| \cup |D_-|$ . The orbits over a given point  $a \in |D_+| \cup |D_-|$  are as follows.*

- (a) *If  $D_+(a) + D_-(a) = 0$  then  $\pi^*(a) = m_+ O$  consists of one orbit  $O$  of type  $(m_+, q_+)$  with multiplicity  $m_+$ . Moreover,  $O$  appears with coefficient  $-e_+$  in  $\text{div } u$ .*
- (b) *If  $D_+(a) + D_-(a) < 0$  then  $\pi^{-1}(a)$  contains two orbits  $O^+$  and  $O^-$  of types  $(m_+, q_+)$  and  $(-m_-, q_-)$ , respectively. Their closures  $\bar{O}^\pm$  intersect in the unique fixed point of the fiber, and  $\pi^*(a) = m_+ \bar{O}^+ - m_- \bar{O}^-$ . Moreover,  $\bar{O}^\pm$  appears with multiplicity  $\mp e_\pm$  in  $\text{div } u$ .*

**Proof.** With the same reasoning as in the proof of Proposition 3.8(b) it is sufficient to treat the case where  $A_0 = \mathbb{C}[t]$  and  $D_\pm$  are supported on  $a = 0 \in \mathbb{A}_{\mathbb{C}}^1$ , i.e.  $D_\pm = \mp e_\pm / m_\pm [0]$ . Note that in this case  $m_+ = d(A_{\geq 0})$  and  $m_- = -d(A_{\leq 0})$ .

(a) If  $D_+ + D_- = 0$ , so that  $e_+ = -e_- =: e$  and  $m_+ = -m_- =: m$  then  $A$  is the semigroup algebra  $\mathbb{C}[\Gamma \cap \mathbb{Z}^2]$ , where  $\Gamma$  is the cone generated over  $\mathbb{R}$  by the vectors  $\pm(e, m)$  and  $(1, 0)$ . Obviously  $\Gamma$  is the half space of all  $(x, y) \in \mathbb{R}^2$  satisfying  $mx - ey \geq 0$ . If we choose  $p, q \in \mathbb{Z}$  with  $|\begin{smallmatrix} p & e \\ q & m \end{smallmatrix}| = 1$  then the vectors  $(p, q)$  and  $(e, m)$  form a basis of  $\mathbb{Z}^2$ , and  $(p, q)$  lies in the half space  $\Gamma$ . Thus

$$\Gamma \cap \mathbb{Z}^2 = \mathbb{Z} \cdot (e, m) + \mathbb{N} \cdot (p, q),$$

and so  $A$  is the algebra of Laurent polynomials

$$(15) \quad A = \mathbb{C}[x, x^{-1}, y], \quad \text{where} \quad x := t^e u^m \in A_m \quad \text{and} \quad y := t^p u^q \in A_q.$$

Clearly then

$$(16) \quad t = x^{-q}y^m \quad \text{and} \quad u = x^p y^{-e}.$$

The action of  $\mathbb{C}^*$  is given by  $\lambda \cdot x = \lambda^m x$  and  $\lambda \cdot y = \lambda^q y$ , whence there is only one orbit  $O$  over  $t = 0$ , and it is given by the equation  $y = 0$ . By (16) we have

$$\pi^*(0) = \text{div } t = m \cdot O \quad \text{and} \quad \text{div } u = -e \cdot O.$$

The stabilizer of any point of  $O$  is the group  $E_m \subseteq \mathbb{C}^*$  of  $m$ -th roots of unity, and the type of the orbit is  $(m, q) = (m_+, q_+)$ , as required in (a).

(b) Let now  $D_+ + D_- < 0$ . Consider a generator  $v_{\pm} = t^{e_{\pm}} u^{m_{\pm}}$  of  $A_{m_{\pm}}$  as  $A_0$ -module (cf. the proof of Theorem 4.15(c)). The localization  $A_{v_{\pm}} = A[t^{-e_{\pm}} u^{-m_{\pm}}]$  is the subring  $A_0[D_+, -D'_-]$  of  $\text{Frac}(A_0)[u, u^{-1}]$  with  $D'_- := \min(D_-, -D_+)$  (see Lemma 4.6). As  $D_+ + D_- \leq 0$  we have  $D'_- = -D_+$ , so by (a) the open subset  $\text{Spec } A_{v_+}$  of  $V$  contains an orbit  $O^+$  of type  $(m_+, q_+)$ , and it has multiplicities  $m_+$  and  $-e_+$  in  $\pi^*(0)$  and  $\text{div } u$ , respectively. Similarly,  $\text{Spec } A_{v_-}$  contains an orbit  $O^-$  of type  $(-m_-, q_-)$ , which has multiplicities  $-m_-$  and  $e_-$  in  $\pi^*(0)$  and  $\text{div } u$ , respectively. We have  $\text{div}(v_+ v_-) = \Delta \cdot (\bar{O}^+ + \bar{O}^-)$ , where by our assumption  $\Delta = m_+ m_- (D_+(0) + D_-(0)) > 0$  (see (14)). Thus the fiber of  $\pi$  over  $t = 0$  can be given by  $v_+ \cdot v_- = 0$ , where the functions  $v_+, v_-$  vanish on  $\bar{O}^-$  and  $\bar{O}^+$ , respectively. The intersection  $\bar{O}^+ \cap \bar{O}^-$  is given by  $v_+ = v_- = 0$ , and so is the unique fixed point of the fiber. □

EXAMPLE 4.19. In the example of the algebra  $A = A_{d,P}$  treated in Corollary 4.16 we have  $D_+ = 0$  and  $D_- = -\text{div}(P)/d = \sum_i -(r_i/d)[a_i]$  (see Example 4.10). The exceptional orbits are located over the points  $a_i \in \mathbb{A}_{\mathbb{C}}^1$ , and  $\pi^{-1}(a_i) = O_i^+ \cup \{a'_i\} \cup O_i^-$ , where  $a'_i$  is the unique fixed point of the fiber (located over the point  $(0, 0, a_i)$  of  $\text{Spec } B_{d,P} \subseteq \mathbb{C}^3$ ). Applying Theorem 4.18, the orbit  $O_i^+$  is principal, and if we write  $r_i/d = e_i/m_i$  with  $\text{gcd}(e_i, m_i) = 1$  then  $O_i^-$  is of type  $(m_i, q_i)$ , where

$$q_i e_i \equiv -1 \pmod{m_i} \quad \text{with} \quad 0 \leq q_i < m_i.$$

REMARK 4.20. We can now precise the character of the affine modifications  $\sigma_{\pm}: V \rightarrow V_{\pm}$  as in Proposition 4.1. Doing this locally we assume first that  $A_0 = \mathbb{C}[t]$  and  $D_{\pm}$  is supported on  $a = 0 \in \mathbb{A}_{\mathbb{C}}^1$ . If  $D_+ + D_- = 0$  then  $A = A_{\geq 0}[v_{\pm}^{-1}] = (A_{\geq 0})_{v_{\pm}}$ , whence  $\sigma_+: V \rightarrow V_+$  is an open embedding and  $V_+ \setminus V$  is the divisor  $\text{div } v_+ = m_+ \iota_+(C)$ . In case  $D_+ + D_- < 0$ , letting in the proof of Proposition 4.1  $f_0 := v_+^{-m_-}$ , we obtain that  $\sigma_+: V \rightarrow V_+$  consists in blowing up a graded ideal  $I \subseteq (t, v_+)$  of the algebra  $A_{\geq 0}$  supported at a fixed point and deleting the proper transform of the divisor  $\text{div } v_+ = m_+ \iota_+(C)$ . The exceptional curve in  $V$  is the orbit closure  $\bar{O}^- = \{v_+ = 0\}$ .

Globalizing we see that  $\sigma_{\pm}: V \rightarrow V_{\pm}$  blows up a graded ideal with support at the fixed points  $b'_1, \dots, b'_l \in \iota_{\pm}(C)$  over the points  $b_i := \pi_{\pm}(b'_i) \in C$  with  $D_+(b_i) +$



$D_-(b_i) < 0$ , and deleting the proper transform of the fixed point curve  $\iota_{\pm}(C) \subseteq V_{\pm}$ . Moreover the exceptional set of  $\sigma_{\pm}$  is  $\bar{O}_1^{\mp} \cup \dots \cup \bar{O}_l^{\mp}$ .

**4.21.** We let as before  $C = \text{Spec } A_0$  be a smooth affine curve with function field  $K_0 = \text{Frac } A_0$ , and we let  $D_+, D_-$  be  $\mathbb{Q}$ -divisors on  $C$ . In what follows we compute the Picard group and the divisor class group of  $A := A_0[D_+, D_-]$  (see also [18, Thm. 5.1] and [26, Cor. 1.7] for the elliptic case). We denote by  $a_1, \dots, a_k$  the points in  $C$  for which  $D_+(a) = -D_-(a) \neq 0$ , and we let  $b_1, \dots, b_l \in C$  be the points with  $D_+(b) + D_-(b) < 0$ . Let us write

$$D_{\pm}(a_i) = \mp \frac{e_i}{m_i}, \quad D_+(b_j) = -\frac{e_j^+}{m_j^+} \quad \text{and} \quad D_-(b_j) = \frac{e_j^-}{m_j^-}$$

with the conventions as in Theorem 4.15. If  $\pi: V := \text{Spec } A \rightarrow C$  denotes the canonical map then the preimage  $\pi^{-1}(a_i)$  consists of only one orbit  $O_i$ , and  $\pi^{-1}(b_j)$  consists of two orbit closures  $\bar{O}_j^+ \cup \bar{O}_j^-$ , so that

$$(17) \quad \pi^*(a_i) = m_i O_i \quad \text{and} \quad \pi^*(b_j) = m_j^+ \bar{O}_j^+ - m_j^- \bar{O}_j^-$$

as divisors on  $V$ , see Theorem 4.18.

**Theorem 4.22.** *The divisor class group  $\text{Cl } A$  of  $A$  is the group*

$$\pi^*(\text{Cl } A_0) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l \left( \mathbb{Z}[\bar{O}_j^+] \oplus \mathbb{Z}[\bar{O}_j^-] \right)$$

*modulo the relations*

$$\begin{aligned} \pi^*(a_i) &= m_i [O_i], \quad i = 1, \dots, k, \\ \pi^*(b_j) &= m_j^+ [\bar{O}_j^+] - m_j^- [\bar{O}_j^-], \quad j = 1, \dots, l, \\ 0 &= \sum_{j=1}^l e_j^+ [\bar{O}_j^+] - \sum_{j=1}^l e_j^- [\bar{O}_j^-]. \end{aligned}$$

*Proof.* Let  $\text{Div}_h A \subseteq \text{div } A$  be the subgroup of all Weil divisors on  $V$  that are homogeneous, i.e. finite sums of irreducible divisors given by homogeneous prime ideals. The homogeneous principal divisors  $\text{Prin}_h A$  form a subgroup of  $\text{Div}_h A$ , which consists of all divisors  $\text{div } f$ , where  $f = g/h \in \text{Frac } A$  is a quotient of homogeneous elements. By [8, §1, Ex. 16]

$$\text{Cl } A \cong \text{Cl}_h A := \text{Div}_h A / \text{Prin}_h A.$$

The group  $\text{Div}_h A$  is freely generated by all  $\mathbb{C}^*$ -invariant subvarieties of codimension 1 in  $V$ , that is by all irreducible components of the fibers of  $\pi: V \rightarrow C$ . If  $D_+(a) =$

$D_-(a) = 0$  then the fiber over  $a$  is the prime divisor  $\pi^*(a)$ . If  $a = a_i$  for some  $i$  then the fiber over  $a$  consists of just one orbit  $O_i$  of type  $(m_i, q_i)$ , and by (17)  $\pi^*(a_i) = m_i O_i$  as divisors on  $V$ . If  $a = b_j$  for some  $j$  then by (17)  $\pi^*(b_j) = m_j^+ \bar{O}_j^+ - m_j^- \bar{O}_j^-$ . Thus the natural map  $\pi^* : \text{div } A_0 \rightarrow \text{Div}_h A$  is injective, and

$$(18) \quad \text{Div}_h A \cong \frac{\pi^*(\text{div } A_0) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l (\mathbb{Z}[\bar{O}_j^+] \oplus \mathbb{Z}[\bar{O}_j^-])}{(\pi^*(a_i) - m_i[O_i], \pi^*(b_j) - m_j^+[\bar{O}_j^+] + m_j^-[\bar{O}_j^-])}.$$

The group  $\text{Prin}_h A$  is generated by all divisors  $\text{div}(fu^k) = \text{div } f + k \text{div } u$ , where  $f \in K_0^\times$  is non-zero. Dividing out  $\pi^*(\text{Prin } A_0) = \pi^* \text{div}(K_0^\times)$  in (18) gives the group

$$(19) \quad \frac{\pi^*(\text{Cl } A_0) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l (\mathbb{Z}[\bar{O}_j^+] \oplus \mathbb{Z}[\bar{O}_j^-])}{(\pi^*(a_i) - m_i[O_i], \pi^*(b_j) - m_j^+[\bar{O}_j^+] + m_j^-[\bar{O}_j^-])}.$$

By Theorem 4.18 the divisor of  $u$  is given by

$$\text{div } u = - \sum_{j=1}^k e_j [O_j] + \sum_{j=1}^l (-e_j^+ [\bar{O}_j^+] + e_j^- [\bar{O}_j^-]).$$

Hence, taking (19) modulo this relation leads to the divisor class group, as required. □

**Corollary 4.23.** *A is factorial if and only if  $C \subseteq \mathbb{A}_\mathbb{C}^1$  (i.e.  $A_0$  is a localization of  $\mathbb{C}[t]$ ) and one of the following two conditions is satisfied.*

- (i)  $l = 0$  and  $\text{gcd}(m_i, m_j) = 1$  for  $1 \leq i < j \leq k$ .
- (ii)  $l = 1, m_i = 1$  for all  $i$  and  $|\frac{e^+}{m^+} \frac{e^-}{m^-}| = \pm 1$ , where  $e^\pm := e_1^\pm$  and  $m^\pm := m_1^\pm$ .

*Proof.* If  $C$  is a curve of genus  $g \geq 1$  then the group  $\text{Cl } A$  is not finitely generated. Thus assuming that  $A$  is factorial,  $C$  is isomorphic to an open subset of  $\mathbb{A}_\mathbb{C}^1$ . By Theorem 4.22 the group  $\text{Cl } A$  has then  $k + 2l$  generators and  $k + l + 1$  independent relations, whence necessarily  $l \leq 1$ . In the case  $l = 1$  the number of generators and the number of relations are equal, and so the order of  $\text{Cl } A$  is the absolute value of the determinant

$$\begin{vmatrix} e^+ & e^- & e_1 & e_2 & \cdots & e_k \\ m^+ & m^- & 0 & 0 & \cdots & 0 \\ 0 & 0 & m_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & m_k \end{vmatrix} = \begin{vmatrix} e^+ & e^- \\ m^+ & m^- \end{vmatrix} \cdot m_1 \cdot m_2 \cdot \cdots \cdot m_k.$$

Thus, if  $\text{Cl } A = 0$  then all the factors of this product are equal to 1, and we are in case (ii). If  $l = 0$  then  $\text{Cl } A$  is the group  $\bigoplus_{i=1}^k \mathbb{Z}_{m_i} \cdot [O_i]$  modulo the relation

$\sum_i e_i[O_i] = 0$ . As  $e_i$  and  $m_i$  are coprime, this group is trivial if and only if (i) holds. Conversely, if (i) or (ii) is satisfied then the discussion above shows that  $\text{Cl } A$  is trivial, finishing the proof.  $\square$

Finally, we determine the Picard group and the canonical divisor of  $A$ . The local divisor class group at the point  $b_j$  is generated by  $\bar{O}_j^\pm$  modulo the relations

$$R_j := e_j^+ \bar{O}_j^+ - e_j^- \bar{O}_j^- = 0 \quad \text{and} \quad S_j := m_j^+ \bar{O}_j^+ - m_j^- \bar{O}_j^- = 0.$$

Since the Picard group  $\text{Pic } A$  is the kernel of the map of  $\text{Cl } A$  into the direct product of all local divisor class groups, this group is the subgroup of  $\text{Cl } A$  generated by  $\pi^*(\text{Cl } A_0)$ ,  $[O_i]$ ,  $R_j$  and  $S_j$ . As  $S_j = \pi^*(b_j)$ , we obtain the following result.

**Corollary 4.24.** *Pic  $A$  is the group*

$$\pi^*(\text{Cl } A_0) \oplus \bigoplus_{i=1}^k \mathbb{Z}[O_i] \oplus \bigoplus_{j=1}^l \mathbb{Z}R_j$$

*modulo the relations*

$$\begin{aligned} \pi^*(a_i) &= m_i[O_i], \quad i = 1, \dots, k, \\ 0 &= \sum_{j=1}^k e_j[O_j] + \sum_{j=1}^l R_j. \end{aligned}$$

*In particular, Pic  $A$  vanishes if and only if  $C \subseteq \mathbb{A}_{\mathbb{C}}^1$  and case (i) in Corollary 4.23 is satisfied or  $l = 1$  and  $m_i = 1$  for all  $1 \leq i \leq k$ .*

**Corollary 4.25.** <sup>2</sup> *The canonical divisor of the surface  $V = \text{Spec } A$  is given by*

$$K_V = \pi^*(K_C) + \sum_{i=1}^k (m_i - 1)[O_i] + \sum_{j=1}^l \left( (m_j^+ - 1)[\bar{O}_j^+] + (-m_j^- - 1)[\bar{O}_j^-] \right).$$

**Proof.** We claim that multiplication by the meromorphic differential form  $du/u$  on  $V$  gives an isomorphism

$$\frac{du}{u} \wedge - : \pi^*(\omega_C) \left( \sum_{j=1}^k (m_j - 1)[O_j] + \sum_{j=1}^l \left( (m_j^+ - 1)[\bar{O}_j^+] + (-m_j^- - 1)[\bar{O}_j^-] \right) \right) \xrightarrow{\cong} \omega_V.$$

This is a local problem, so with the same arguments as in the proof of Theorem 4.18 we can reduce to the case that  $A_0 \cong \mathbb{C}[t]$  and  $D_+ = -D_- = -(e/d)[0]$ , where  $e, m$  are

<sup>2</sup>cf. [26, Thm. 2.8] and [19, Lemma 2.6].

coprime. In this case (15) in the proof of Theorem 4.18 shows that  $A = \mathbb{C}[x, x^{-1}, y]$  with  $x := t^e u^m$  and  $y := t^p u^q$ , where  $p, q$  are integers with  $|\begin{smallmatrix} p & e \\ q & m \end{smallmatrix}| = 1$ . Moreover by (16)  $t = x^{-q} y^m$  and  $u = x^p y^{-e}$ . By an elementary calculation  $(du/u) \wedge dt = x^{-q-1} y^{m-1} dx \wedge dy$ , whence the result follows.  $\square$

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