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# A CRITERION OF EXACTNESS OF THE CLEMENS-SCHMID SEQUENCES ARISING FROM SEMI-STABLE FAMILIES OF OPEN CURVES

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## 1. Introduction

Let $\pi\colon X \to \Delta$ be a semistable family of projective algebraic varieties over the unit disk. In other words, $\pi$ is flat and projective over $\Delta$, smooth over $\Delta^* = \Delta - \{0\}$, and the central fiber $Y = \pi^{-1}(0)$ is a divisor with normal crossings without multiple components. We write $X_t := \pi^{-1}(t)$ for $t \in \Delta^*$ and $X^* := X - Y$. There is the Wang exact sequence

$$\cdots \to H^q(X^*, \mathbf{Q}) \to H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \to H^{q+1}(X^*, \mathbf{Q}) \to \cdots, \tag{1.1}$$

and the localization exact sequence

$$\cdots \to H^q_Y(X, \mathbf{Q}) \to H^q(X, \mathbf{Q}) \to H^q(X^*, \mathbf{Q}) \to H^{q+1}_Y(X, \mathbf{Q}) \to \cdots. \tag{1.2}$$

Here $N$ denotes the log monodromy around $\Delta^* = \Delta - \{0\}$. Combining those sequences and the natural isomorphism $H^q(X, \mathbf{Q}) \simeq H^q(Y, \mathbf{Q})$, we obtain a sequence

(1.3)
$$\cdots \to H^{q-2}(X_t, \mathbf{Q}) \to H^q_Y(X, \mathbf{Q}) \to H^q(Y, \mathbf{Q}) \to H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \to \cdots.$$

This is called the *Clemens-Schmid sequence*. A theorem of Clemens and Schmid says that the sequence (1.3) is exact ([1, §3]). In particular, the first piece

$$H^q(Y, \mathbf{Q}) \longrightarrow H^q(X_t, \mathbf{Q}) \xrightarrow{N} H^q(X_t, \mathbf{Q}) \tag{1.4}$$

is called the *local invariant cycle theorem* ([3, (5.12)]).

In [2, (12.3.1)], S. Usui et al. proposed a problem whether the Clemens-Schmid sequence (1.3) or (1.4) is exact when we remove the assumption that $\pi$ is proper. In this paper, we give a necessary and sufficient condition for that the sequence (1.4) is exact when $\pi$ is a semistable family of open curves. In particular, we see that the Clemens-Schmid sequences of non-proper families are not exact in general.

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## 2. A Criterion and an example

In this section, we assume that $\pi$ is a family of curves. Let $D$ be a divisor on $X$ which is smooth and projective over $\Delta$, and such that $D + Y$ is a normal crossing divisor. We write $D_t = D \cap X_t$ for $t \in \Delta^*$ and $D^* = D \cap X^*$.

We discuss when the following sequence

$$H^1(Y - D \cap Y) \xrightarrow{i} H^1(X_t - D_t) \xrightarrow{N_0} H^1(X_t - D_t) \tag{2.1}$$

is exact, where $N_0$ denotes the log monodromy on $H^1(X_t - D_t)$. (We omit to write the coefficient $\mathbf{Q}$.) The sequence (2.1) fits into the commutative diagram

$$\begin{array}{ccccccc}
 & & H^0(D \cap Y) & \xrightarrow{\simeq} & H^0(D_t) & & H^0(D_t) \\
 & & r_0 \uparrow & & \uparrow r_t & & \uparrow \\
 & & H^1(Y - D \cap Y) & \xrightarrow{i} & H^1(X_t - D_t) & \xrightarrow{N_0} & H^1(X_t - D_t) \\
 & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(X_t) & \xrightarrow{N} & H^1(X_t) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0,
\end{array} \tag{2.2}$$

in which each vertical sequence is exact. The bottom sequence is exact by the local invariant cycle theorem ([3, (5.12)]). Therefore, the map $i$ is injective.

The following lemma is the key of our criterion.

**Lemma 2.1.** *The image of $N$ coincides with the image of $N_0$.*

Proof. Image $N \subset$ Image $N_0$ is trivial. We show the converse. The cohomology groups $H^1(X_t, \mathbf{Q})$ and $H^1(X_t - D_t, \mathbf{Q})$ carry the limit mixed Hodge structures due to Steenbrink and Zucker ([3], [4]). The weight filtration $W_\bullet$ on $H^1(X_t, \mathbf{Q})$ is the weight monodromy filtration, which is explicitly given as follows:

$$W_2 H^1(X_t) = H^1(X_t), \quad W_1 H^1(X_t) = \ker N,$$
$$W_0 H^1(X_t) = \mathrm{Image}\, N, \quad W_{-1} H^1(X_t) = 0.$$

We put the weight filtration on $H^0(D_t)$ by $\mathrm{Gr}_2^W H^0(D_t) = H^0(D_t)$. Then the weight filtration $W_\bullet$ on $H^1(X_t - D_t)$ (called the relative monodromy filtration) is explicitly given by

$$W_2 H^1(X_t - D_t) = H^1(X_t - D_t), \quad W_1 H^1(X_t - D_t) = W_1 H^1(X_t),$$
$$W_0 H^1(X_t - D_t) = W_0 H^1(X_t), \quad W_{-1} H^1(X_t - D_t) = 0.$$

The map $N_0$ is a morphism of the mixed Hodge structure of type $(-1,-1)$ ([4, (3.13).iii)]). Therefore $\operatorname{Image} N_0$ is contained in $W_0H^1(X_t - D_t, \mathbf{Q}(0)) = W_0H^1(X_t)$. However, this is the image of $N$. Thus we have $\operatorname{Image} N_0 \subset \operatorname{Image} N$. □

Since the map $i$ is injective, the sequence (2.1) is exact if and only if

$$h^1(Y - D \cap Y) = \dim H^1(Y - D \cap Y) = \dim \ker N_0. \tag{2.3}$$

By Lemma 2.1, we have

$$\begin{aligned}\dim \ker N_0 &= h^1(X_t - D_t) - \dim \operatorname{Image} N\\ &= h^1(X_t - D_t) - h^1(X_t) + h^1(Y)\\ &= \dim \operatorname{Image} r_t + h^1(Y).\end{aligned}$$

On the other hand,

$$h^1(Y - D \cap Y) = \dim \operatorname{Image} r_0 + h^1(Y).$$

Therefore, (2.3) holds if and only if

$$\dim \operatorname{Image} r_0 = \dim \operatorname{Image} r_t. \tag{2.4}$$

Let $n$ be the number of the irreducible components of $Y$, and $e$ the number of the components $Y_i$ of $Y$ such that $D \cap Y_i \neq \emptyset$. Then we have

$$\begin{aligned}\dim \operatorname{Image} r_t &= \dim \ker\bigl(H^0(D_t) \to H^2(X_t)\bigr)\\ &= \deg D_t - 1,\end{aligned}$$

and

$$\begin{aligned}\dim \operatorname{Image} r_0 &= \dim \ker\bigl(H^0(D \cap Y) \to H^2(Y)\bigr)\\ &= h^0(D \cap Y) - h^2(Y) + h^2(Y - D \cap Y)\\ &= \deg D_t - n + (n - e)\\ &= \deg D_t - e.\end{aligned}$$

Thus we have the following criterion:

**Theorem 2.2.** *The Clemens-Schmid sequence* (2.1) *is exact if and only if* $e = 1$, *that is, the support of* $D \cap Y$ *is contained in one irreducible component of* $Y$.

EXAMPLE 2.3. We denote by $(x, y, z)$ homogeneous coordinates of the projective plane $\mathbf{P}^2_{\mathbf{C}}$. Let $X$ be a complex submanifold of $\mathbf{P}^2(\mathbf{C}) \times \Delta$ defined by a equation

$$zy^2 = zx^2 + t(x^3 - z^3),$$

and $\pi\colon X \to \Delta$ the projection. The morphism $\pi$ is smooth and proper over $\Delta^*$, and the central fiber $Y = \pi^{-1}(0)$ is a reduced divisor with normal crossing. Let $Y_1$, $Y_2$ and $Y_3$ be the irreducible components of $Y$ given by

$$Y_1 : z = 0,\ \ t = 0,$$
$$Y_2 : y + x = 0,\ \ t = 0,$$
$$Y_3 : y - x = 0,\ \ t = 0$$

respectively. Let $D = D_1 + D_2 + D_3$ be flat sections of $\pi$ given by

$$\begin{aligned} D_1 &= \{(0, 1, 0)\} \times \Delta, \\ D_2 &= \{(1, -1, \omega)\} \times \Delta, \\ D_3 &= \{(1, 1, \omega')\} \times \Delta, \end{aligned}$$

where $\omega$ and $\omega'$ are 3rd roots of unity. The divisor $D + Y$ is normal crossing.

The divisor $D_1$ (resp. $D_2$, $D_3$) meets only with $Y_1$ (resp. $Y_2$, $Y_3$). Thus, the Clemens-Schmid sequence (2.1) is not exact by Theorem 2.2.

A family $\pi\colon X \times X' \to \Delta$ with $X' \to \Delta$ a constant family of a projective nonsingular variety gives a non-exact Clemens-Schmid sequence (1.4) for all $q \geq 1$.

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