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INTERFACE REGULARITY FOR MAXELL AND STOKES SYSTEMS

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1. Introduction

The purpose of the present paper is to study the interface regularity of three dimensional Maxwell and Stokes systems. To our knowledge, not so much regards have been taken in this topic, but actually the solenoidal condition provides the regularity across interface to a specified component of the unknown vector field.

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and $\mathcal{M} \subset \mathbf{R}^3$ be a C^2 hypersurface cutting Ω transversally. Then, it holds that

$$\begin{aligned} &\mathcal{M} \cap \Omega \neq \emptyset \\ &\Omega = \Omega_+ \cup (\Omega \cap \mathcal{M}) \cup \Omega_- \quad \text{(disjoint union)} \end{aligned} \tag{1}$$

with the open subsets $\Omega_\pm$ of Ω. First, we take the Maxwell system in magnetostatics,

$$\left.\begin{aligned} \nabla \times B = J \\ \nabla \cdot B = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_\pm, \tag{2}$$

where $B = (B^1(x), B^2(x), B^3(x))$ and $J = (J^1(x), J^2(x), J^3(x))$ stand for the three dimensional vector fields, indicating the magnetic field and the total current density, respectively. Here and henceforth, $\nabla = {}^T(\partial_1, \partial_2, \partial_3)$ denotes the gradient operator and $\times$ and $\cdot$ are the outer and the inner products in $\mathbf{R}^3$, so that $\nabla\times$ and $\nabla\cdot$ are the operations of the rotation and the divergence, respectively.

In the context of magnetoencephalography, Suzuki, Watanabe, and Shimogawara [2] studied the case when the interface is given by the boundary ∂D of a smooth bounded domain $D \subset \mathbf{R}^3$. Namely, from the properties of the layer potential, it showed that if J is piecewise continuous on $\mathbf{R}^3 \setminus \partial D$ and system (2) has a solution $B \in C(\mathbf{R}^3)^3 \cap C^1(\mathbf{R}^3 \setminus \partial D)^3$ for $\Omega_- = D$ and $\Omega_+ = \mathbf{R}^3 \setminus D$, then

$$[\nabla(n \cdot B)]_-^+ = 0 \quad \text{on} \quad \partial D$$

follows, regardless with the continuity of J across ∂D. Here, n denotes the outer unit normal vector to ∂D, $[A]_-^+ = A_+ - A_-$, and

$$A_+(\xi) = \lim_{x\to\xi,\, x\in\mathbf{R}^3\setminus D} A(x), \qquad A_-(\xi) = \lim_{x\to\xi,\, x\in D} A(x)$$

for $\xi \in \partial D$. In this paper we study its local version, that is, the case where the bounded domain Ω is given with the interface $\mathcal{M}\cap\Omega$ as in (1).

To state the result, we take preliminaries on function spaces from Girault and Raviart [1]. Namely, let $D \subset \mathbf{R}^3$ be a bounded domain with Lipschitz boundary ∂D and n be the unit normal vector to ∂D. For $p \in [1,\infty]$, $L^p(D)$ denotes the standard L^p space on D provided with the norm $\|\cdot\|_{L^p(D)}$, and the Sobolev space $H^m(D)$ is defined by

$$H^m(D) = \left\{u \in L^2(D) \mid \partial^\alpha u \in L^2(D) \quad \text{for} \quad |\alpha| \le m\right\}$$

for a positive integer m, where $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ for the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Given $\sigma \in (0,1)$, we say that $u \in H^{m+\sigma}(D)$ if $u \in H^m(D)$ and

$$\int_D\int_D \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{n+2\sigma}}\,dx\,dy < +\infty$$

for any α in $|\alpha| = m$ and $n = 3$. The space $H^s(\Gamma)$ is defined similarly with $n = 2$ through the local chart of Γ, where $s \in [0,1]$ and $\Gamma \subset \partial D$ is a relatively open connected set. Then, we set $H^{-s}(\Gamma) = H_0^s(\Gamma)'$, where $H_0^s(\Gamma)$ denotes the closure in $H^s(\Gamma)$ of the space composed of Lipschitz continuous functions on Γ with compact supports. Thus, we have $H_0^s(\Gamma) = H^s(\Gamma)$ if $\Gamma \subset \partial D$ is a closed surface, and in particular, it holds that $H^{1/2}(\partial D) = H_0^{1/2}(\partial D)$. We also put

$$H(\mathrm{div}, D) = \left\{u \in L^2(D)^3 \mid \nabla\cdot u \in L^2(D)\right\}$$

and

$$H(\mathrm{rot}, D) = \left\{u \in L^2(D)^3 \mid \nabla\times u \in L^2(D)^3\right\}.$$

Then, any $v \in H(\mathrm{div}, D)$ admits the trace $n\cdot v|_{\partial D} \in H^{-1/2}(\partial D)$, and Green's formula

$$\big((v, \nabla\varphi)\big)_D + (\nabla\cdot v, \varphi)_D = \langle n\cdot v, \varphi\rangle_{\partial D}$$

holds for $\varphi \in H^1(D)$. Here and henceforth, $(\cdot,\cdot)_D$ and $((\cdot,\cdot))_D$ denote $L^2(D)$ and $L^2(D)^3$ inner products, respectively, and $\langle\cdot,\cdot\rangle_{\partial D}$ the duality pairing between $H^{-1/2}(\partial D)$ and $H^{1/2}(\partial D) = H_0^{1/2}(\partial D)$. Let us note here that the standard trace theorem guarantees $\varphi|_{\partial D} \in H^{1/2}(\partial D)$ for $\varphi \in H^1(D)$. Similarly, any $v \in H^1(\mathrm{rot}, D)$ admits the trace $n\times v|_{\partial D} \in H^{-1/2}(\partial D)^3$, and the Stokes formula

$$\big((\nabla\times v, w)\big)_D - \big((v, \nabla\times w)\big)_D = \langle\langle n\times v, w\rangle\rangle_{\partial D}$$

holds for $w \in H^1(D)^3$, where $\langle\langle \cdot, \cdot \rangle\rangle_{\partial D}$ denotes the duality pairing between $H^{-1/2}(\partial D)^3$ and $H^{1/2}(\partial D)^3$.

Now, to discuss the interface regularity of the solution B to the Maxwell system (2), we take that

$$\Gamma_\pm = \partial\Omega_\pm \cap M$$

with $\partial\Omega_\pm$ being the boundary of $\Omega_\pm$. This means that Γ_+ and Γ_- coincide as sets, but they are regarded as the parts of the boundaries of Ω_+ and Ω_-, respectively. Henceforth, n denotes the outer unit normal vector to Γ_- so that $-n$ is the outer unit normal vector to Γ_+. Henceforth, C^2 extension of the vector field n defined on $\Gamma = \mathcal{M} \cap \Omega$ is always taken to Ω. Furthermore, given a function $A(x)$ on $\Omega_\pm$, we set

$$[A]_-^+ = A_+ - A_- \qquad \text{on} \qquad \Gamma,$$

where $A_\pm(\xi) = \lim_{x\to\xi,\, x\in\Omega_\pm} A(x)$ for $\xi \in \Gamma$ are usually taken in the sense of traces to $\Gamma_\pm$.

Suppose that B and J are in $L^2(\Omega_\pm)^3$ and satisfy (2). This means that those relations hold piecewisely in $\Omega_\pm$ in the sense of distributions $\mathcal{D}'(\Omega_\pm)$, that is,

$$\int_{\Omega_\pm} B \cdot \nabla \times C = \int_{\Omega_\pm} J \cdot C \qquad \text{and} \qquad \int_{\Omega_\pm} B \cdot \nabla\varphi = 0$$

for any $C \in C_0^\infty(\Omega_\pm)^3$ and $\varphi \in C_0^\infty(\Omega_\pm)$. Unless otherwise stated, those vector fields $B \in L^2(\Omega_\pm)$ and $J \in L^2(\Omega_\pm)^3$ are identified with the elements in $L^2(\Omega)^3$.

Relation (2) for $B \in L^2(\Omega_\pm)^3$ and $J \in L^2(\Omega_\pm)^3$ implies that $B \in H(\mathrm{rot}, \Omega_\pm) \cap H(\mathrm{div}, \Omega_\pm)$, which assures the well-definedness of

$$n \times B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3 \qquad \text{and} \qquad n \cdot B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm),$$

and hence $B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ follows. Furthermore,

$$[n \times B]_-^+ = 0 \qquad \text{and} \qquad [n \cdot B]_-^+ = 0 \tag{3}$$

if and only if

$$\nabla \times B = J \in L^2(\Omega)^3 \qquad \text{and} \qquad \nabla \cdot B = 0 \in L^2(\Omega) \tag{4}$$

as distributions in Ω, respectively. If both relations of (3) are satisfied, then $B \in H^1_{loc}(\Omega)^3$ follows, because $B \in H^1_{loc}(\Omega)^3$ is equivalent to $[B]_-^+ = 0$ on Γ for $B \in H^1(\Omega_\pm)^3$. This fact is also obtained by Corollary I.2.10 of [1],

$$H(\mathrm{rot}, \Omega) \cap H(\mathrm{div}, \Omega) \subset H^1_{loc}(\Omega)^3, \tag{5}$$

as (4) for $B \in L^2(\Omega)^3$ implies $B \in H^1_{loc}(\Omega)^3$.

Our first result is stated as follows. Let us note again that n defined on $\mathcal{M}$ is extended to a C^2 vector field in Ω, and $n \cdot B \in H^1_{loc}(\Omega)$ follows from $B \in H^1(\Omega)^3$.

Theorem 1. *If* $B \in H^1(\Omega)^3$ *and* $J \in H(\mathrm{rot}, \Omega_\pm)$ *satisfy* (2), *then it holds that* $n \cdot B \in H^2_{loc}(\Omega)$.

In the above theorem, B solves (2) in Ω as a distribution, because it is assumed to be in $H^1(\Omega)^3$. That is,

$$\int_\Omega B \cdot \nabla \times C = \int_\Omega J \cdot C \qquad \text{and} \qquad \int_\Omega B \cdot \nabla \varphi = 0$$

hold for any $C \in C_0^\infty(\Omega)^3$ and $\varphi \in C_0^\infty(\Omega)$. On the other hand, $J \in H(\mathrm{rot}, \Omega_\pm)$ belongs to $J \in H(\mathrm{rot}, \Omega)$ if and only if $[n \times J]_-^+ = 0$ on Γ. If this condition is satisfied furthermore, then it holds that

$$-\Delta B = \nabla \times J \in L^2(\Omega)^3$$

(as distributions in Ω), because $\nabla \times B = J \in H(\mathrm{rot}, \Omega)$ and $\nabla \cdot B = 0 \in L^2(\Omega)^3$ are valid similarly in Ω. Then, $B \in H^2_{loc}(\Omega)^3$ is obtained from the elliptic regularity. Thus, Theorem 1 says, in contrast, that even if $n \times J$ has an interface on $\Gamma = \mathcal{M} \cap \Omega$, the normal component $n \cdot B$ of B gains the regularity in one rank. It is not difficult to suspect that the solenoidal condition $\nabla \cdot B = 0$ in Ω plays an essential role in such a regularity.

In this connection, it may be worth noting that the assumption of Theorem 1 does not permit the interface to $n \cdot J$. In fact, equation (2) holds in Ω as we have seen, and therefore,

$$\nabla \cdot J = \nabla \cdot (\nabla \times B) = 0$$

follows there. This implies $J \in H(\mathrm{div}, \Omega)$, and hence we have $[n \cdot J]_-^+ = 0$ on Γ in particular.

Theorem 1 can be applicable to the stationary Stokes system;

$$\left.\begin{aligned} -\Delta v + \nabla p &= f \\ \nabla \cdot v &= 0 \end{aligned}\right\} \qquad \text{in} \qquad \Omega_\pm \tag{6}$$

and the stationary Navier-Stokes system;

$$\left.\begin{aligned} -\Delta v + (v \cdot \nabla)\, v + \nabla p &= f \\ \nabla \cdot v &= 0 \end{aligned}\right\} \qquad \text{in} \qquad \Omega_\pm, \tag{7}$$

where $v = (v^1(x), v^2(x), v^3(x))$ denotes the velocity of fluid, $p = p(x)$ the pressure, and

$f(x) = (f^1(x), f^2(x), f^3(x))$ the external force. We have the following theorem, where $\omega = \nabla \times v$ indicates the vorticity of fluid.

Theorem 2. *If* $v \in H^2(\Omega_\pm)^3$, $p \in H^1(\Omega_\pm)$, *and* $f \in H(\mathrm{rot}, \Omega_\pm)$ *satisfy* (6) *or* (7) *and if* $\omega = \nabla \times v$ *is in* $H^1(\Omega)^3$, *then it holds that* $n \cdot \omega \in H^2_{loc}(\Omega)$.

We note that $v \in H^2(\Omega_\pm)^3$ implies $\omega = \nabla \times v \in H^1(\Omega_\pm)^3$, and hence the assumption $\omega \in H^1(\Omega)^3$ means $[\omega]^+_- = 0$ on Γ. It is equivalent to saying that $\omega = \nabla \times v \in H^1(\Omega)^3$ as a distribution in Ω, with v regarded as an element in $L^2(\Omega)^3$.

In the above theorem, system of equations is supposed to hold piecewisely in $\Omega_\pm$, and v, ∇v, p, and f may have interfaces on $\Gamma = \mathcal{M} \cap \Omega$. Neverthless, it says that the normal component $n \cdot \omega$ of vorticity ω gains the regularity in one rank if $[\omega]^+_- = 0$ holds on $\Gamma = \Omega \cap \mathcal{M}$ for $\omega = \nabla \times v \in H^1(\Omega_\pm)^3$.

On the other hand, all components of ω gain the interface regularity, if v, p, f are free from the interface, so that if $v \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, and $f \in H(\mathrm{rot}, \Omega)$ hold in (6), then $\omega \in H^2_{loc}(\Omega)^3$ follows. In fact, in this case system (6) holds in Ω, and hence

$$\nabla \times \nabla p = 0 \in L^2(\Omega), \qquad \text{and} \qquad -\Delta\omega = \nabla \times f \in L^2(\Omega)$$

follow in turn as distributions in Ω. Then, the elliptic regularity guarantees for $\omega \in H^1(\Omega)^3$ to be in $\omega \in H^2_{loc}(\Omega)$ from the last inclusion.

The interface regularity of p, the pressure of fluid, follows similarly from the standard regularity. Namely, if $v \in H^2(\Omega)^3$, $p \in H^1(\Omega)$, and $f \in H(\mathrm{div}, \Omega)$ satisfy (6), then it follows that $p \in H^2_{loc}(\Omega)$. In fact, then we have

$$\nabla p = \Delta v + f \qquad \text{and} \qquad \nabla \cdot v = 0$$

in Ω (as distributions again), and hence

$$\Delta p = \nabla \cdot f \in L^2(\Omega)$$

follows similarly. Thus, we obtain $p \in H^2_{loc}(\Omega)$ from the elliptic regularity.

Those standard regularities are valid even to (7), because $v \in H^2(\Omega)^3$ implies $v \in L^\infty(\Omega)^3$ and $\partial v/\partial x_j \in L^4(\Omega)^3$ for $j = 1, 2, 3$ by Sobolev's imbedding theorem, and therefore, $(v \cdot \nabla)v \in H^1(\Omega)^3$ follows from

$$\frac{\partial}{\partial x_j}(v \cdot \nabla v) = \frac{\partial v}{\partial x_j} \cdot \nabla v + v \cdot \nabla \frac{\partial v}{\partial x_j} \in L^2(\Omega).$$

In other words, even in (7), $v \in H^2(\Omega)$, $p \in H^1(\Omega)$, and $f \in H(\mathrm{rot}, \Omega)$ imply $v \in H^3_{loc}(\Omega)^3$ and $v \in H^2(\Omega)$, $p \in H^1(\Omega)$, and $f \in H(\mathrm{div}, \Omega)$ imply $p \in H^2_{loc}(\Omega)$.

We confirm again that, in contrast with those standard results, Theorem 2 assures the interface regularity gain in one rank for $n \cdot \omega$, only from the piecewise regularity of the data. This is actually the case even for the velocity itself as the following theorem shows, where C^3 extension of n is taken to Ω. See the remark after the following theorem concerning the non-standard interface regularity for p.

Theorem 3. *If $\mathcal{M} \subset \mathbf{R}^3$ is C^3 and $v \in H^2(\Omega)^3$, $p \in H^2(\Omega_\pm)$, and $f \in H^1(\Omega_\pm)^3$ satisfy* (6) *or* (7)*, then it holds that $n \cdot v \in H^3_{loc}(\Omega)$.*

The corresponding standard regularity to the above theorem is obvious, so that $v \in H^2(\Omega)^3$, $p \in H^2(\Omega)$, and $f \in H^1(\Omega)^3$ imply $v \in H^3_{loc}(\Omega)$ in (6) or (7).

In this theorem, similarly to the previous one, (6) or (7) does not hold in Ω as a system, because $p \in H^1(\Omega)$ is not required in spite of $v \in H^2(\Omega)^3$. However, if we add $f \in H(\mathrm{div}, \Omega)$ in (6) to the assumptions of Theorem 3, then

$$\left[\frac{\partial p}{\partial n}\right]_-^+ = 0 \qquad \text{on} \qquad \Gamma \tag{8}$$

follows from

$$\nabla p = f + \Delta v \qquad \text{in} \qquad \Omega_\pm,$$

becuase $v \in H^2(\Omega)^3$ and $n \cdot v \in H^3_{loc}(\Omega)$ imply $n \cdot \Delta v \in H^1_{loc}(\Omega)$. The same fact holds similarly to (7), as $(v \cdot \nabla)v \in H^1(\Omega)^3$ holds by $v \in H^2(\Omega)^3$. Later, we shall show that (8) is valid under the assumptions of Theorem 2 and $f \in H(\mathrm{div}, \Omega)$.

Relation (8) implies $\nabla\cdot(\nabla p) \in L^2(\Omega)$ if $\nabla p \in H^1(\Omega_\pm)^3$ is regarded as an element in $L^2(\Omega)^3$. However, in constrast with the standard case described before Theorem 3, this does not mean $\Delta p \in L^2(\Omega)$ because $p \in H^2(\Omega_\pm)$ itself may have the interface, and $\nabla p \in L^2(\Omega)^3$ does not hold in Ω when the distributional derivative is taken to $p \in L^2(\Omega)$ in Ω.

This paper is composed of three sections. In Section 2, a key lemma is provided. Then, Theorems 1, 2, 3 are proven in Section 3. Those theorems have component-wise versions and Sections 4 are devoted to that topic. The final section is the concluding remark.

2. Key lemma

In this section, we are concentrated on the Maxwell system (2) and show the following lemma. It is a fundamental tool for the proof of theorems.

Lemma 2.1. *If $B \in L^2(\Omega_\pm)^3$ and $J \in H(\mathrm{rot}, \Omega_\pm)$ satisfy* (2)*, then*

$$\nabla(n \cdot B)|_\pm \in H^{-1/2}(\Gamma_\pm)^3$$

is well-defined and it holds that

$$\begin{aligned} &\langle\langle \nabla(n\cdot B), C\rangle\rangle_-^+ - \langle\langle (\nabla\cdot n)B, C\rangle\rangle_-^+ \\ &= \langle\langle B, (n\cdot\nabla)C\rangle\rangle_-^+ - \langle\langle n\times B, \nabla\times C\rangle\rangle_-^+ - \langle n\cdot B, \nabla\cdot C\rangle_-^+ \end{aligned} \tag{9}$$

for any $C \in C_0^\infty(\Omega)^3$, *where* $\langle\langle\ ,\ \rangle\rangle_-^+ = \langle\langle\ ,\ \rangle\rangle_{\Gamma_+} - \langle\langle\ ,\ \rangle\rangle_{\Gamma_-}$.

Proof. As is described in introduction, it follows from (2) and B, $J \in L^2(\Omega_\pm)^3$ that $B \in H(\mathrm{rot}, \Omega_\pm)$, $n \times B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$, $B \in H(\mathrm{div}, \Omega_\pm)$, and $n\cdot B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)$. It also holds by (5) that $B \in H^1_{loc}(\Omega_\pm)^3$.

Now, in use of $J \in H(\mathrm{rot}, \Omega_\pm)$, we have

$$n \times J|_{\Gamma_\pm} = n \times (\nabla \times B)|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3.$$

Furthermore, $\nabla \cdot B = 0$ in $\Omega_\pm$ implies $-\Delta B = \nabla \times J \in L^2(\Omega_\pm)^3$, and hence

$$\frac{\partial B}{\partial n} = (n\cdot\nabla)B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$$

is well-defined for $B \in H^1_{loc}(\Omega_\pm)^3$ by Corollary I.2.6 of [1]. Thus, through the (distributional) identity

$$(n\cdot\nabla)B + n\times(\nabla\times B) = \nabla(n\cdot B) - (\nabla\cdot n)B, \tag{10}$$

valid for $n \in C^1(\Omega)$ and $B \in L^2(\Omega)^3$, it follows that

$$\nabla(n\cdot B)|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3.$$

Henceforth, we set

$$\nabla B \otimes \nabla C = \sum_{i,j=1}^{3} \frac{\partial B^i}{\partial x_j}\frac{\partial C^i}{\partial x_j}$$

for $B = (B^1, B^2, B^3)$ and $C = (C^1, C^2, C^3) \in C_0^\infty(\Omega)^3$. Then, it holds that

$$\int_\Omega \nabla B\otimes\nabla C = -\langle\langle (n\cdot\nabla)B, C\rangle\rangle_-^+ - ((\Delta B, C))_\Omega. \tag{11}$$

In fact, because $\mp n$ is the outer unit normal vector to $\Gamma_\pm$, Green's formula, described in the previous section, guarantees that

$$(\nabla B^i, \nabla C^i)_{\Omega_\pm} = -\left\langle \frac{\partial B^i}{\partial n}, C^i \right\rangle_{\Gamma_\pm} - (\Delta B^i, C^i)_{\Omega_\pm}$$

for $i = 1, 2, 3$. This implies (11).

Here, equality (10) is applied to the first term of the right-hand side of (11). We have

$$-\langle\langle (n\cdot\nabla)B, C\rangle\rangle_-^+ = \langle\langle n\times(\nabla\times B), C\rangle\rangle_-^+ - \langle\langle \nabla(n\cdot B), C\rangle\rangle_-^+ + \langle\langle (\nabla\cdot n)B, C\rangle\rangle_-^+ .$$

Since $-\nabla\times(\nabla\times B) = \Delta B$ holds in $\Omega_\pm$, the Stokes formula now gives that

$$\begin{aligned}\langle\langle n\times(\nabla\times B), C\rangle\rangle_-^+ &= -\big((\nabla\times(\nabla\times B), C)\big)_\Omega + \big((\nabla\times B, \nabla\times C)\big)_\Omega \\ &= \big((\Delta B, C)\big)_\Omega + \big((\nabla\times B, \nabla\times C)\big)_\Omega.\end{aligned}$$

Those relations are summarized as

$$\begin{aligned}&\langle\langle (\nabla\cdot n)B, C\rangle\rangle_-^+ - \langle\langle \nabla(n\cdot B), C\rangle\rangle_-^+ \\ &\quad= \int_\Omega \nabla B\otimes\nabla C - \big((\nabla\times B, \nabla\times C)\big)_\Omega.\end{aligned} \tag{12}$$

On the other hand, we have

$$\int_\Omega \nabla B\otimes\nabla C = -\langle\langle B, (n\cdot\nabla)C\rangle\rangle_-^+ - \big((B, \Delta C)\big)_\Omega$$

similarly to (11). Combining this with (12), we obtain

$$\begin{aligned}&\langle\langle \nabla(n\cdot B), C\rangle\rangle_-^+ - \langle\langle (\nabla\cdot n)B, C\rangle\rangle_-^+ \\ &\quad= \langle\langle B, (n\cdot\nabla)C\rangle\rangle_-^+ + \big((B, \Delta C)\big)_\Omega + \big((\nabla\times B, \nabla\times C)\big)_\Omega.\end{aligned} \tag{13}$$

Now, we take the Helmholtz decomposition of C. We put

$$C = C_0 + \nabla p, \tag{14}$$

where p is a scalar field defined in Ω satisfying

$$\begin{aligned}-\Delta p &= \nabla\cdot C \ \text{ in } \Omega \\ \frac{\partial p}{\partial n} &= n\cdot C (= 0) \ \text{ on } \partial\Omega.\end{aligned}$$

First, we have $p\in C^\infty(\Omega)$ and $\nabla\times C_0 = \nabla\times C \in C_0^\infty(\Omega)^3$. This implies that

$$\big((\nabla\times B, \nabla\times C)\big)_\Omega = \big((\nabla\times B, \nabla\times C_0)\big)_\Omega.$$

On the other hand, we have $\Delta p = -\nabla\cdot C \in C_0^\infty(\Omega)$ and hence

$$\big((B, \Delta C)\big)_\Omega = \big((B, \Delta C_0 + \nabla(\Delta p))\big)_\Omega$$

$$= \big((B, \Delta C_0)\big)_\Omega - (\nabla \cdot B, \Delta p)_\Omega + \langle n \cdot B, \Delta p\rangle_-^+$$
$$= \big((B, \Delta C_0)\big)_\Omega - \langle n \cdot B, \nabla \cdot C\rangle_-^+.$$

Finally, we have for $L = \nabla \times C_0 \in C_0^\infty(\Omega)^3$ that $\Delta C_0 = -\nabla \times L$ by $\nabla \cdot C_0 = 0$ in Ω and hence

$$\begin{aligned}\big((B, \Delta C_0)\big)_\Omega &= -\big((B, \nabla \times L)\big)_\Omega \\ &= -\big((\nabla \times B, L)\big)_\Omega - \langle\langle n \times B, L\rangle\rangle_-^+ \\ &= -\big((\nabla \times B, \nabla \times C_0)\big)_\Omega - \langle\langle n \times B, \nabla \times C_0\rangle\rangle_-^+.\end{aligned}$$

Those relations are summarized as

$$\begin{aligned}&\big((B, \Delta C)\big)_\Omega + \big((\nabla \times B, \nabla \times C)\big)_\Omega \\ &\quad = \big((B, \Delta C_0)\big)_\Omega + \big((\nabla \times B, \nabla \times C_0)\big)_\Omega - \langle n \cdot B, \nabla \cdot C\rangle_-^+ \\ &\quad = -\langle\langle n \times B, \nabla \times C_0\rangle\rangle_-^+ - \langle n \cdot B, \nabla \cdot C\rangle_-^+ \\ &\quad = -\langle\langle n \times B, \nabla \times C\rangle\rangle_-^+ - \langle n \cdot B, \nabla \cdot C\rangle_-^+.\end{aligned}$$

Therefore, (9) follows from (13). The proof is complete. □

3. Proof of Theorems

First, we give the following.

Proof of Theorem 1. Since $[B]_-^+ = 0$ on Γ, we have by making use of (9) that

$$\langle\langle \nabla(n \cdot B), C\rangle\rangle_-^+ = 0$$

for any $C \in \{C_0^\infty(\Omega)\}^3$ by (9). This implies that

$$[\nabla(n \cdot B)]_-^+ = 0 \quad \text{on} \quad \Gamma. \tag{15}$$

On the other hand, we have $B \in H^1_{loc}(\Omega)^3$ and $-\Delta B = \nabla \times J \in L^2(\Omega_\pm)$. Hence $\Delta(n \cdot B) \in L^2(\Omega_\pm)$ follows in $\Omega_\pm$ as distributions. Combining this with (15), we get $\Delta(n \cdot B) \in L^2(\Omega)$ with $n \cdot B \in H^1(\Omega)$, and the elliptic regularity guarantees that $n \cdot B \in H^2_{loc}(\Omega)$.

More precisely, because $\nabla(n \cdot B)|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ satisfies (15), Green's formula now gives that

$$\int_\Omega \Delta(n \cdot B)\psi\, dx = \int_\Omega (n \cdot B)\Delta\psi\, dx$$

for any $\psi \in C_0^\infty(\Omega)$. This means that for $f \in L^2(\Omega)$ defined by

$$f = \begin{cases} \Delta\,(n \cdot B)|_{\Omega_+} & \text{in } \Omega_+ \\ \Delta\,(n \cdot B)|_{\Omega_-} & \text{in } \Omega_-, \end{cases}$$

it follows that $\Delta(n \cdot B) = f \in L^2(\Omega)$ in Ω (as distributions). The proof is complete. $\square$

Now, we study the Stokes system (6);

$$\left.\begin{aligned} -\Delta v + \nabla p = f \\ \nabla \cdot v = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_{\pm}.$$

We give the following.

Proof of Theorem 2 to (6). Recall that $v \in H^2(\Omega_{\pm})^3$, $\nabla p \in L^2(\Omega_{\pm})^3$, and $f \in H(\text{rot}, \Omega_{\pm})$ satisfy (6), and that $\omega = \nabla \times v$ is in $H^1(\Omega)^3$. Then, we have

$$\left.\begin{aligned} \nabla \times \omega = J \\ \nabla \cdot \omega = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_{\pm} \tag{16}$$

for

$$J = f - \nabla p \in L^2(\Omega_{\pm})^3.$$

Here, we have $\nabla \times J = \nabla \times f \in L^2(\Omega_{\pm})^3$, and hence $J \in H(\text{rot}, \Omega_{\pm})$ follows. Then, Theorem 2 for (6) is a direct consequence of Theorem 1. $\square$

Under the assumption of Theorem 2, relation (16) holds with $\omega \in H^1(\Omega)^3$ and $J = f - \nabla p \in H(\text{rot}, \Omega_{\pm})$. As is noticed in introduction, this implies $[n \cdot J]_-^+ = 0$ on Γ as a compatibility condition. Therefore, $\left[\partial p/\partial n\right]_-^+ = 0$ on Γ is obtained if $f \in H(\text{div}, \Omega)$ is imposed furthermore. Namely, relation (8) holds with the well-definedness of $\partial p/\partial n \in H^{-1/2}(\Gamma_{\pm})$ under the assumptions of Theorem 2 and $f \in H(\text{div}, \Omega)$. The same fact is true for (7), from the proof of this theorem to that case.

Proof of Theorem 3 to (6). Recall that $v \in H^2(\Omega)^3$, $\nabla p \in H^1(\Omega_{\pm})^3$, and $f \in H^1(\Omega_{\pm})^3$ satisfy (6). Then, we have

$$\left.\begin{aligned} \nabla \times \left(\frac{\partial v}{\partial x_j}\right) = \frac{\partial \omega}{\partial x_j} \\ \nabla \cdot \left(\frac{\partial v}{\partial x_j}\right) = 0 \end{aligned}\right\} \quad \text{in} \quad \Omega_{\pm} \tag{17}$$

for $j = 1, 2, 3$, where $\omega = \nabla \times v$.

Now, we shall show that

$$\frac{\partial \omega}{\partial x_j} \in H(\text{rot}, \Omega_{\pm}). \tag{18}$$

In fact, if this is the case, then Theorem 1 applied to (17) guarantees that $n \cdot (\partial v/\partial x_j) \in H^2_{loc}(\Omega)$, and then, the desired conclusion, $n \cdot v \in H^3_{loc}(\Omega)$ follows.

For this purpose, first, we note that $\omega = \nabla \times v \in H^1(\Omega)^3$ holds by $v \in H^2(\Omega)^3$, which implies that $\partial\omega/\partial x_j \in L^2(\Omega)^3$. On the other hand, from (6) we have

$$\begin{aligned}\nabla \times \left(\frac{\partial \omega}{\partial x_j}\right) &= \frac{\partial}{\partial x_j}(\nabla \times \omega)\\ &= -\frac{\partial}{\partial x_j}\Delta v = \frac{\partial}{\partial x_j}(f - \nabla p) \qquad \text{in} \qquad \Omega_\pm\end{aligned}$$

and hence $\nabla \times \left(\partial\omega/\partial x_j\right) \in L^2(\Omega_\pm)^3$ holds by $f \in H^1(\Omega_\pm)^3$ and $p \in H^2(\Omega_\pm)$. This means (18), and thus the proof is complete. □

As is noticed in the above proof, relation (17) holds for $v \in H^2(\Omega)$ and $\omega = \nabla \times v \in H^1(\Omega)$. Then, we have $\partial\omega/\partial x_j \in H(\mathrm{div}, \Omega)$ similarly to $J \in H(\mathrm{rot}, \Omega)$ for (2), and $\left[n \cdot \partial\omega/\partial x_j\right]_-^+ = 0$ follows on Γ together with the well-definedness of $n \cdot \partial\omega/\partial x_j \in H^{-1/2}(\Gamma_\pm)^3$. Thus, if $\mathcal{M}$ is C^1, $v \in H^2(\Omega)^3$, and $\omega = \nabla \times v$, then it holds that $[\nabla(n \cdot \omega)]_-^+ = 0$ on Γ with $\nabla(n \cdot \omega) \in H^{-1/2}(\Gamma_\pm)^3$.

This section is concluded by the study of the Navier-Stokes system (7);

$$\left.\begin{aligned} -\Delta v + (v \cdot \nabla)v + \nabla p = f\\ \nabla \cdot v = 0\end{aligned}\right\} \qquad \text{in} \qquad \Omega_\pm.$$

Proof of Theorem 2 to (7). System (7) is identified with (6) if f is replaced by $f - (v \cdot \nabla)v$. Therefore, we have only to show that the condition

$$F \equiv f - (v \cdot \nabla)v \in H(\mathrm{rot}, \Omega_\pm) \tag{19}$$

follows from the assumption for this theorem to prove.

In fact, we have $v \in H^2(\Omega_\pm)^3 \subset L^\infty(\Omega_\pm)^3$ and hence $(v \cdot \nabla)v \in L^2(\Omega_\pm)^3$ holds. Furthermore, $\partial v/\partial x_j \in H^1(\Omega_\pm)^3 \subset L^4(\Omega_\pm)^3$ implies that

$$\frac{\partial}{\partial x_j}\big((v \cdot \nabla)v\big) = \frac{\partial v}{\partial x_j} \cdot \nabla v + v \cdot \nabla \frac{\partial v}{\partial x_j} \in L^2(\Omega_\pm)^3.$$

Those relations guarantee that $(v \cdot \nabla)v \in H^1(\Omega_\pm)^3$, and (19) follows from the assumption to f. □

Proof of Theorem 3 to (7). Similarly, we have only to show that

$$\frac{\partial}{\partial x_j}\big((v \cdot \nabla)v\big) \in L^2(\Omega_\pm)^3$$

holds for $j = 1, 2, 3$. However, this follows actually from the proof of the previous theorem, and the proof is complete. □

4. Component-wise Regularity

In this section, we suppose that $\mathcal{M}$ is flat.

First, we take the Maxwell system (2). As we have seen in Lemma 2.1, in this case $B \in H^1(\Omega_\pm)^3$ and $J \in H(\mathrm{rot}, \Omega_\pm)$ imply $\nabla(n \cdot B) \in H^{-1/2}(\Gamma_\pm)^3$. Then, Theorem 1 splits into component-wise versions described in the following theorem. In this connection, we confirm that the traces to $\Gamma_\pm$ of the first derivatives of any component of B are also well-defined in this system with $B \in H^1(\Omega_\pm)^3$ and $J \in H(\mathrm{rot}, \Omega_\pm)$. In fact, $(n \cdot \nabla)B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ is well-defined by $-\Delta B = \nabla \times J \in L^2(\Omega_\pm)^3$ and $B \in H^1(\Omega_\pm)^3$ as is indicated in the proof of Lemma 2.1. Next, $B \in H^1(\Omega_\pm)^3$ implies $B|_{\Gamma_\pm} \in H^{1/2}(\Gamma_\pm)^3$ and hence $(n \times \nabla)\, B|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)^3$ is also well-defined through the local chart. Those traces are compatible to the ones taken in the proof of Lemma 2.1 and that of the next theorem.

Theorem 4. *Suppose that the interface $\mathcal{M}$ is flat, and that $B \in H^1(\Omega_\pm)^3$ and $J \in H(\mathrm{rot}, \Omega_\pm)$ satisfy* (2). *Then, if $[n \cdot B]_-^+ = 0$ on Γ it holds that $[(n \times \nabla)(n \cdot B)]_-^+ = 0$ on Γ. Similarly, if $[n \times B]_-^+ = 0$ on Γ we have $[(n \cdot \nabla)(n \cdot B)]_-^+ = 0$ on Γ.*

Proof. In this case n is a constant vector and we have

$$B \cdot (n \cdot \nabla)\, C - n \times B \cdot \nabla \times C = B \cdot \nabla\, (n \cdot C)$$

for $C \in C_0^\infty(\Omega)^3$. Therefore, equality (9) is reduced to

$$\langle\langle \nabla\, (n \cdot B)\, , C \rangle\rangle_-^+ = \langle\langle B, \nabla\, (n \cdot C) \rangle\rangle_-^+ - \langle n \cdot B, \nabla \cdot C \rangle_-^+ \tag{20}$$

Without loss of generality, we assume $\mathcal{M} = \{(x_1, x_2, x_3) \mid x_3 = 0\}$ and $n = {}^T(0, 0, 1)$. Then, if $[n \cdot B]_-^+ = 0$ on Γ we have

$$\begin{aligned}\langle\langle B, \nabla(n \cdot C) \rangle\rangle_-^+ &= \left\langle B^1, \frac{\partial C^3}{\partial x_1} \right\rangle_-^+ + \left\langle B^2, \frac{\partial C^3}{\partial x_2} \right\rangle_-^+ \\ &= -\left\langle \frac{\partial B^1}{\partial x_1} + \frac{\partial B^2}{\partial x_2}, C^3 \right\rangle_-^+ = \left\langle \frac{\partial B^3}{\partial x_3}, C^3 \right\rangle_-^+ .\end{aligned}$$

Therefore, it follows from (20) that

$$\left\langle \frac{\partial B^3}{\partial x_1}, C^1 \right\rangle_-^+ + \left\langle \frac{\partial B^3}{\partial x_2}, C^2 \right\rangle_-^+ = 0$$

for any $C^1, C^2 \in C_0^\infty(\Omega)$. This implies $\left[\partial B^3/\partial x_1\right]_-^+ = \left[\partial B^3/\partial x_2\right]_-^+ = 0$, or equivalently, $[(n \times \nabla)(n \cdot B)]_-^+ = 0$ on Γ.

If $[n \times B]_{-}^{+} = 0$ on Γ, equality (20) is reduced to

$$\begin{aligned}\langle\langle \nabla(n \cdot B), C\rangle\rangle_{-}^{+} &= \langle\langle B, (n \cdot \nabla)C\rangle\rangle_{-}^{+} - \langle n \cdot B, \nabla \cdot C\rangle_{-}^{+} \\ &= \left\langle B^3, \frac{\partial C^3}{\partial x_3}\right\rangle_{-}^{+} - \left\langle B^3, \nabla \cdot C\right\rangle_{-}^{+} \\ &= -\left\langle B^3, \frac{\partial C^1}{\partial x_1} + \frac{\partial C^2}{\partial x_2}\right\rangle_{-}^{+} \\ &= \left\langle \frac{\partial B^3}{\partial x_1}, C^1\right\rangle_{-}^{+} + \left\langle \frac{\partial B^3}{\partial x_1}, C^2\right\rangle_{-}^{+}.\end{aligned}$$

This implies $\left[\partial B^3/\partial x_3\right]_{-}^{+} = 0$, or equivalently, $[(n \cdot \nabla)(n \cdot B)]_{-}^{+} = 0$ on Γ. The proof is complete. □

Now, we proceed to the Stokes system (6). We continue to suppose that $\mathcal{M}$ is flat and take $n = {}^T(0, 0, 1)$ without loss of generality. The following propositions are obtained by applying Theorem 4 to systems (17) with $j = 3$ and (16), respectively, and the traces to $\Gamma_\pm$ in their conclusions are well-defined in $H^{-1/2}(\Gamma_\pm)^3$ or $H^{-1/2}(\Gamma_\pm)$.

Proposition 4.1. *Assume that $\mathcal{M}$ is flat and system* (6) *holds with* $(n \cdot \nabla)v \in H^1(\Omega_\pm)^3$, $(n \cdot \nabla)\omega \in L^2(\Omega_\pm)^3$, $(n \cdot \nabla)p \in H^1(\Omega_\pm)$, *and* $(n \cdot \nabla)f \in L^2(\Omega_\pm)^3$ *for* $\omega = \nabla \times v$. *Then, the conditions*

$$[(n \cdot \nabla)(n \cdot v)]_{-}^{+} = 0 \qquad \text{and} \qquad [(n \cdot \nabla)(n \times v)]_{-}^{+} = 0$$

imply

$$[(n \times \nabla)(n \cdot \nabla)(n \cdot v)]_{-}^{+} = 0 \qquad \text{and} \qquad \left[(n \cdot \nabla)^2(n \cdot v)\right]_{-}^{+} = 0,$$

respectively, on Γ.

Proof. In fact, we have (6) and (17) as distributions in $\Omega_\pm$. In the latter relation with $j = 3$, we have $B = \partial v/\partial x_3 \in H^1(\Omega_\pm)^3$ and $J = \partial\omega/\partial x_3 \in H(\mathrm{rot}, \Omega_\pm)$ because

$$\nabla \times \left(\frac{\partial \omega}{\partial x_3}\right) = \frac{\partial}{\partial x_3}(-\Delta v) = \frac{\partial}{\partial x_3}(f - \nabla p)$$

holds by the former. Then the assertion is obtained from the previous theorem. □

Proposition 4.2. *Assume that $\mathcal{M}$ is flat and system* (6) *holds with* $\omega = \nabla \times v \in H^1(\Omega_\pm)^3$ *and* $f \in H(\mathrm{rot}, \Omega_\pm)$. *Then, the conditions*

$$[n \cdot (\nabla \times v)]_{-}^{+} = 0 \qquad \text{and} \qquad [n \times (\nabla \times v)]_{-}^{+} = 0$$

imply

$$[(n \times \nabla)\,(n \cdot (\nabla \times v))]_-^+ = 0 \qquad \textit{and} \qquad [(n \cdot \nabla)\,(n \cdot (\nabla \times v))]_-^+ = 0,$$

respectively, on Γ.

Proof. We have (16) with $J = f - \nabla p$, and the assertion is obtained by Theorem 4 and $\nabla \times J = \nabla \times f$ in $\Omega_\pm$. □

Those propositions assure the extra reguality on the interface to the tangential components of the solution, and in particular, the following theorems hold.

Theorem 5. *Let* $\mathcal{M}$ *be flat, and assume that* $v \in H^2(\Omega_\pm)^3$, $p \in H^1(\Omega_\pm)$, *and* $f \in L^2(\Omega_\pm)^3$ *satisfy the Stokes system* (6). *Assume, furthermore, that* $(n \cdot \nabla)p \in H^1(\Omega_\pm)$, $f \in H(\mathrm{rot}, \Omega_\pm)$, *and* $(n \cdot \nabla)f \in L^2(\Omega_\pm)$ *hold true. Then, if the conditions*

$$[n \times v]_-^+ = 0 \qquad \textit{and} \qquad [(n \cdot \nabla)(n \cdot v)]_-^+ = [n \cdot (\nabla \times v)]_-^+ = 0$$

are satisfied on Γ, *it holds that* $n \times v|_\Gamma \in H^{5/2}_{loc}(\Gamma)^3$.

Proof. In fact, all requirements of piecewise regularity in Propositions 4.1 and 4.2 are satisfied, and therefore, from the assumption across the interface regularity we have

$$[(n \times \nabla)(n \cdot \nabla)(n \cdot v)]_-^+ = [(n \times \nabla)\,(n \cdot (\nabla \times v))]_-^+ = 0$$

on Γ. Without loss of generality, we continue to take

$$\mathcal{M} = \{(x_1, x_2, x_3) \mid x_3 = 0\} \qquad \text{and} \qquad n = {}^T(0, 0, 1).$$

Let

$$g = \frac{\partial v^1}{\partial x_1} + \frac{\partial v^2}{\partial x_2} \qquad \text{and} \qquad h = \frac{\partial v^2}{\partial x_1} - \frac{\partial v^1}{\partial x_2}.$$

Then, we have $g = -\partial v_3/\partial x_3 = -(n \cdot \nabla)(n \cdot v)$ and hence

$$\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \in H^1_{loc}(\Omega)$$

follows from $[(n \times \nabla)(n \cdot \nabla)(n \cdot v)]_-^+ = 0$. On the other hand, we have $h = n \cdot (\nabla \times v)$ and hence

$$\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2} \in H^1_{loc}(\Omega)$$

follows from $[(n \times \nabla)(n \cdot (\nabla \times v))]_-^+ = 0$. Those relations imply

$$\frac{\partial g}{\partial x_1} - \frac{\partial h}{\partial x_2} = \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) v^1 \in H^1_{loc}(\Omega)$$

and

$$\frac{\partial g}{\partial x_2} + \frac{\partial h}{\partial x_1} = \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) v^2 \in H^1_{loc}(\Omega).$$

Therefore,

$$\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) (n \times v)\Big|_\Gamma \in H^{1/2}_{loc}(\Gamma)^3$$

holds.

On the other hand, we have $n \times v|_\Gamma \in H^{1/2}_{loc}(\Gamma)^3$ from the assumption, and hence $n \times v|_\Gamma \in H^{5/2}_{loc}(\Gamma)^3$ is obtained by the elliptic regularity. The proof is complete. □

Theorem 6. *Suppose, similarly, that $\mathcal{M}$ is flat, that $v \in H^2(\Omega_\pm)^3$, $p \in H^1(\Omega_\pm)$, and $f \in L^2(\Omega_\pm)^3$ satisfy the Stokes system* (6), *and that $(n \cdot \nabla)p \in H^1(\Omega_\pm)$, $f \in H(\mathrm{rot}, \Omega_\pm)$, and $(n \cdot \nabla) f \in L^2(\Omega_\pm)$ hold true. Then, if the conditions*

$$[(n \cdot \nabla)(n \times v)]_-^+ = [n \times (\nabla \times v)]_-^+ = 0$$

are satisfied on Γ, it holds that $(n \cdot \nabla)(n \times v)|_\Gamma \in H^{3/2}_{loc}(\Gamma)^3$.

Proof. Under the same notations as in the proof of the previous theorem, we have $\partial g/\partial x_3$, $\partial h/\partial x_3 \in H^1_{loc}(\Omega)$ in this case by Proposition 4.2 and Theorem 5. This implies

$$\frac{\partial g}{\partial x_3}\Big|_\Gamma, \frac{\partial h}{\partial x_3}\Big|_\Gamma \in H^{1/2}_{loc}(\Gamma),$$

and hence

$$\left(\frac{\partial^2 g}{\partial x_1 \partial x_3} - \frac{\partial^2 h}{\partial x_2 \partial x_3}\right)\Big|_\Gamma = \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) \frac{\partial v^1}{\partial x_3}\Big|_\Gamma \in H^{-1/2}_{loc}(\Gamma)$$

and

$$\left(\frac{\partial^2 g}{\partial x_2 \partial x_3} - \frac{\partial^2 h}{\partial x_1 \partial x_3}\right)\Big|_\Gamma = \left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right) \frac{\partial v^2}{\partial x_3}\Big|_\Gamma \in H^{-1/2}_{loc}(\Gamma)$$

follows. On the other hand, we have

$$\frac{\partial v^1}{\partial x_3}\Big|_\Gamma, \frac{\partial v^2}{\partial x_3}\Big|_\Gamma \in H^{1/2}_{loc}(\Gamma)$$

by $[(n\cdot\nabla)(n\times v)]_-^+ = 0$ on Γ, and hence $(n\cdot\nabla)(n\times v)|_\Gamma \in H^{3/2}_{loc}(\Gamma)^3$ follows from the elliptic regularity. The proof is complete. □

The Navier-Stokes system (7) is treated similarly. Actually, this system is reduced to (6) with f replaced by $f-(v\cdot\nabla)v$. Then, the nonlinear term $(v\cdot\nabla)v$ is in $H^1(\Omega_\pm)$ in the case of $v\in H^2(\Omega_\pm)^3$. Thus, we get the following theorem.

Theorem 7. Theorems 5 *and* 6 *hold similarly even to system* (7).

Now, we shall examine the assumptions and conclusions of Theorems 5 and 6. First, assumptions on the piecewise regularity of those theorems are summarized as

$$p\in H^1(\Omega_\pm),\ (n\cdot\nabla)p\in H^1(\Omega_\pm),\\ f\in H(\mathrm{rot},\Omega_\pm),\ (n\cdot\nabla)f\in L^2(\Omega_\pm)^3, \tag{21}$$

and

$$v\in H^2(\Omega_\pm)^3. \tag{22}$$

On the other hand, the assumptions across interface of Theorems 5 and 6 are

$$\left[v^1\right]_-^+ = \left[v^2\right]_-^+ = 0,\quad \left[\frac{\partial v^3}{\partial x_3}\right]_-^+ = 0,\quad \left[\frac{\partial v^2}{\partial x_1}-\frac{\partial v^1}{\partial x_2}\right]_-^+ = 0 \tag{23}$$

and

$$\left[\frac{\partial v^1}{\partial x_3}\right]_-^+ = \left[\frac{\partial v^2}{\partial x_3}\right]_-^+ = 0,\qquad \left[\frac{\partial v^3}{\partial x_2}-\frac{\partial v^2}{\partial x_3}\right]_-^+ = \left[\frac{\partial v^1}{\partial x_3}-\frac{\partial v^3}{\partial x_1}\right]_-^+ = 0,$$

respectively. The latter means that

$$\left[\frac{\partial v^2}{\partial x_3}\right]_-^+ = \left[\frac{\partial v^1}{\partial x_3}\right]_-^+ = 0,\qquad \left[\frac{\partial v^3}{\partial x_2}\right]_-^+ = \left[\frac{\partial v^3}{\partial x_1}\right]_-^+ = 0. \tag{24}$$

The second relations of (23) and (24) are summarized as $n\cdot v\in H^2(\Omega)$. Then, the first relation of (23) means $v\in H^1(\Omega)^3$, and the rest are equivalent to $\omega=\nabla\times v\in H^1(\Omega)^3$ and $(n\cdot\nabla)v\in H^1(\Omega)^3$. Namely, we have

$$v\in H^1(\Omega)^3,\qquad \omega=\nabla\times v\in H^1(\Omega)^3 \tag{25}$$

and

$$n\cdot v\in H^2(\Omega),\qquad (n\cdot\nabla)v\in H^1(\Omega)^3 \tag{26}$$

as the regularity assumption across the interface. However, (25) implies $v \in H^2_{loc}(\Omega)$, because $\nabla \cdot v = 0$ and $-\Delta v = \nabla \times \omega \in L^2(\Omega)^3$ holds in Ω. Thus, we replace those assumptions on the interface regularity, (25) and (26), simply by $v \in H^2(\Omega)^3$.

On the other hand, the conclusions of Theorems 5, 6 assure for $\psi = v^1$, v^2 that

$$\frac{\partial^2\psi}{\partial x_1^2},\ \frac{\partial^2\psi}{\partial x_1\partial x_2},\ \frac{\partial^2\psi}{\partial x_2^2},\ \frac{\partial^2\psi}{\partial x_3\partial x_1},\ \frac{\partial^2\psi}{\partial x_3\partial x_2} \in H^1_{loc}(\Omega).$$

Thus, we obtain the following.

Theorem 8. *Let the interface $\mathcal{M}$ be flat and $v \in H^2(\Omega)^3$, $p \in H^1(\Omega_\pm)$, $(n\cdot\nabla)p \in H^1(\Omega_\pm)$, $f \in H(\mathrm{rot}, \Omega_\pm)$, and $(n\cdot\nabla)f \in L^2(\Omega_\pm)^3$ hold in the Stokes or the Navier-Stokes system* (6), (7), *and let ψ be any tangential component of v. Then, $\left(\partial\psi/\partial n\right)^2\Big|_{\Gamma_\pm} \in H^{-1/2}(\Gamma_\pm)$ is well-defined, and ψ belongs to $H^3_{loc}(\Omega)$ if and only if*

$$\left[\left(\frac{\partial}{\partial n}\right)^2 \psi\right]_-^+ = 0 \tag{27}$$

holds on Γ.

Proof. We shall describe only on the Stokes system (6), because the Navier-Stokes system (7) is treated similarly. In fact, we have (17) with $j = 3$,

$$\nabla \times \left(\frac{\partial v}{\partial x_3}\right) = \frac{\partial \omega}{\partial x_3} \quad \text{and} \quad \nabla \cdot \left(\frac{\partial v}{\partial x_3}\right) = 0 \qquad \text{in} \qquad \Omega,$$

with $\partial v/\partial x_3 \in H^1(\Omega)^3$, $\partial\omega/\partial x_3 \in L^2(\Omega)^3$, and

$$\nabla \times \left(\frac{\partial \omega}{\partial x_3}\right) = -\Delta\left(\frac{\partial v}{\partial x_3}\right) = \frac{\partial}{\partial x_3}(f - \nabla p) \in L^2(\Omega_\pm)^3 \tag{28}$$

(as distributions) in $\Omega_\pm$. This implies $\partial\omega/\partial x_3 \in L^2(\mathrm{rot}, \Omega_\pm)$ and therefore, $\nabla(\partial v_j/\partial x_3) \in H^{-1/2}(\Gamma_\pm)^3$ is well-defined for $j = 1, 2, 3$, as is noticed at the begining of §4.

Then, the assumption (27) implies $-\Delta(\partial\psi/\partial x_3) \in L^2(\Omega)$ as distributions in Ω with $\partial\psi/\partial x_3 \in H^1(\Omega)^3$, because $-\Delta(\partial\psi/\partial x_3) \in L^2(\Omega_\pm)$ holds in $\Omega_\pm$ by (28). Therefore, $\psi \in H^3_{loc}(\Omega)$ follows from the elliptic regularity. The only if part is obvious, and the proof is complete. □

Theorem 3 guarantees the interface regularity of the normal component of v. Because all assumptions of Theorems 3 and 8 are satisfied if $v \in H^2(\Omega)^3$, $p \in H^2(\Omega_\pm)$, and $f \in H^1(\Omega_\pm)^3$, we get the following.

Theorem 9. *If the interface $\mathcal{M}$ is flat and $v \in H^2(\Omega)^3$, $p \in H^2(\Omega_\pm)$, $f \in H^1(\Omega_\pm)^3$ satisfy the Stokes or the Navier-Stokes system* (6), (7), *then $H^3(\Omega)$ interface of v can occur only in the normal direction of tangential components. Namely, any second derivative of any component of v is well-defined as an element in $H^{-1/2}(\Gamma_\pm)$, and*

$$\left[\left(\frac{\partial}{\partial n} \right)^2 (n \times v) \right]_-^+ = 0 \qquad \text{on} \qquad \Gamma \tag{29}$$

implies $v \in H^3_{loc}(\Omega)^3$.

The assumptions of Theorem 9 holds if $v \in H^2(\Omega)^3 \cap H^3(\Omega_\pm)^3$ and $p \in H^2(\Omega_\pm)$. Actually, if $v \in H^2(\Omega)^3 \cap H^3(\Omega_\pm)^3$ satisfies $\nabla \cdot v = 0$ in Ω, then the Stokes system (6) arises for $p = 0$ and $f = -\Delta v$. Then, we can apply Theorem 9 and obtain the following.

Theorem 10. *If $\mathcal{M}$ is flat and $v \in H^2(\Omega)^3 \cap H^3(\Omega_\pm)^3$ satisfies $\nabla \cdot v = 0$ in Ω, then the condition* (29) *implies $v \in H^3(\Omega)$.*

In this connection, it should be noted that the assumptions of Theorems 8 or 9 without (27) or (29) does not induce even the piecewise regularity indicated as $v \in H^3(\Omega_\pm)^3$. In fact, they assure only $v \in H^2(\Omega)^3$ and $-\Delta v \in H^1(\Omega_\pm)^3$, which guarantees only $v \in H^3_{loc}(\Omega_\pm)^3$. Thus, the H^3 regularity up to $\Gamma_\pm$ of v is missed without (29).

5. Concluding Remarks

Generizations of Theorems 8, 9, and 10 to the case of the non-flat interface are quite inetersting. Actually, they will be regarded as a counter part or the natural extension of Theorem 3 and will be studied in the forthcoming paper.

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