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THE FUNDAMENTAL GROUP OF THE COMPLEMENT OF THE BRANCH CURVE OF $T \times T$ IN $\mathbb{C}^2$

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1. Background

The concentration on the fundamental group of a complement of a branch curve of an algebraic surface X with respect to a generic projection onto $\mathbb{CP}^2$, leads us to the computation of $\pi_1(X_{Gal})$ the fundamental group of the Galois cover of X with respect to this generic projection. Galois covers are surfaces of a general type.

Bogomolov conjectured that the Galois covers corresponding to generic projections of algebraic surfaces to $\mathbb{CP}^2$ have infinite fundamental groups.

In [7] we justify Bogomolov's conjecture by proving that $\pi_1(T \times T)_{Gal}$ is an infinite group.

In order to compute $\pi_1(T \times T)_{Gal}$, we have to enclode the braid monodromy factorization of the branch curve S of $T \times T$. Then we have to apply the van Kampen Theorem on the factors in the factorization in order to get relations for $\pi_1(\mathbb{C}^2 - S, *)$ the fundamental group of the complement of S in $\mathbb{C}^2$.

The fundamental group of the Galois cover X_{Gal} is known to be a quotient of a certain subgroup of the fundamental group of the complement of S.

We recall shortly the computations from [6].

Let $X = T \times T$ be an algebraic surface (where T is a complex torus) embedded in $\mathbb{CP}^5$, and $f\colon X \to \mathbb{CP}^2$ be a generic projection. We degenerate X to a union of 18 planes X_0([6, Section 3]). We numerate the lines and vertices as shown in Fig. 1.

We have a generic projection $f_0\colon X_0 \to \mathbb{CP}^2$. We get a degenerated branch curve S_0 which is a line arrangement and compounds nine 6-points. We regenerate each 6-point separately.

We concentrate for example in a regeneration in a neighbourhood of V_2. We consider the local numeration of lines meeting at V_2 ([6, Figure 6]). First, the diagonal lines 4 and 5 become conics which are tangent to the lines 2, 3 and 1, 6 respectively, see Fig. 2.

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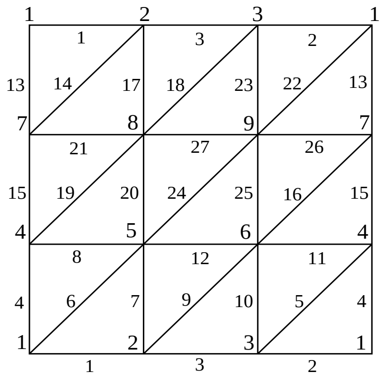


Fig. 1.

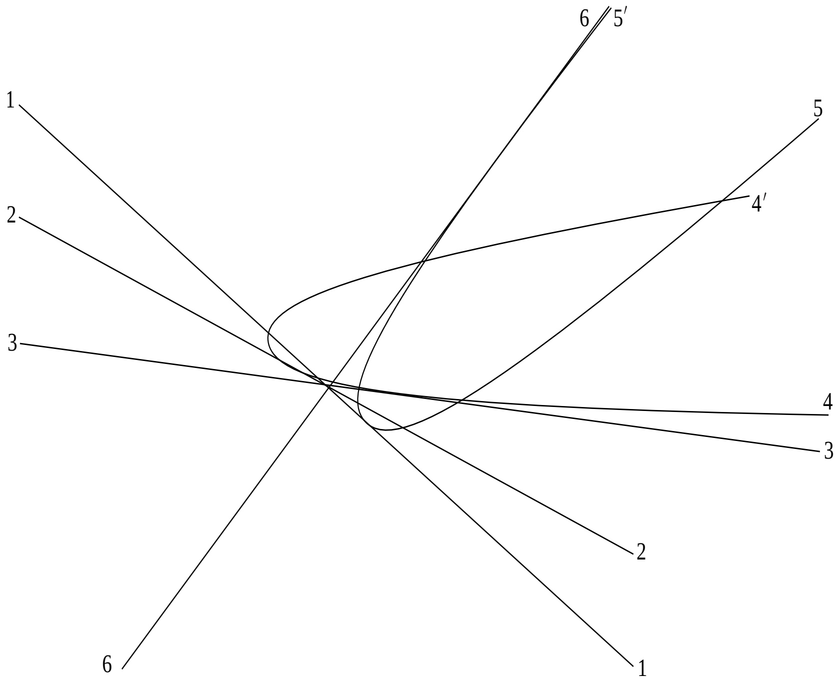


Fig. 2.

Now, concentrating in a neighbourhood of the left 4-point (the intersection point of 1, 2, 3, 6), the two horizontal lines become a hyperbola. Each one of the two vertical lines is replaced by two parallel lines, which are tangent to the hyperbola, see Fig. 3.

Finally, the hyperbola is doubled. Each one of the tangent points is replaced by

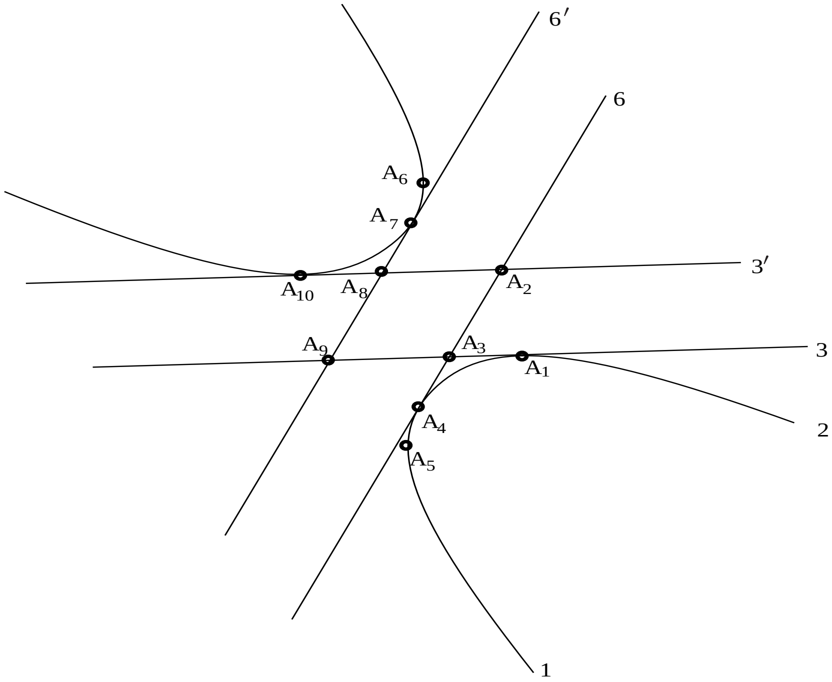


Fig. 3.

three cusps ([6, Theorem 13]). Naturally, it occurs that each node is replaced by two or four ones.

We end up with a regenerated cuspidal curve, which has a degree of 12.

We do so to each one of the 6-points and get regenerated curves S_i, $1 \leq i \leq 9$. The union of these curves is the regenerated branch curve S. Deg $S_0 = 27$, thus $\deg S = 54$. The reason is that each intersection point q_j of the curve S_0 with the typical fiber was replaced by two close points $\{q_j, q_{j'}\}$.

In order to define a g-base for the fundamental group of the complement of S in $\mathbb{C}^2$, we present the following situation (following Fig. 4 to understand the below notions).

S is an algebraic curve in $\mathbb{C}^2$, $54 = \deg S$. $\pi\colon \mathbb{C}^2 \to \mathbb{C}$ a generic projection on the first coordinate. $K(x) = \{y \mid (x, y) \in S\}$ is the projection to the y-axis of $\pi^{-1}(x) \cap S$. Let $N = \{x \mid \#K(x) < 54\}$ and $M' = \{x \in S \mid \pi_{|x}$ is not étale at $x\}$ such that $\pi(M') = N$. Assume $\#(\pi^{-1}(x) \cap M') = 1$, $\forall x \in N$. Let E (resp. D) be a closed disk on x-axis (resp. y-axis), such that $M' \subset E \times D$, $N \subset \mathrm{Int}(E)$. We choose $u \in \partial E$, $x \ll u \quad \forall x \in N$. $\mathbb{C}_u = \{q_1, q_{1'}, \ldots, q_{27}, q_{27'}\}$.

We now specify a standard set of generators for the fundamental group $\pi_1(\mathbb{C}^2 - S, M)$, where M is a point outside S.

Write $S \cap \mathbb{C}_u = \{q_1, \ldots, q_{27'}\}$. Let γ_j be paths from M to q_j $\forall j$, such that

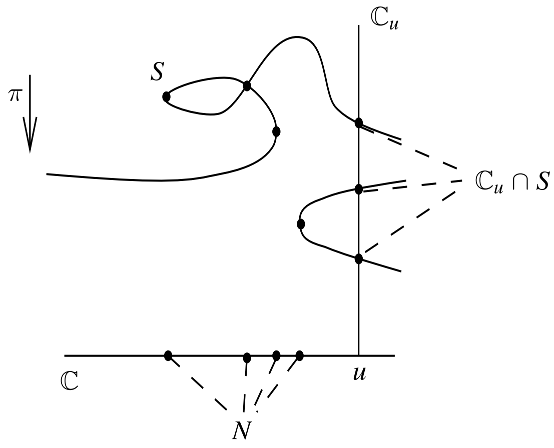


Fig. 4.

the γ_j do not meet each other in any point except M. Let η_j be a small oriented circle around q_j. Let γ'_j be the part of γ_j outside η_j, and take $\Gamma_j = \gamma'_j \eta_j (\gamma'_j)^{-1}$. In the same way we specify generators $\Gamma_{j'}$. The set $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^{27}$ freely generates $\pi_1(\mathbb{C}_u - S, M)$ [40]. Such a set is called a g-base for $\pi_1(\mathbb{C}_u - S, M)$.

By Lemma 12 we have a surjection $\pi_1(\mathbb{C}_u - S, M) \xrightarrow{\nu} \pi_1(\mathbb{C}^2 - S, M) \to 0$, so the set $\{\nu(\Gamma_j)\}$ generates $\pi_1(\mathbb{C}^2 - S, M)$. By abuse of notation, we shall denote $\nu(\Gamma_j)$ by Γ_j. A presentation for $\pi_1(\mathbb{C}^2 - S, M)$ is obtained by the van Kampen Theorem, from a list of braids in $B_{54}[\mathbb{C}_u, \mathbb{C}_u \cap S]$.

The group $\pi_1(\mathbb{C}^2 - S, M)$ acts on the points in $\mathbb{C}_u$. This leads to a permutation representaion $\psi\colon \pi_1(\mathbb{C}^2 - S, M) \to S_{18}$, 18 is the number of planes in X_0.

Let $\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle$ denote the normal subgroup generated by Γ_j^2, $\Gamma_{j'}^2$.

DEFINITION 1. Define

$$\tilde{\pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, M)}{\langle \Gamma_j^2, \Gamma_{j'} \rangle}.$$

Since f is stable, Γ_j and $\Gamma_{j'}^2$ induce a transposition in S_{18} so that $\langle \Gamma_j^2 \rangle \subseteq \ker \psi$. The map $\tilde{\pi}_1 \to S_{18}$ is also denoted ψ.

By the isomorphism theorems, we have an exact sequence

$$1 \to \ker \psi \to \tilde{\pi}_1 \xrightarrow{\psi} S_{18} \to 1. \tag{1}$$

DEFINITION 2. Consider the fibered product

$$\underbrace{X \times_f \cdots \times_f X}_{n} = \{(x_1, \ldots, x_n) \in X^n : f(x_1) = \cdots = f(x_n)\},$$

and the diagonal

$$\Delta = \{(x_1, \ldots, x_n) \in X \times_f \cdots \times_f X \mid x_i = x_j \text{ for some } i \neq j\}.$$

The surface X_{Gal} is the Galois cover of X with respect to the generic projection $f : X \to \mathbb{CP}^2$. That is the Zariski closure of the complement of Δ:

$$X_{Gal} = \overline{X \times_f \cdots \times_f X - \Delta}.$$

Let X_{Gal}^{Aff} be the part of X_{Gal} lying over $\mathbb{C}^2$ ($\subseteq \mathbb{CP}^2$). There is a surjective map $X_{Gal}^{Aff} \to X_{Gal}$.

Theorem 3 ([16, Secion 0.3]). *$\pi_1(X_{Gal}^{Aff})$ is isomorphic to the kernel of $\psi : \tilde{\pi}_1 \to S_{18}$.*

We denote $\mathcal{A} = \pi_1(X_{Gal}^{Aff})$.

The exact sequence (1) gets the form

$$1 \to \mathcal{A} = \ker \psi \to \tilde{\pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, M)}{\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle} \xrightarrow{\psi} S_{18} \to 1. \tag{2}$$

The plan is to use the van Kampen Theorem and the braid monodromy technique to obtain a presentation of $\pi_1(\mathbb{C}^2 - S, M)$ and by adding the relations $\Gamma_j^2 = \Gamma_{j'}^2 = 1$ to get a presentation of $\tilde{\pi}_1$. Then we use the Reidemeister Schreier method to obtain a presentation of $\mathcal{A}$ (see [7]).

1.1. Braid monodromy. Recall that S is the branch curve of X, $\deg S = 54$. Recall the above N, M', $K(x)$, u, $\mathbb{C}_u$, E, D. Let $B_{54}[D, \mathbb{C}_u]$ be the braid group, and $H_1, \ldots, H_{53}$ be its frame. Let $M \in \partial D$ and $\pi_1(\mathbb{C}^2 - S, M)$ is the fundamental group of the complement of S, with a g-base $\Gamma_1, \ldots, \Gamma_{27'}$.

The braid monodromy of S is a map $\varphi : \pi_1(E - N, u) \to B_{54}[D, \mathbb{C}_u]$ defined as follows: every loop in $E - N$ starting at u has liftings to a system of 54 paths in $(E - N) \times D$ starting at $q_1, \ldots, q_{27'}$. Projecting them to D we get 54 paths in D defining a motion $\{q_1(t), \ldots, q_{27'}(t)\}$ of 54 points in D starting and ending at $\mathbb{C}_u$, $0 \leq t \leq 1$. This motion defines a braid in $B_{54}[D, \mathbb{C}_u]$.

Theorem 4 (The Artin Theorem). *Let S be a curve and let $\delta_1, \ldots, \delta_q$ be a g-base of $\pi_1(E - N, u)$. Assume that the singularities of S are cusps, nodes and branch* *points. Let* $\varphi\colon \pi_1(E - N, u) \to B_{54}[D, \mathbb{C}_u]$ *be the braid monodromy. Then for all* i*, there exist a halftwist* $V_i \in B_{54}[D, \mathbb{C}_u]$ *and* $r_i \in \mathbb{Z}$*, such that* $\varphi(\delta_i) = V_i^{r_i}$ *and* r_i *depends on the type of the singularity*: $r_i = 1, 2, 3$*, for branch point, node, cusp respectively.*

Proposition 5 ([17, Proposition VI. 2.1]). *Let* S *be the regenerated branch curve of degree* 54 *in* $\mathbb{CP}^2$*. Let* π*,* u*,* D*,* E*,* $\mathbb{C}_u$ *be as above. Let* φ *be the braid monodromy of* S *with respect to* π*,* u*. Let* $\delta_1, \ldots, \delta_q$ *be a g-base of* $\pi_1(E - N, u)$*. Then* $\prod_{i=1}^{q} \varphi(\delta_i) = \Delta^2_{54}$.

2. The braid monodromy factorization Δ^2_{54}

2.1. $\boldsymbol{C_i}$**.** There are some lines in X_0 which do not meet, but when projecting them to $\mathbb{CP}^2$, they may intersect. These intersections $\tilde{C}_i$, $i = 1, \ldots, 9$ are called parasitic intersections.

Recall from [6, Section 5]: $\tilde{C}_1 = \prod_{t=1,2,4,6,13,22} D_t$, $\tilde{C}_2 = \prod_{t=3,7,9,14,17} D_t$, $\tilde{C}_3 = \prod_{t=5,10,18,23} D_t$, $\tilde{C}_4 = \prod_{t=8,11,15,19} D_t$, $\tilde{C}_5 = \prod_{t=12,20,24} D_t$, $\tilde{C}_6 = \prod_{t=16,25} D_t$, $\tilde{C}_7 = \prod_{t=21,26} D_t$, $\tilde{C}_8 = D_{27}$, $\tilde{C}_9 = Id$.

D_t, $1 \leq t \leq 27$, are regenerated. We use the Complex Conjugation [6, Subsection 7.2] and obtain the following results (denoted as above):

$$D_1 = D_2 = D_3 = Id,\ D_4 = Z^2_{33',44'},\ D_5 = \overset{(4)(4')}{\underline{Z}^2_{11',55'}},\ D_6 = \underline{Z}^2_{33',66'} \cdot Z^2_{55',66'},$$

$$D_7 = \prod_{i=2,4} \overset{(6)(6')}{\underline{Z}^2_{ii',77'}} \cdot \underline{Z}^2_{55',77'},\ D_8 = \prod_{i=1}^{3} \overset{(6)-(7')}{\underline{Z}^2_{ii',88'}},\ D_9 = \prod_{i=2,4-6,8} \underline{Z}^2_{ii',99'},$$

$$D_{10} = \prod_{i=1,4,6,7} \overset{(9)(9')}{\underline{Z}^2_{ii',10\,10'}} \cdot \underline{Z}^2_{88',10\,10'},\ D_{11} = \prod_{i=1-3,6,7} \overset{(9)-(10')}{\underline{Z}^2_{ii',11\,11'}},$$

$$D_{12} = \prod_{i=1}^{5} \overset{(9)-(11')}{\underline{Z}^2_{ii',12\,12'}},\ D_{13} = \prod_{\substack{i=3\\ i\neq 4,6}}^{12} \underline{Z}^2_{ii',13\,13'},\ D_{14} = \prod_{\substack{i=2\\ i\neq 3,7,9}}^{11} \overset{(13)(13')}{\underline{Z}^2_{ii',14\,14'}} \cdot \underline{Z}^2_{12\,12',14\,14'},$$

$$D_{15} = \prod_{\substack{i=1\\ i\neq 4,5,8}}^{10} \overset{(13)-(14')}{\underline{Z}^2_{ii',15\,15'}} \cdot \underline{Z}^2_{12\,12',\,15\,15'},\ D_{16} = \prod_{i=1}^{8} \overset{(13)-(15')}{\underline{Z}^2_{ii',16\,16'}},\ D_{17} = \prod_{\substack{i=2\\ i\neq 3,7,9,14}}^{16} \underline{Z}^2_{ii',17\,17'},$$

$$D_{18} = \prod_{\substack{i=1\\ i\neq 2,3,5,10}}^{15} \overset{(17)(17')}{\underline{Z}^2_{ii',18\,18'}} \cdot \underline{Z}^2_{16\,16',\,18\,18'},\ D_{19} = \prod_{\substack{i=1\\ i\neq 4,5,8,11}}^{14} \overset{(17)-(18')}{\underline{Z}^2_{ii',19\,19'}} \cdot \underline{Z}^2_{16\,16',\,19\,19'},$$

$$D_{20} = \prod_{\substack{i=1\\ i\neq 6-8,12}}^{15} \overset{(17)-(19')}{\underline{Z}^2_{ii',20\,20'}} \cdot \underline{Z}^2_{16\,16',\,20\,20'},\ D_{21} = \prod_{i=1}^{12} \overset{(17)-(20')}{\underline{Z}^2_{ii',21\,21'}},$$

$$D_{22} = \prod_{\substack{i=3 \\ i\neq 4,6,13}}^{21} \underline{Z}^2_{ii',22\,22'},\ D_{23} = \prod_{\substack{i=1 \\ i\neq 2,3,5,10,18}}^{20} \overset{(22)(22')}{\underline{Z}^2_{ii',23\,23'}} \cdot \underline{Z}^2_{21\,21',23\,23'},$$

$$D_{24} = \prod_{\substack{i=1 \\ i\neq 6-8,12}}^{19} \overset{(22)-(23')}{\underline{Z}^2_{ii',24\,24'}} \cdot \underline{Z}^2_{21\,21',24\,24'},\ D_{25} = \prod_{\substack{i=1 \\ i\neq 9-12,16}}^{20} \overset{(22)-(24')}{\underline{Z}^2_{ii',25\,25'}} \cdot \underline{Z}^2_{21\,21',25\,25'},$$

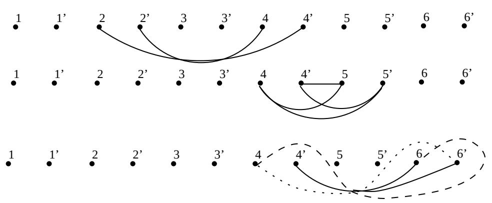
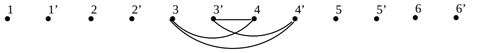
$$D_{26} = \prod_{\substack{i=1 \\ i\neq 13-16}}^{19} \overset{(22)-(25')}{\underline{Z}^2_{ii',26\,26'}} \cdot \overset{(21)(21')}{\underline{Z}^2_{20\,20',26\,26'}},\ D_{27} = \prod_{i=1}^{16} \overset{(22)-(26')}{\underline{Z}^2_{ii',27\,27'}}.$$

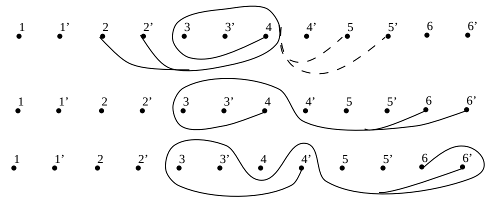
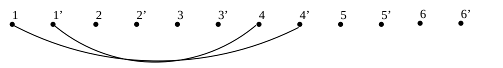
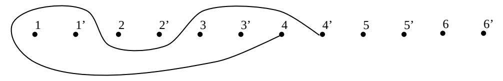
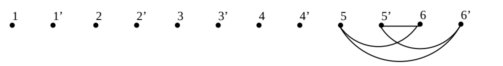
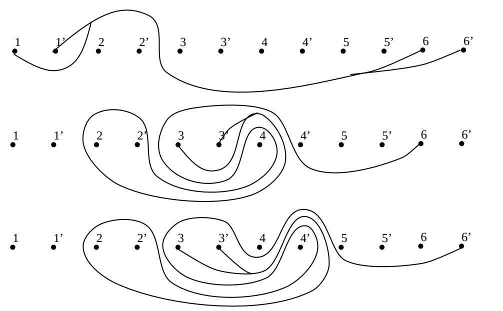
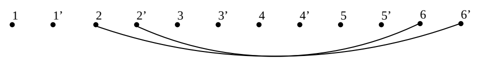
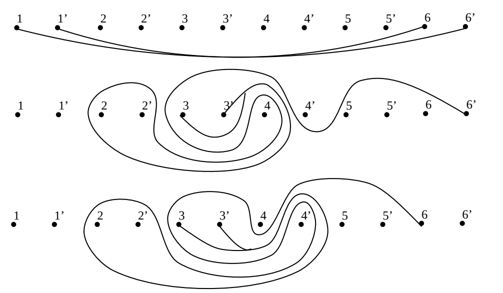
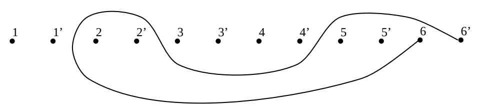
During the regeneration, each $\tilde{C}_i$ is regenerated to C_i, $1 \leq i \leq 9$.
Each C_i is now a product of the certain regenerated D_t (as shown for $\tilde{C}_i$).

2.2. H_{V_i}. Recall that the regenerated branch curve compounds nine 6-points V_i, $i = 1, \ldots, 9$. We regenerate in their neighbourhood. H_{V_i} are the resulting regenerated braid monodromies. We computed in detail H_{V_i} for $i = 1, 4, 7$ in [6, Sections 8, 9, 10]. In this section we present the resulting H_{V_i} and the paths which correspond to their different factors. We have proved in [4] an Invariance Property to each H_{V_i}, therefore in the following tables, one can see expressions such as $\rho_4^j \rho_3^i z_{34} \rho_3^{-i} \rho_4^{-j}$ where $\rho_4^j = Z^j_{44'}$ and $\rho_4^i = Z^i_{33'}$, $i, j \in \mathbb{Z}$. The meaning is that we can consider any relation we need for calculations, but setting $i = j = \pm 1$ is enough in order to derive all other possibilities.

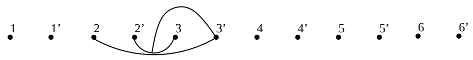
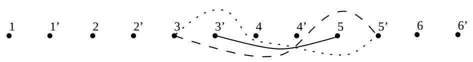
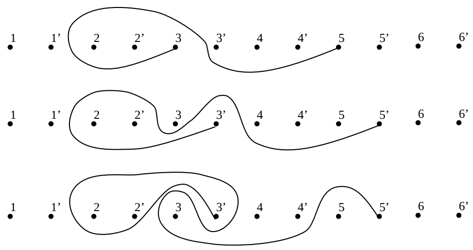
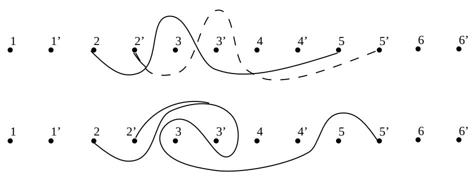
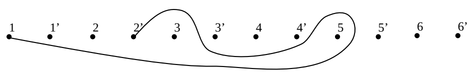
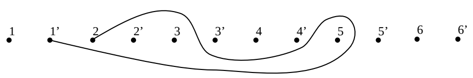
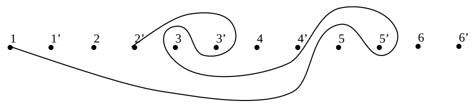
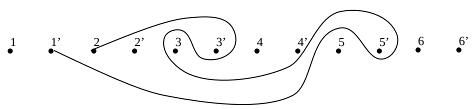
There are two tables which correspond to each H_{V_i}. The first table presents the paths which correspond to the factors in H_{V_i} and some complex conjugates, the braids themselves and the exponent of the braids according to the Emil Artin Theorem. The second table presents the same ones but for the factors of the form $\hat{F}_1 \cdot (\hat{F}_1)^{\rho^{-1}}$.

$$H_{V_1} = \left(\underline{Z}^2_{4'i}\right)_{i=22',55',6,6'} \cdot Z^3_{33',4} \cdot \left(\underline{Z}^2_{4i}\right)^{Z^2_{33',4}}_{i=22',55',6,6'} \cdot \underline{Z}^3_{11',4} \cdot \left(Z_{44'}\right)^{Z^2_{33',4}\underline{Z}^2_{11',4}} \cdot Z^3_{55',6} \cdot \left(\underline{Z}^2_{i6}\right)^{Z^2_{i,22'}Z^2_{33',4}}_{i=11',33'} \cdot \underline{Z}^3_{22',6} \cdot \left(\underline{Z}^2_{i6'}\right)^{\underline{Z}^2_{i6}\underline{Z}^2_{i,55'}Z^2_{i,22'}Z^2_{33',4}}_{i=11',33'} \cdot \left(Z_{66'}\right)^{Z^2_{55',6}\underline{Z}^2_{22',6}} \cdot \left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{33'}Z^{-1}_{55'}}\right)^{Z^2_{55',6}Z^2_{33',4}}.$$

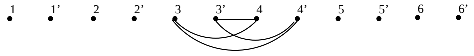
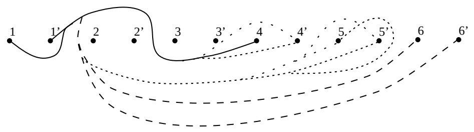
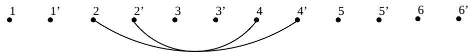
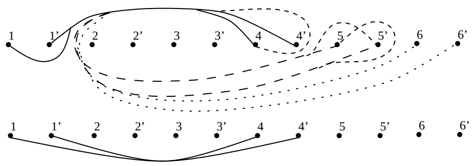
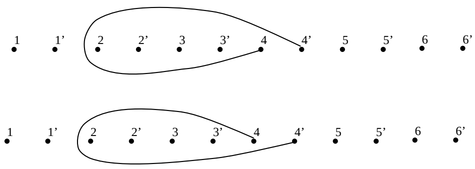
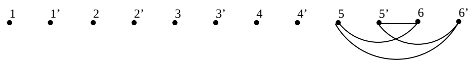
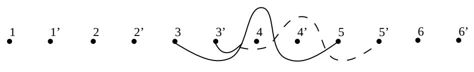
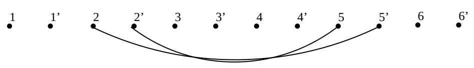
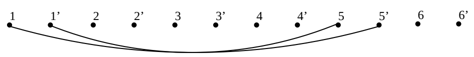
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_4^j \rho_m^i \underline{z}_{m4} \rho_m^{-i} \rho_4^{-j}$ $m = 2, 5$ $\rho_6^i \rho_4^i \underline{z}_{4'6} \rho_4^{-i} \rho_6^{-i}$	2
(2)		$\rho_4^j \rho_3^i z_{34} \rho_3^{-i} \rho_4^{-j}$	3

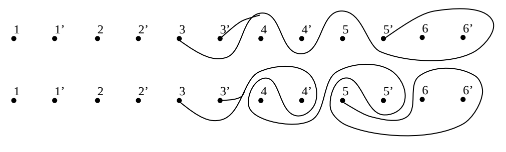
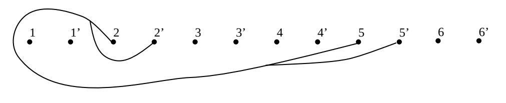
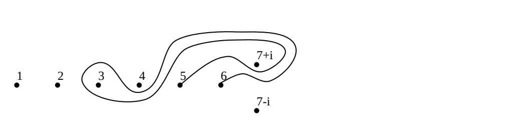
(3)		$\rho_4^j \rho_m^i \tilde{z}_{4m} \rho_m^{-i} \rho_4^{-j}$ $m = 2, 5$ $\rho_6^i \rho_4^i \tilde{z}_{46} \rho_4^{-i} \rho_6^{-i}$	2
(4)		$\rho_4^j \rho_1^i \underline{z}_{14} \rho_1^{-i} \rho_4^{-j}$	3
(5)		$\tilde{z}_{44'}$	1
(6)		$\rho_6^j \rho_5^i z_{56} \rho_5^{-i} \rho_6^{-j}$	3
(7)		$\rho_6^j \rho_m^i \tilde{z}_{m6} \rho_m^{-i} \rho_6^{-j}$ $m = 1, 3$	2
(8)		$\rho_6^j \rho_2^i \underline{z}_{26} \rho_2^{-i} \rho_6^{-j}$	3
(9)		$\rho_6^j \rho_m^i \tilde{z}_{m6} \rho_m^{-i} \rho_6^{-j}$ $m = 1, 3$	2
(10)		$\tilde{z}_{66'}$	1

$$\left(\hat{F}_1\cdot(\hat{F}_1)^{Z_{33'}^{-1}Z_{55'}^{-1}}\right)^{Z^2_{55',6}Z^2_{33',4}}$$

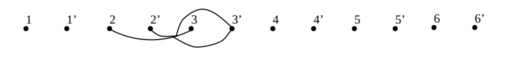
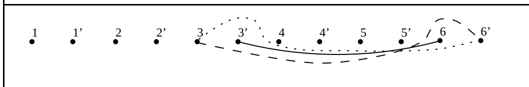
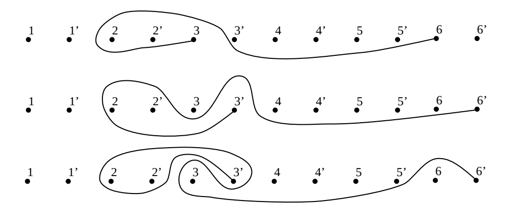
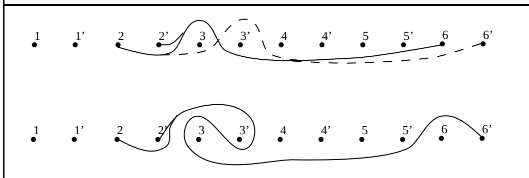
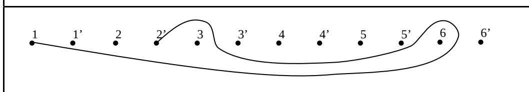
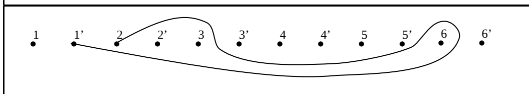
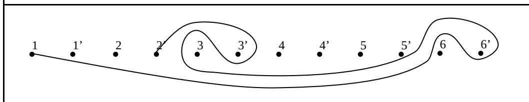
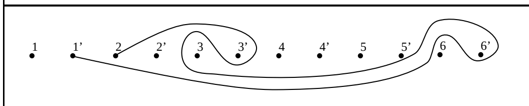
	$\rho_3^j\rho_2^i\underline{z}_{23}\rho_2^{-i}\rho_3^{-j}$	3
	$\rho_5^i\rho_3^i\underline{z}_{3'5}\rho_3^{-i}\rho_5^{-i}$	2
	$\rho_5^i\rho_3^i\tilde{z}_{35}\rho_3^{-i}\rho_5^{-i}$	2
	$\rho_5^j\rho_2^i\tilde{z}_{25}\rho_2^{-i}\rho_5^{-j}$	3
	$z_{12'}$	1
	$z_{1'2}$	1
	$z_{12'}^{\rho^{-1}}$	1
	$z_{1'2}^{\rho^{-1}}$	1

$$H_{V_2} = Z^3_{33',4} \cdot \left(\underline{Z}^2_{i4}\right)^{Z^2_{i,22'}}_{i=11',5,5',66'} \cdot \underline{Z}^3_{22',4} \cdot \left(\underline{Z}^2_{i4'}\right)^{\underline{Z}^2_{i4} \underline{Z}^2_{i,33'} \underline{Z}^2_{i,22'}}_{i=11',5,5',66'} \cdot \left(Z_{44'}\right)^{Z^2_{33',4} \underline{Z}^2_{22',4}} \cdot Z^3_{5',66'} \cdot \left(\underline{Z}^2_{33',5}\right)^{Z^2_{33',4}} \cdot \underline{Z}^2_{22',5} \cdot \underline{Z}^3_{11',5} \cdot \left(\underline{Z}^2_{33',5'}\right)^{Z^2_{33',5} Z^2_{5',66'} Z^2_{33',4}} \cdot \left(\underline{Z}^2_{22',5}\right)^{\underline{Z}^{-2}_{11',22'}} \cdot \left(Z_{55'}\right)^{\underline{Z}^2_{22',5} \underline{Z}^2_{11',5} Z^2_{5',66'}} \cdot \left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{33'} Z^{-1}_{66'}}\right)^{\underline{Z}^{-2}_{5,66'} Z^2_{33',4}}.$$

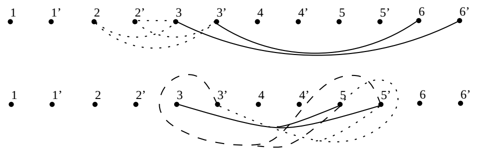
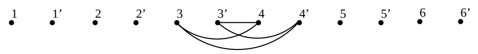
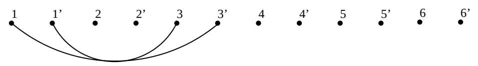
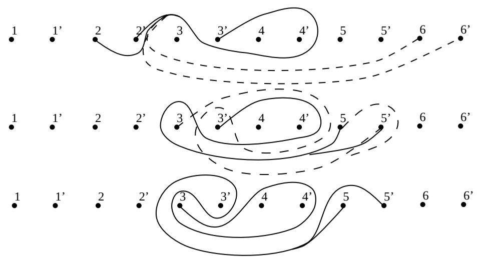
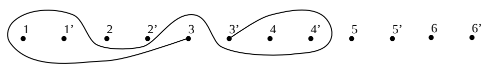
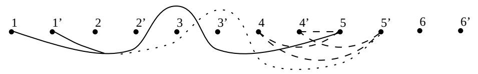
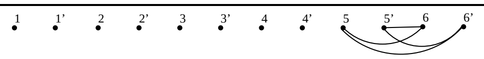
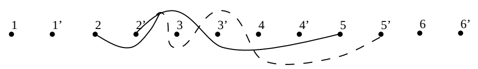
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_4^j \rho_3^i \underline{z}_{34} \rho_3^{-i} \rho_4^{-j}$	3
(2)		$\rho_4^j \rho_m^i \tilde{z}_{m4} \rho_m^{-i} \rho_4^{-j}$ $m = 1, 6$ $\rho_5^i \rho_4^i \tilde{z}_{45} \rho_4^{-i} \rho_5^{-i}$	2
(3)		$\rho_4^j \rho_2^i \underline{z}_{24} \rho_2^{-i} \rho_4^{-j}$	3
(4)		$\rho_4^j \rho_m^i \tilde{z}_{m4} \rho_m^{-i} \rho_4^{-j}$ $m = 1, 6$ $\rho_5^i \rho_4^i \tilde{z}_{45} \rho_4^{-i} \rho_5^{-i}$	2
(5)		$\tilde{z}_{44'}$	1
(6)		$\rho_6^j \rho_5^i z_{56} \rho_5^{-i} \rho_6^{-j}$	3
(7)		$\rho_5^j \rho_3^i \tilde{z}_{35} \rho_3^{-i} \rho_5^{-j}$	2
(8)		$\rho_5^j \rho_2^i \underline{z}_{25} \rho_2^{-i} \rho_5^{-j}$	2
(9)		$\rho_5^j \rho_1^i \underline{z}_{15} \rho_1^{-i} \rho_5^{-j}$	3

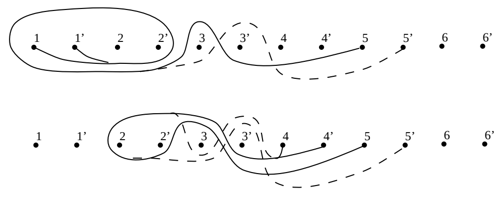
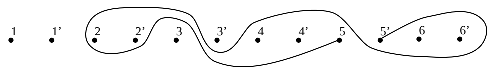
(10)		$\rho_5^j\rho_3^i\tilde{z}_{35'}\rho_3^{-i}\rho_5^{-j}$	2
(11)		$\rho_5^j\rho_2^i\tilde{z}_{25}\rho_2^{-i}\rho_5^{-j}$	2
(12)		$\tilde{z}_{55'}$	1

$$\left(\hat{F}_1\cdot(\hat{F}_1)^{Z_{33'}^{-1}Z_{66'}^{-1}}\right)^{\underline{Z}_{5,66'}^{-2}Z_{33',4}^{2}}$$

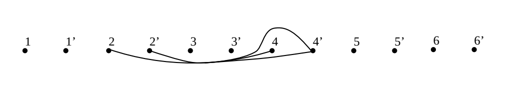
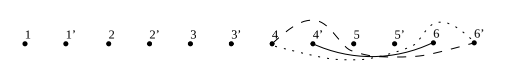
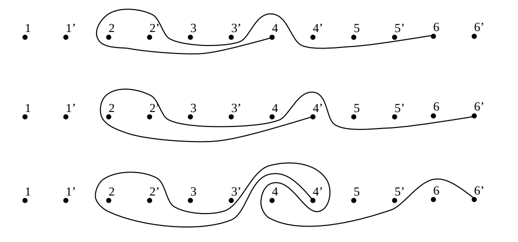
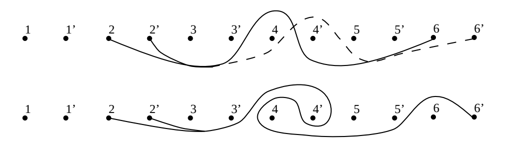
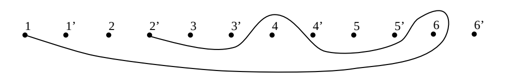
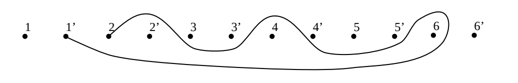
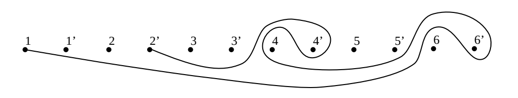
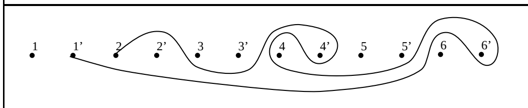
	$\rho_3^j\rho_2^i\underline{z}_{23}\rho_2^{-i}\rho_3^{-j}$	3
	$\rho_6^i\rho_3^i\underline{z}_{3'6}\rho_3^{-i}\rho_6^{-i}$	2
	$\rho_6^i\rho_3^i\tilde{z}_{36}\rho_3^{-i}\rho_6^{-i}$	2
	$\rho_6^j\rho_2^i\tilde{z}_{26}\rho_2^{-i}\rho_6^{-j}$	3
	$z_{12'}$	1
	$z_{1'2}$	1
	$z_{12'}^{\rho^{-1}}$	1
	$z_{1'2}^{\rho^{-1}}$	1

$H_{V_3} = \left(\underline{Z}^2_{3i}\right)_{i=22',5,5',66'} \cdot Z^3_{3',44'} \cdot \underline{Z}^3_{11',3} \cdot \left(\underline{Z}^2_{3'i}\right)^{\underline{Z}^2_{3i} Z^2_{3',44'}}_{i=22',5,5',66'} \cdot \left(Z_{33'}\right)^{\underline{Z}^2_{11',3} Z^2_{3',44'}} \cdot \left(\underline{Z}^2_{i5}\right)^{Z^2_{22',3}\underline{Z}^2_{11',3}}_{i=11',44'} \cdot Z^3_{5',66'} \cdot \left(\underline{Z}^3_{22',5}\right)^{Z^2_{22',3}\underline{Z}^2_{11',3}} \cdot \left(\underline{Z}^2_{i5}\right)^{\underline{Z}^{-2}_{22',i} Z^2_{22',3}\underline{Z}^2_{11',3}}_{i=11',44'} \cdot \left(Z_{55'}\right)^{Z^2_{5',66'} Z^2_{44',5}\underline{Z}^2_{22',5} Z^2_{22',3}\underline{Z}^2_{11',3}} \cdot \left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{44'} Z^{-1}_{66'}}\right)^{Z^2_{22',3}\underline{Z}^2_{11',3}\underline{Z}^{-2}_{5,66'}}.$

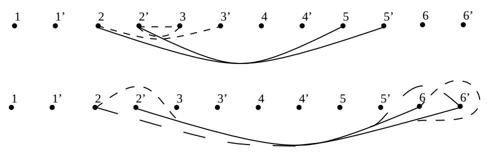
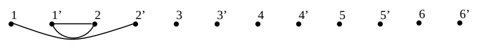
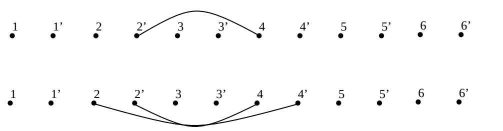
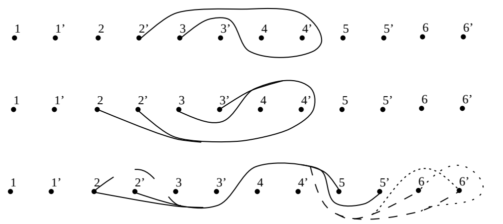
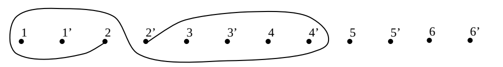
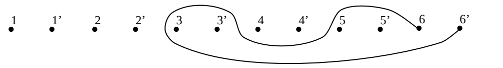
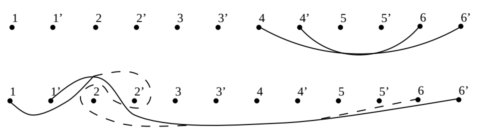
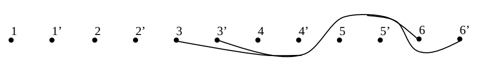
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_m^j \rho_3^i \underline{z}_{3m} \rho_3^{-i} \rho_m^{-j}$ $m = 2, 6$ $\rho_5^i \rho_3^i \underline{z}_{35} \rho_3^{-i} \rho_5^{-i}$	2
(2)		$\rho_4^j \rho_3^i \underline{z}_{3'4} \rho_3^{-i} \rho_4^{-j}$	3
(3)		$\rho_3^j \rho_1^i \underline{z}_{13} \rho_1^{-i} \rho_3^{-j}$	3
(4)		$\rho_m^j \rho_3^i z_{3'm} \rho_3^{-i} \rho_m^{-j}$ $m = 2, 6$ $\rho_5^i \rho_3^i \tilde{z}_{3'5} \rho_3^{-i} \rho_5^{-i}$	2
(5)		$\tilde{z}_{33'}$	1
(6)		$\rho_5^j \rho_m^i \tilde{z}_{m5} \rho_m^{-i} \rho_5^{-j}$ $m = 1, 4$	2
(7)		$\rho_6^j \rho_5^i \underline{z}_{5'6} \rho_5^{-i} \rho_6^{-j}$	3
(8)		$\rho_5^j \rho_2^i \tilde{z}_{25} \rho_2^{-i} \rho_5^{-j}$	3

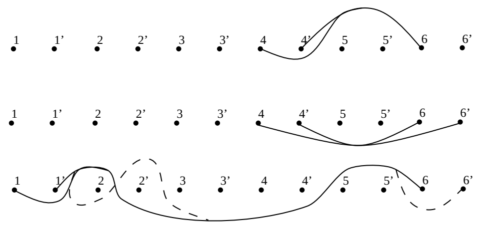
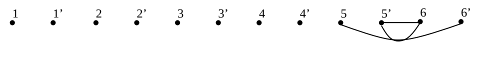
(9)		$\rho_5^j \rho_m^i \tilde{z}_{m5} \rho_m^{-i} \rho_5^{-j}$ $m = 1, 4$	2
(10)		$\tilde{z}_{55'}$	1

$$\left(\hat{F}_1 \cdot (\hat{F}_1)^{Z_{44'}^{-1} Z_{66'}^{-1}}\right)^{Z^2_{22',3} \underline{Z}^2_{11',3} \underline{Z}^{-2}_{5,66'}}$$

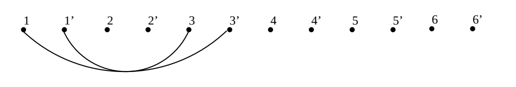
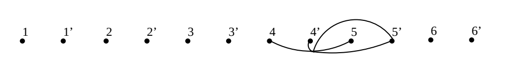
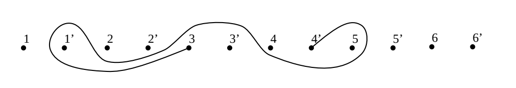
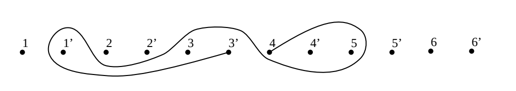
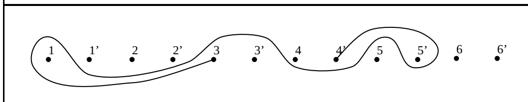
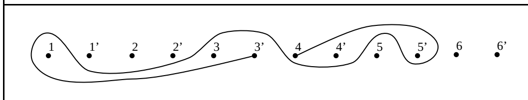
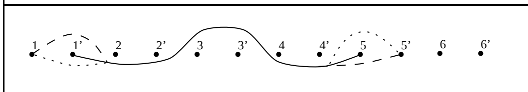
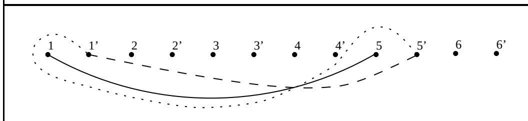
	$\rho_4^j \rho_2^i \underline{z}_{24} \rho_2^{-i} \rho_4^{-j}$	3
	$\rho_6^i \rho_4^i \underline{z}_{4'6} \rho_4^{-i} \rho_6^{-i}$	2
	$\rho_6^i \rho_4^i \tilde{z}_{46} \rho_4^{-i} \rho_6^{-i}$	2
	$\rho_6^j \rho_2^i \tilde{z}_{26} \rho_2^{-i} \rho_6^{-j}$	3
	$z_{12'}$	1
	$z_{1'2}$	1
	$z_{12'}^{\rho^{-1}}$	1
	$z_{1'2}^{\rho^{-1}}$	1

$$H_{V_4} = \left(\underline{Z}^2_{2'i}\right)_{i=33',55',6,6'} \cdot Z^3_{11',2} \cdot \overline{Z}^3_{2',44'} \cdot \left(\underline{Z}^2_{2'i}\right)^{\underline{Z}^{-2}_{i,44'}}_{i=33',55',6,6'} \cdot \left(Z_{22'}\right)^{Z^2_{11',2}\underline{Z}^2_{2',44'}Z^2_{2',33'}} \cdot$$
$$\left(Z_{66'}\right)^{\underline{Z}^{-2}_{33',6}Z^{-2}_{55',6}} \cdot \left(\underline{Z}^2_{i6'}\right)^{Z^2_{11',2}}_{i=11',44'} \cdot \left(\underline{Z}^3_{33',6}\right)^{Z^{-2}_{55',6}} \cdot \left(\underline{Z}^2_{i6}\right)^{Z^{-2}_{55',6}Z^2_{11',2}}_{i=11',44'} \cdot Z^3_{55',6} \cdot \left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{11'}Z^{-1}_{55'}}\right)^{Z^2_{11',2}Z^{-2}_{55',6}}.$$

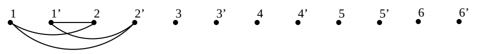
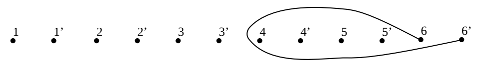
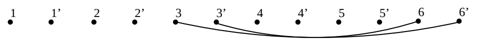
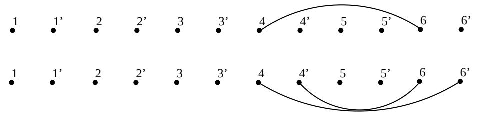
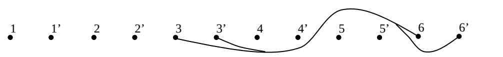
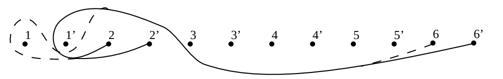
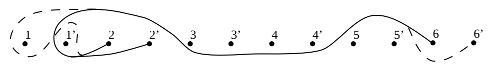
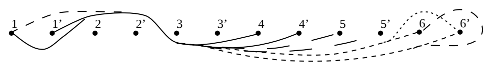
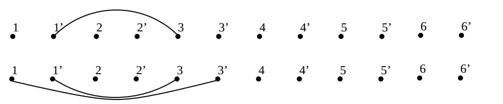
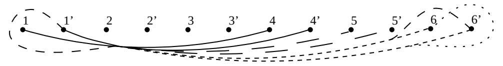
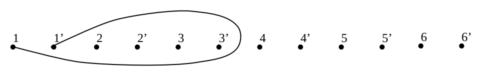
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_m^j\rho_2^i\underline{z}_{2'm}\rho_2^{-i}\rho_m^{-j}$ $m = 3, 5$ $\rho_6^i\rho_2^i\underline{z}_{2'6}\rho_2^{-i}\rho_6^{-i}$	2
(2)		$\rho_2^j\rho_1^i\underline{z}_{12}\rho_1^{-i}\rho_2^{-j}$	3
(3)		$\rho_4^j\rho_2^i\underline{z}_{2'4}\rho_2^{-i}\rho_4^{-j}$	3
(4)		$\rho_m^j\rho_2^i\tilde{z}_{2'm}\rho_2^{-i}\rho_m^{-j}$ $m = 3, 5$ $\rho_6^i\rho_2^i\tilde{z}_{2'6}\rho_2^{-i}\rho_6^{-i}$	2
(5)		$\tilde{z}_{22'}$	1
(6)		$\tilde{z}_{66'}$	1
(7)		$\rho_6^j\rho_m^i\tilde{z}_{m6'}\rho_m^{-i}\rho_6^{-j}$ $m = 1, 4$	2
(8)		$\rho_6^j\rho_3^i\tilde{z}_{36}\rho_3^{-i}\rho_6^{-j}$	3

(9)		$\rho_6^j \rho_m^i \tilde{z}_{m6} \rho_m^{-i} \rho_6^{-j}$ $m = 1, 4$	2
(10)		$\rho_6^j \rho_5^i z_{56} \rho_5^{-i} \rho_6^{-j}$	3

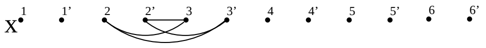
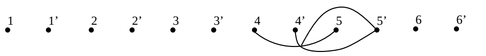
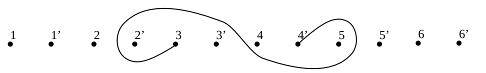
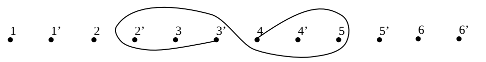
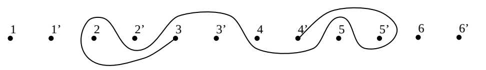
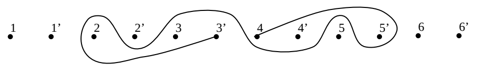
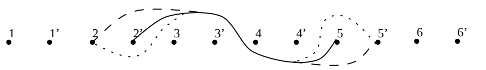
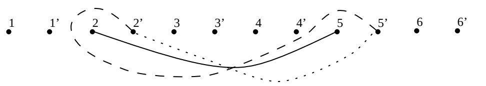
$$\left(\hat{F}_1 \cdot (\hat{F}_1)^{Z_{11'}^{-1} Z_{55'}^{-1}}\right)^{Z^2_{11',2} Z^{-2}_{55',6}}$$

	$\rho_3^j \rho_1^i \underline{z}_{1'3} \rho_1^{-i} \rho_3^{-j}$	3
	$\rho_5^j \rho_4^i \underline{z}_{45} \rho_4^{-i} \rho_5^{-j}$	3
	$z_{34'}$	1
	$z_{3'4}$	1
	$z_{34'}^{\rho^{-1}}$	1
	$z_{3'4}^{\rho^{-1}}$	1
	$\rho_5^i \rho_1^i \tilde{z}_{1'5} \rho_1^{-i} \rho_5^{-i}$	2
	$\rho_5^i \rho_1^i \underline{z}_{15} \rho_1^{-i} \rho_5^{-i}$	2

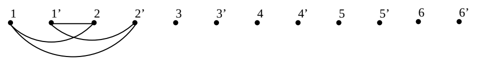
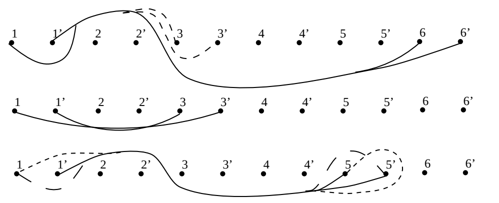
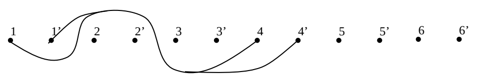
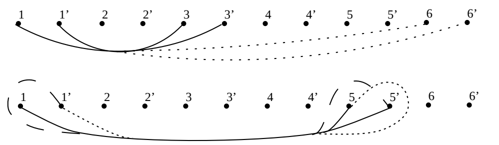
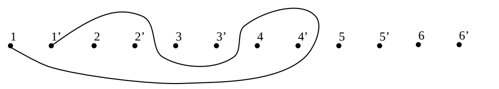
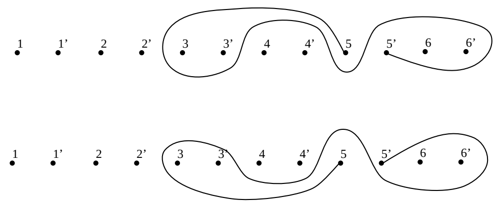
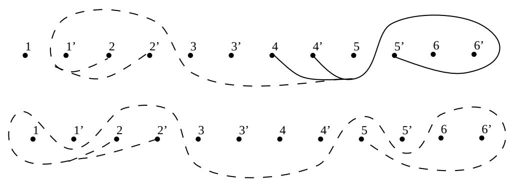
$$H_{V_5} = Z^3_{1',22'} \cdot \left(Z_{66'}\right)^{\underline{Z}^{-2}_{44',6} Z^{-2}_{55',6}} \cdot \underline{Z}^2_{33',6'} \cdot \bar{Z}^3_{44',6} \cdot \left(\underline{Z}^2_{33',6}\right)^{Z^{-2}_{55',6}} \cdot \left(\underline{Z}^2_{22',6'}\right)^{Z^2_{1',22'}} \cdot \left(\underline{Z}^2_{22',6}\right)^{Z^2_{1',22'} Z^{-2}_{55',6}} \cdot$$
$$\left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{22'} Z^{-1}_{55'}}\right)^{Z^2_{1',22'} Z^{-2}_{55',6}} \cdot \underline{Z}^3_{55',6} \cdot \left(\underline{Z}^2_{1'i}\right)^{Z^2_{1',22'}}_{i=44',55',6,6'} \cdot \bar{Z}^3_{1',33'} \cdot \left(\underline{Z}^2_{1i}\right)_{i=44',55',6,6'} \cdot \left(Z_{11'}\right)^{\underline{Z}^2_{1',33'} Z^2_{1',22'}}.$$

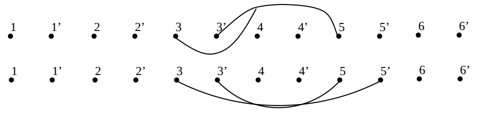
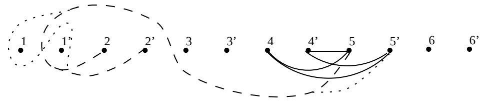
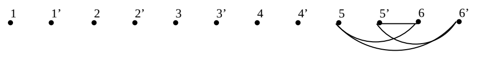
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_2^j \rho_1^i z_{1'2} \rho_1^{-i} \rho_2^{-j}$	3
(2)		$\tilde{z}_{66'}$	1
(3)		$\rho_6^j \rho_3^i \underline{z}_{36'} \rho_3^{-i} \rho_6^{-j}$	2
(4)		$\rho_6^j \rho_4^i \underline{z}_{46} \rho_4^{-i} \rho_6^{-j}$	3
(5)		$\rho_6^j \rho_3^i \tilde{z}_{36} \rho_3^{-i} \rho_6^{-j}$	2
(6)		$\rho_6^j \rho_2^i \tilde{z}_{26'} \rho_2^{-i} \rho_6^{-j}$	2
(7)		$\rho_6^j \rho_2^i \tilde{z}_{26} \rho_2^{-i} \rho_6^{-j}$	2
(8)		$\rho_6^j \rho_5^i z_{56} \rho_5^{-i} \rho_6^{-j}$	3
(9)		$\rho_m^j \rho_1^i \tilde{z}_{1'm} \rho_1^{-i} \rho_m^{-j}$ $m = 4, 5$ $\rho_6^i \rho_1^i \tilde{z}_{1'6} \rho_1^{-i} \rho_6^{-i}$	2
(10)		$\rho_3^j \rho_1^i \underline{z}_{1'3} \rho_1^{-i} \rho_3^{-j}$	3
(11)		$\rho_m^j \rho_1^i \underline{z}_{1m} \rho_1^{-i} \rho_m^{-j}$ $m = 4, 5$ $\rho_6^i \rho_1^i \underline{z}_{16} \rho_1^{-i} \rho_6^{-i}$	2
(12)		$\tilde{z}_{11'}$	1

$$\left(\hat{F}_1 \cdot (\hat{F}_1)^{Z^{-1}_{22'} Z^{-1}_{55'}}\right)^{\underline{Z}^2_{1',33'} Z^2_{1',22'}}$$

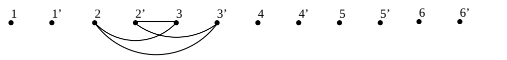
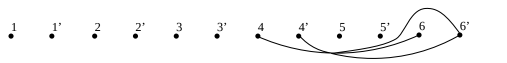
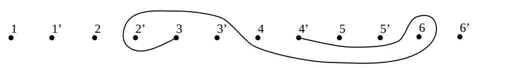
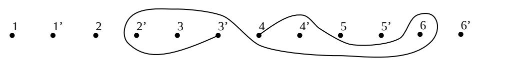
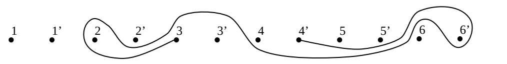
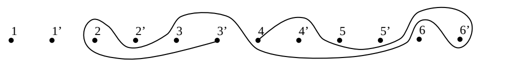
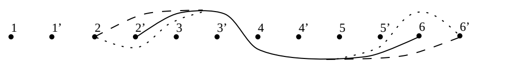
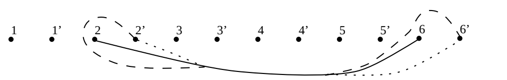
	$\rho_3^j \rho_2^i \underline{z}_{23} \rho_2^{-i} \rho_3^{-j}$	3
	$\rho_5^j \rho_4^i \underline{z}_{45} \rho_4^{-i} \rho_5^{-j}$	3
	$z_{34'}$	1
	$z_{3'4}$	1
	$z_{34'}^{\rho^{-1}}$	1
	$z_{3'4}^{\rho^{-1}}$	1
	$\rho_5^i \rho_2^i \tilde{z}_{2'5} \rho_2^{-i} \rho_5^{-i}$	2
	$\rho_5^i \rho_2^i \underline{z}_{25} \rho_2^{-i} \rho_5^{-i}$	2

$$H_{V_6} = Z^3_{1',22'} \cdot \left(\underline{Z}^2_{1'i}\right)^{Z^2_{1',22'}}_{i=33',5,5',66'} \cdot \left(\underline{Z}^3_{1',44'}\right)^{Z^2_{1',22'}} \cdot \left(\underline{Z}^2_{1i}\right)_{i=33',5,5',66'} \cdot \left(Z_{11'}\right)^{\underline{Z}^2_{1',44'} Z^2_{1',22'}} \cdot$$
$$\left(Z_{55'}\right)^{\bar{Z}^{-2}_{33',5} Z^{-2}_{5',66'}} \cdot \left(\underline{Z}^2_{i5'}\right)^{Z^{-2}_{5',66'} Z^2_{1',22'}}_{i=22',44'} \cdot \bar{Z}^3_{33',5} \cdot \left(\underline{Z}^2_{i5}\right)^{Z^2_{1',22'}}_{i=22',44'} \cdot Z^3_{5',66'} \cdot \left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{22'} Z^{-1}_{66'}}\right)^{Z^2_{1',22'} Z^{-2}_{5',66'}}.$$

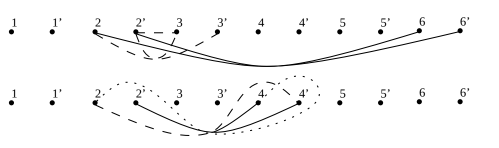
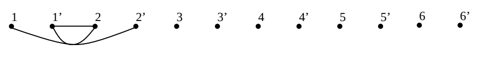
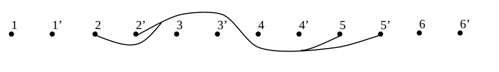
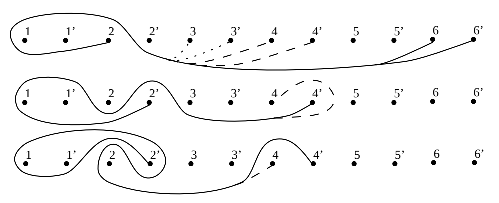
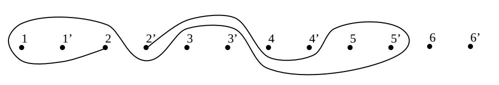
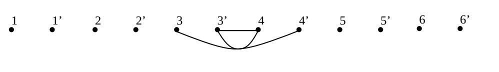
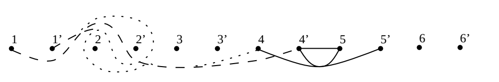
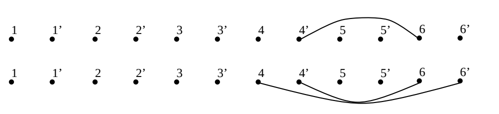
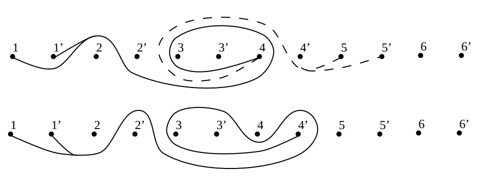
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_2^j\rho_1^i z_{1'2}\rho_1^{-i}\rho_2^{-j}$	3
(2)		$\rho_m^j\rho_1^i\tilde{z}_{1'm}\rho_1^{-i}\rho_m^{-j}$ $m = 3, 6$ $\rho_5^i\rho_1^i\tilde{z}_{1'5}\rho_1^{-i}\rho_5^{-i}$	2
(3)		$\rho_4^j\rho_1^i\tilde{z}_{1'4}\rho_1^{-i}\rho_4^{-j}$	3
(4)		$\rho_m^j\rho_1^i\underline{z}_{1m}\rho_1^{-i}\rho_m^{-j}$ $m = 3, 6$ $\rho_5^i\rho_1^i\underline{z}_{15}\rho_1^{-i}\rho_5^{-i}$	2
(5)		$\tilde{z}_{11'}$	1
(6)		$\tilde{z}_{55'}$	1
(7)		$\rho_5^j\rho_m^i\tilde{z}_{m5'}\rho_m^{-i}\rho_5^{-j}$ $m = 2, 4$	2

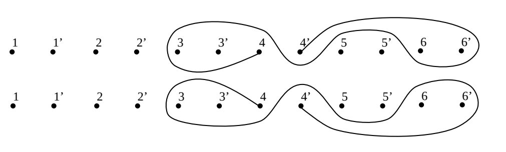
(8)		$\rho_5^j \rho_3^i \underline{z}_{35} \rho_3^{-i} \rho_5^{-j}$	3
(9)		$\rho_5^j \rho_m^i \tilde{z}_{m5} \rho_m^{-i} \rho_5^{-j}$ $m = 2, 4$	2
(10)		$\rho_6^j \rho_5^i z_{5'6} \rho_5^{-i} \rho_6^{-j}$	3

$$\left(\hat{F}_1(\hat{F}_1)^{Z_{22'}^{-1} Z_{66'}^{-1}}\right)^{Z^2_{1',22'} Z^{-2}_{5',66'}}$$

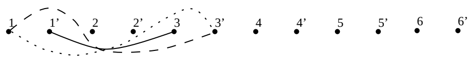
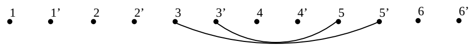
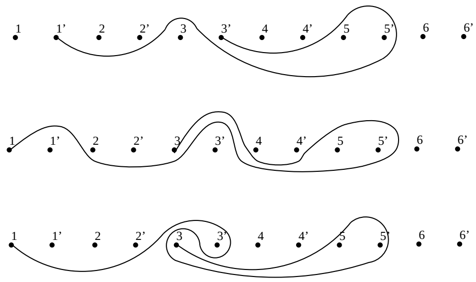
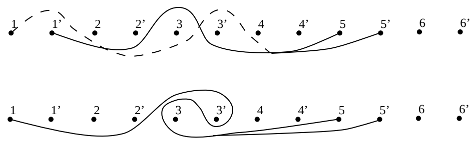
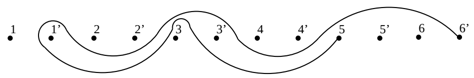
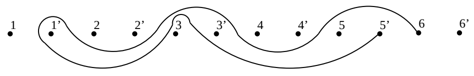
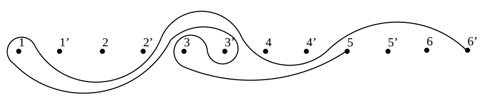
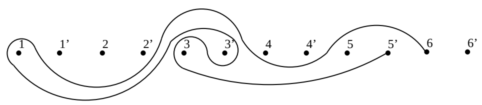
	$\rho_3^j \rho_2^i z_{2'3} \rho_2^{-i} \rho_3^{-j}$	3
	$\rho_6^j \rho_4^i \underline{z}_{46} \rho_4^{-i} \rho_6^{-j}$	3
	$z_{34'}$	1
	$z_{3'4}$	1
	$z_{34'}^{\rho^{-1}}$	1
	$z_{3'4}^{\rho^{-1}}$	1
	$\rho_6^i \rho_2^i \tilde{z}_{2'6} \rho_2^{-i} \rho_6^{-i}$	2
	$\rho_6^i \rho_2^i \underline{z}_{26} \rho_2^{-i} \rho_6^{-i}$	2

$$H_{V_7} = \left(\underline{Z}^2_{2'i}\right)_{i=33',4,4',66'} \cdot Z^3_{11',2} \cdot \left(\underline{Z}^3_{2',55'}\right)^{Z^2_{2',33'}} \cdot \left(\underline{Z}^2_{2i}\right)^{Z_{11',2}}_{i=33',4,4',66'} \cdot \left(Z_{22'}\right)^{Z^2_{11',2}\bar{Z}^2_{2',55'}Z^2_{44',55'}} \cdot Z^3_{33',4} \cdot \left(\underline{Z}^2_{i4'}\right)^{Z^2_{11',2}}_{i=11',55'} \cdot \bar{Z}^3_{4',66'} \cdot \left(\underline{Z}^2_{i4}\right)^{Z^2_{33',4}Z^2_{11',2}}_{i=11',55'} \cdot \left(Z_{44'}\right)^{\bar{Z}^2_{4',66'}Z^2_{33',4}} \cdot \left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{11'}Z^{-1}_{33'}}\right)^{\underline{Z}^{-2}_{4,55'}\underline{Z}^{-2}_{4,66'}Z^2_{11',2}}.$$

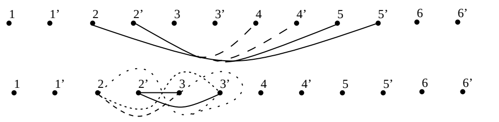
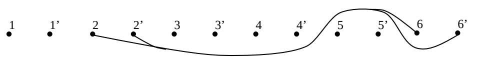
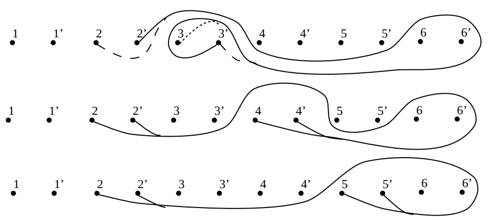
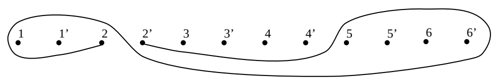
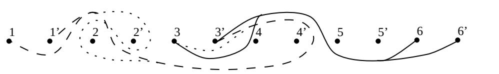
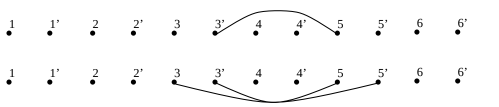
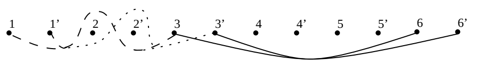
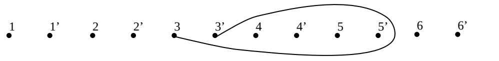
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_m^j\rho_2^i\underline{z}_{2'm}\rho_2^{-i}\rho_m^{-j}$ $m = 3, 6$ $\rho_4^i\rho_2^i\underline{z}_{2'4}\rho_2^{-i}\rho_4^{-i}$	2
(2)		$\rho_2^j\rho_1^i\underline{z}_{12}\rho_1^{-i}\rho_2^{-j}$	3
(3)		$\rho_5^j\rho_2^i\tilde{z}_{2'5}\rho_2^{-i}\rho_5^{-j}$	3
(4)		$\rho_m^j\rho_2^i\tilde{z}_{2m}\rho_2^{-i}\rho_m^{-j}$ $m = 3, 6$ $\rho_4^i\rho_2^i\tilde{z}_{24}\rho_2^{-i}\rho_4^{-i}$	2
(5)		$\tilde{z}_{22'}$	1
(6)		$\rho_4^j\rho_3^i\underline{z}_{34}\rho_3^{-i}\rho_4^{-j}$	3
(7)		$\rho_m^j\rho_4^i z_{4'm}\rho_4^{-i}\rho_m^{-j}$ $m = 1, 5$	2
(8)		$\rho_6^j\rho_4^i\underline{z}_{4'6}\rho_4^{-i}\rho_6^{-j}$	3
(9)		$\rho_4^j\rho_m^i\tilde{z}_{m4}\rho_m^{-i}\rho_4^{-j}$ $m = 1, 5$	2

(10)		$\tilde{z}_{44'}$	1

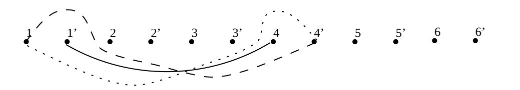
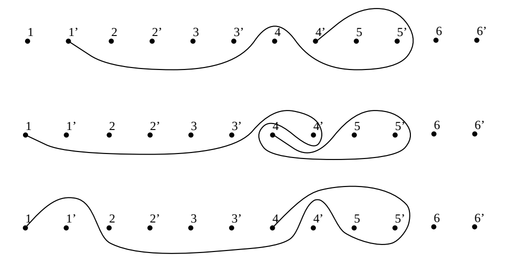
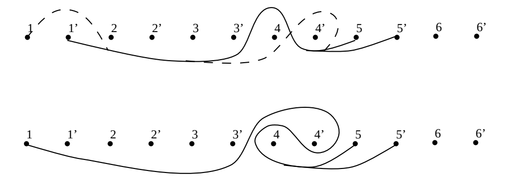
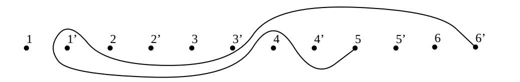
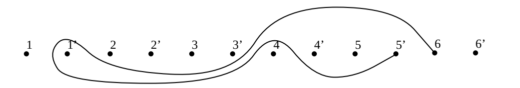
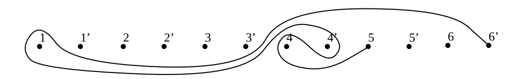
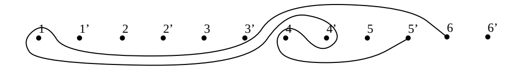
$$\left(\hat{F}_1 \cdot (\hat{F}_1)^{Z_{11'}^{-1} Z_{33'}^{-1}}\right)^{\underline{Z}_{4,55'}^{-2}\underline{Z}_{4,66'}^{-2} Z_{11',2}^{2}}$$

	$\rho_3^i \rho_1^i \underline{z}_{1'3} \rho_1^{-i} \rho_3^{-i}$	2
	$\rho_5^j \rho_3^i \underline{z}_{3'5} \rho_3^{-i} \rho_5^{-j}$	3
	$\rho_3^i \rho_1^i \tilde{z}_{1'3'} \rho_1^{-i} \rho_3^{-i}$	2
	$\rho_5^j \rho_1^i \tilde{z}_{1'5} \rho_1^{-i} \rho_5^{-j}$	3
	$z_{56'}$	1
	$z_{5'6}$	1
	$z_{56'}^{\rho^{-1}}$	1
	$z_{5'6}^{\rho^{-1}}$	1

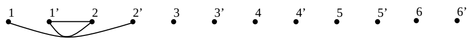
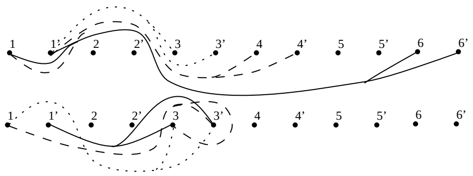
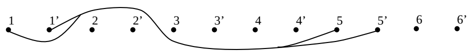
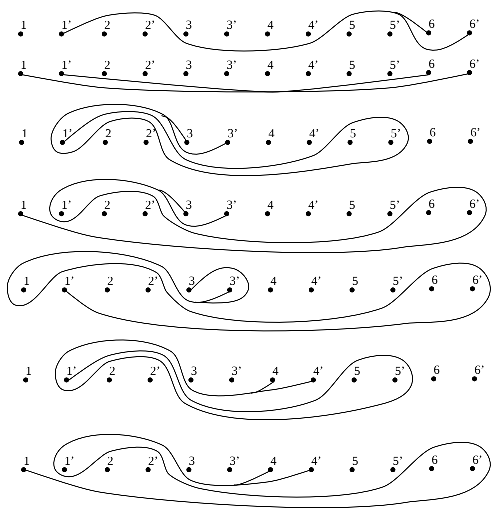
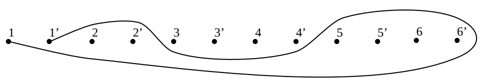
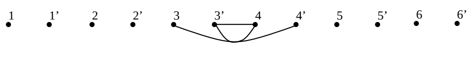
$$H_{V_8} = Z^3_{11',2} \cdot \left(\underline{Z}^2_{2'i}\right)_{i=3,3',44',55'} \cdot \left(\underline{Z}^3_{2',66'}\right)^{Z^{-2}_{55',66'}} \cdot \left(\underline{Z}^2_{2'i}\right)^{\underline{Z}^{-2}_{i,66'}}_{i=3,3',44',55'} \cdot \left(Z_{22'}\right)^{\underline{Z}^2_{2',66'}\underline{Z}^2_{2',55'}Z^2_{11',2}} \cdot Z^3_{3',44'} \cdot$$
$$\left(\underline{Z}^2_{3'i}\right)^{Z^2_{3',44'}Z^2_{11',2}}_{i=11',66'} \cdot \bar{Z}^3_{3',55'} \cdot \left(\underline{Z}^2_{3i}\right)^{Z^2_{11',2}}_{i=11',66'} \cdot \left(Z_{33'}\right)^{\underline{Z}^2_{3',55'}Z^2_{3',44'}} \cdot \left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{11'}Z^{-1}_{44'}}\right)^{\underline{Z}^2_{11',3'}Z^2_{3',44'}Z^2_{11',2}}.$$

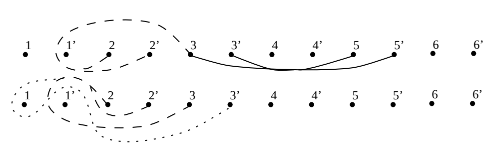
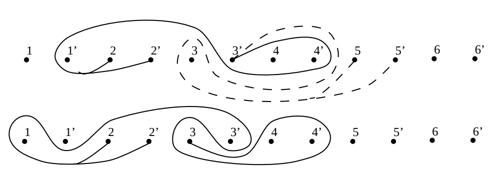
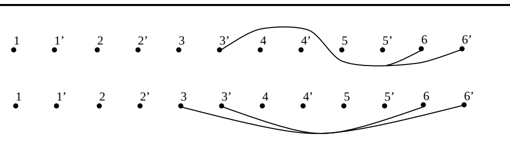
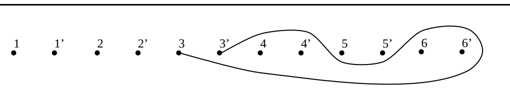
	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_2^j\rho_1^i\underline{z}_{12}\rho_1^{-i}\rho_2^{-j}$	3
(2)		$\rho_m^j\rho_2^i\underline{z}_{2'm}\rho_2^{-i}\rho_m^{-j}$ $m = 4, 5$ $\rho_3^i\rho_2^i\underline{z}_{2'3}\rho_2^{-i}\rho_3^{-i}$	2
(3)		$\rho_6^j\rho_2^i\tilde{z}_{2'6}\rho_2^{-i}\rho_6^{-j}$	3
(4)		$\rho_m^j\rho_2^i\tilde{z}_{2'm}\rho_2^{-i}\rho_m^{-j}$ $m = 4, 5$ $\rho_3^i\rho_2^i\tilde{z}_{23}\rho_2^{-i}\rho_3^{-i}$	2
(5)		$\tilde{z}_{22'}$	1
(6)		$\rho_4^j\rho_3^i\underline{z}_{3'4}\rho_3^{-i}\rho_4^{-j}$	3
(7)		$\rho_m^j\rho_3^i\tilde{z}_{3'm}\rho_3^{-i}\rho_m^{-j}$ $m = 1, 6$	2
(8)		$\rho_5^j\rho_3^i\underline{z}_{3'5}\rho_3^{-i}\rho_5^{-j}$	3
(9)		$\rho_m^j\rho_3^i\tilde{z}_{3m}\rho_3^{-i}\rho_m^{-j}$ $m = 1, 6$	2
(10)		$\tilde{z}_{33'}$	1

$$\left(\hat{F}_1\cdot(\hat{F}_1)^{Z_{11'}^{-1}Z_{44'}^{-1}}\right)^{\underline{Z}^2_{11',3'}Z^2_{3',44'}Z^2_{11',2}}$$

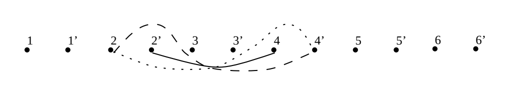
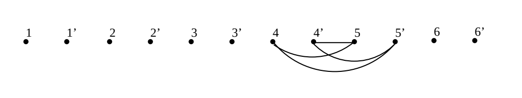
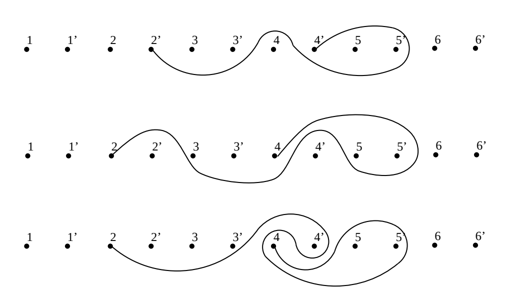
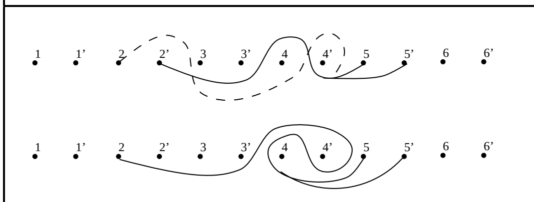
	$\rho_4^i\rho_1^i\underline{z}_{1'4}\rho_1^{-i}\rho_4^{-i}$	2
	$\rho_5^j\rho_4^i\underline{z}_{4'5}\rho_4^{-i}\rho_5^{-j}$	3
	$\rho_4^i\rho_1^i\tilde{z}_{1'4'}\rho_1^{-i}\rho_4^{-i}$	2
	$\rho_5^j\rho_1^i\tilde{z}_{1'5}\rho_1^{-i}\rho_5^{-j}$	3
	$z_{56'}$	1
	$z_{5'6}$	1
	$z_{56'}^{\rho^{-1}}$	1
	$z_{5'6}^{\rho^{-1}}$	1

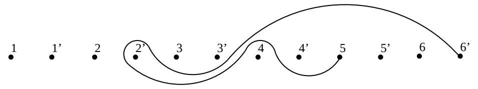
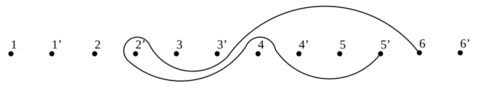
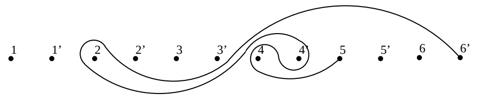
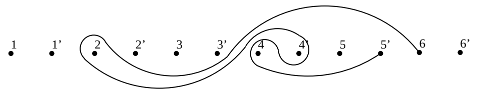
$$H_{V_9} = Z^3_{1',22'} \cdot \left(\underline{Z}^2_{1'i}\right)^{Z^2_{1',22'}}_{i=3,3',44',66'} \cdot \left(\underline{Z}^3_{1',55'}\right)^{Z^2_{1',22'}} \cdot \left(\underline{Z}^2_{1'i}\right)^{\underline{Z}^2_{1',55'} Z^2_{1',22'}}_{i=3,3',44',66'} \cdot \left(Z_{11'}\right)^{\underline{Z}^2_{1',66'} \underline{Z}^2_{1',55'} Z^2_{1',22'}} \cdot$$
$$Z^3_{3',44'} \cdot \left(\underline{Z}^2_{3i}\right)^{Z^2_{1',22'}}_{i=22',55'} \cdot \left(\underline{Z}^2_{i3'}\right)^{Z^2_{3',44'} \underline{Z}^2_{i3} Z^2_{1',22'}}_{i=22',55'} \cdot \left(\underline{Z}^3_{3',66'}\right)^{Z^2_{3',44'}} \cdot \left(Z_{33'}\right)^{\underline{Z}^2_{3',66'} Z^2_{3',44'}} \cdot$$
$$\left(\hat{F}_1(\hat{F}_1)^{Z^{-1}_{22'} Z^{-1}_{44'}}\right)^{Z^2_{22',33'} Z^2_{3',44'} Z^2_{1',22'}}.$$

	The paths/complex conjugates	The braids	The exponent of braids
(1)		$\rho_2^j \rho_1^i z_{1'2} \rho_1^{-i} \rho_2^{-j}$	3
(2)		$\rho_m^j \rho_1^i \tilde{z}_{1'm} \rho_1^{-i} \rho_m^{-j}$ $m = 4, 6$ $\rho_3^i \rho_1^i \underline{z}_{1'3} \rho_1^{-i} \rho_3^{-i}$	2
(3)		$\rho_5^j \rho_1^i \tilde{z}_{1'5} \rho_1^{-i} \rho_5^{-j}$	3
(4)		$\rho_m^j \rho_1^i \tilde{z}_{1'm} \rho_1^{-i} \rho_m^{-j}$ $m = 4, 6$ $\rho_3^i \rho_1^i \tilde{z}_{13} \rho_1^{-i} \rho_3^{-i}$	2
(5)		$\tilde{z}_{11'}$	1
(6)		$\rho_4^j \rho_3^i z_{3'4} \rho_3^{-i} \rho_4^{-j}$	3

(7)		$\rho_m^j \rho_3^i \tilde{z}_{3m} \rho_3^{-i} \rho_m^{-j}$ $m = 2, 5$	2
(8)		$\rho_m^j \rho_3^i \tilde{z}_{3'm} \rho_3^{-i} \rho_m^{-j}$ $m = 2, 5$	2
(9)		$\rho_6^j \rho_3^i \tilde{z}_{3'6} \rho_3^{-i} \rho_6^{-j}$	3
(10)		$\tilde{z}_{33'}$	1

$$\left(\hat{F}_1 \cdot (\hat{F}_1)^{Z_{22'}^{-1} Z_{44'}^{-1}}\right)^{Z^2_{22',33'} Z^2_{3',44'} Z^2_{1',22'}}$$

	$\rho_4^i \rho_2^i \underline{z}_{2'4} \rho_2^{-i} \rho_4^{-i}$	2
	$\rho_5^j \rho_4^i z_{4'5} \rho_4^{-i} \rho_5^{-j}$	3
	$\rho_4^i \rho_2^i \tilde{z}_{2'4'} \rho_2^{-i} \rho_4^{-i}$	2
	$\rho_5^j \rho_2^i \tilde{z}_{2'5} \rho_2^{-i} \rho_5^{-j}$	3

	$z_{56'}$	1
	$z_{5'6}$	1
	$z_{56'}^{\rho^{-1}}$	1
	$z_{5'6}^{\rho^{-1}}$	1

2.3. Properties of Δ^2_{54}. Now we are checking if the braids we have found are the only ones. For that, we assume that $\Delta^2_{54} = \prod_{i=1}^{9} C_i H_{V_i} \prod b_i$ be the braid monodromy factorization. b_i are factors corresponding to singularities that are not covered by $\prod_{i=1}^{9} C_i H_{V_i}$, and each b_i is of the form $Y_i^{t_i}$, Y_i is a positive halftwist and $0 \le t_i \le 3$. We compute degrees of all factors involved. By [6, Theorem 8], $\deg \Delta^2_{54} = 54 \cdot 53 = 2862$.

On the other hand, $\prod_{i=1}^{9} C_i$ compounds 864 factors, so $\deg \prod_{i=1}^{9} C_i = 864 \cdot 2 = 1728$. In the computations of each H_{V_i}, $1 \le i \le 9$, $\deg \hat{F}_1 = \deg(\hat{F}_1)^{\rho^{-1}} = 24$, so $\deg \hat{F}_1(\hat{F}_1)^{\rho^{-1}} = 48$. The factors outside $\hat{F}_1(\hat{F}_1)^{\rho^{-1}}$ in each H_{V_i} are: 20 degree two factors, 12 degee three factors, 2 degree one factor. The sum of the factors' degrees resulting from each H_{V_i} is $20 \cdot 2 + 12 \cdot 3 + 2 \cdot 1 = 78$. Thus $\deg H_{V_i} = 78 + 48 = 126$ for $1 \le i \le 9$, and therefore $\deg \prod_{i=1}^{9} H_{V_i} = 126 \cdot 9 = 1134$. Finally $\deg \prod_{i=1}^{9} C_i H_{V_i} = 1134 + 1728 = 2862$.

Therefore, $\deg \prod_{i=1}^{9} b_i = 2862 \cdot 2862^{-1} = 1$. Since $\forall i$, b_i is a positive exponent of a positive halftwist, we get $b_i = 1$ $\forall i$.

Finally, $\Delta^2_{54} = \prod_{i=1}^{9} C_i H_{V_i}$.

In the following lemma we prove the Invariance Property for C_i.

Lemma 6 (Complex Invariance of C_i). *For every i, $i = 1, \ldots, 9$ and $\forall m_j \in \mathbb{Z}$, $1 \le j \le 27$, C_i is invariant under $\prod_{j=1}^{27} Z_{jj'}^{m_j}$.*

Proof. We apply Invariance Rule II and Invariance Remark (iv) [6, Subsection 7.3] on each of the factors of C_i of the form $Y^2_{ii',jj'}$. □

We can summarize the properties of Δ^2_{54} in the following theorems.

Theorem 7 (Invariance Theorem for Δ^2_{54}). *Let $\rho = \prod_{j=1}^{27} \rho_{m_j}$ for $\rho_{m_j} = Z^{m_j}_{jj'}$ and $1 \leq j \leq 27$. Then Δ^2_{54} is invariant under ρ for every $m_j \in \mathbb{Z}$.*

Proof. By Invariance Properties of H_{V_i} ([4, Sections 3.1–3.9]) and by Lemma 6. □

Theorem 8 (Complex Conjugation Theorem). *Δ^2_{54} is invariant under complex conjugation.*

Proof. The same proof from Lemma 19, [22]. □

2.4. Consequences from the Invariance Theorems. Recall that $S^{(0)} = S$ is the regenerated branch curve. Every factor of a braid monodromy factorization Δ^2_{54} induces a relation on $\pi_1(\mathbb{C}^2 - S, M)$. The Invariance Properties are an essential addition to the van Kampen Theorem, since we get more relations in $\pi_1(\mathbb{C}^2 - S, M)$.

Theorem 9 ([6, Section 3.10]). *Let S, φ, B_{54} be as above. If a sub-factorization $\prod_{i=s}^{r} Z_i$ of Δ^2_{54} is invariant under any element h, and $\prod_{i=s}^{r} Z_i$ induces a relation $\Gamma_{i_1} \cdot \dots \cdot \Gamma_{i_t}$ on $\pi_1(\mathbb{C}^2 - S, M)$ via the van Kampen method, then $(\Gamma_{i_1})_h \cdot \dots \cdot (\Gamma_{i_t})_h$ is also a relation.*

Corollary 10. *If R is any relation in $\pi_1(\mathbb{C}^2 - S, M)$, then R_ρ is also a relation in $\pi_1(\mathbb{C}^2 - S, M)$, where R_ρ is the relation induced from R by replacing Γ_j and $\Gamma_{j'}$ with $(\Gamma_j)_{\rho_j^{m_j}}$ and $(\Gamma_{j'})_{\rho_j^{m_j}}$ respectively for $1 \leq j \leq 27$, where $\rho = \prod_{j=1}^{27} \rho_{m_j}$ and $\rho_{m_j} = Z^{m_j}_{jj'}$.*

Proof. By Theorem 7. □

Corollary 11. *$\pi_1(\mathbb{C}^2 - S, M)$ satisfies all the relations induced in $R(\Delta^2_{54})$, all the relations induced from $(R(\Delta^2_{54}))_\rho$ and all relations induced from the complex conjugation of Δ^2_{54}.*

Proof. By Theorems 7 and 8. □

3. $\pi_1(\mathbb{C}^2 - S, M)$ and $\tilde{\pi}_1$

3.1. The van Kampen Theorem. The van Kampen Theorem induces a finite presentation of the fundamental group of complements of curves by meaning of generators and relations.

We obtained the regenerated braid monodromy factorization $\Delta^2_{54} = \prod_{i=1}^{9} C_i H_{V_i}$. We have to apply the van Kampen Theorem on the paths, which correspond to the factors in Δ^2_{54}. We take any path from k to l, cut it in M, then towards k along the path,

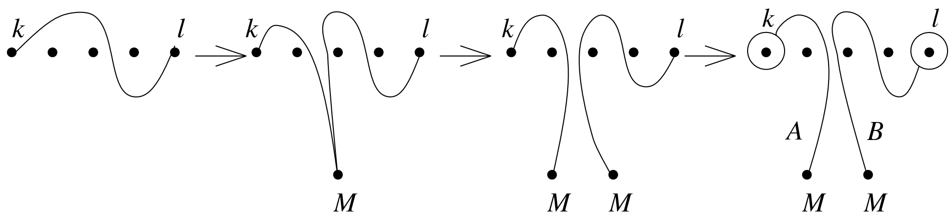


Fig. 5.

around k and coming back the same way. Consider A as an element of the fundamental group. Do the same to l to obtain B. So A, B are conjugations of Γ_k and Γ_l respectively, see Fig. 5.

Lemma 12 ([40]). *$\{A, B\}$ can be extended to a g-base of $\pi_1(\mathbb{C}_u - S, M)$ denoted by $\{\Gamma_j, \Gamma_{j'}\}_{j=1}^{27}$.*
Moreover, there is an epimorphism $\pi_1(\mathbb{C}_u - S, M) \to \pi_1(\mathbb{C}^2 - S, M)$.

Theorem 13 (van Kampen for cuspidal curves). *Let S be the regenerated cuspidal branch curve, u, M, φ, A, B defined as above. Let $\{\delta_i\}$ be a g-base of $\pi_1(E - N, u)$. Let $\varphi(\delta_i) = V_i^{\nu_i}$, V_i be a halftwist, $\nu_i = 1, 2, 3$. Let $\{\Gamma_j, \Gamma_{j'}\}_{i=1}^{27}$ be a g-base for $\pi_1(\mathbb{C}_u - S, M)$. Then: $\pi_1(\mathbb{C}^2 - S, M)$ is generated by the images of Γ_j, $\Gamma_{j'}$ in $\pi_1(\mathbb{C}^2 - S, M)$ (denoted also by Γ_j, $\Gamma_{j'}$) and we get a complete set of relations from those induced from $\varphi(\delta_i) = V_i^{\nu_i}$, as follows (when A, B are expressed in terms of $\{\Gamma_j, \Gamma_{j'}\}$):*
(a). *$A = B$, when $\nu_i = 1$;*
(b). *$[A, B] = 1$, when $\nu_i = 2$;*
(c). *$\langle A, B\rangle = ABABAB = 1$, when $\nu_i = 3$.*

Recall that

$$\tilde{\pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, M)}{\langle \Gamma_j^2, \Gamma_{j'}\rangle}.$$

Let us apply the theorem on three examples, taken from the table corresponding to H_{V_2}. We follow Figs. 6, 7, 8.

Following Fig. 6, A and B are conjugations of Γ_5 and $\Gamma_{5'}$ respectively. All factors in the conjugations are with a positive exponent since $\Gamma_j = \Gamma_j^{-1}$ and $\Gamma_{j'} = \Gamma_{j'}^{-1}$ in $\tilde{\pi}_1$.

We construct B by proceeding from M towards $5'$ above $6'$ and 6 encircling $5'$ counterclockwise and proceeding back above 6 and $6'$. Therefore $B = \Gamma_{6'}\Gamma_6\Gamma_{5'}\Gamma_6\Gamma_{6'} = (\Gamma_{5'})^{\Gamma_6\Gamma_{6'}}$. In a similar way, $A = \Gamma_5\Gamma_{2'}\Gamma_2\Gamma_{1'}\Gamma_1\Gamma_5\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{2'}\Gamma_5 = (\Gamma_5)^{\Gamma_1\Gamma_{1'}\Gamma_2\Gamma_{2'}\Gamma_5}$.

This path is related to a braid which is induced from a branch point. Therefore by Theorem 13, $A = B$. The derived relation is: $(\Gamma_5)^{\Gamma_1\Gamma_{1'}}\Gamma_2\Gamma_{2'}\Gamma_5 = (\Gamma_{5'})^{\Gamma_6\Gamma_{6'}}$.

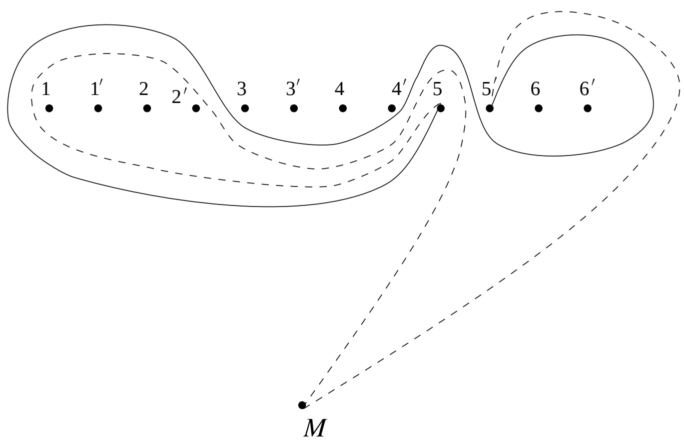


Fig. 6.

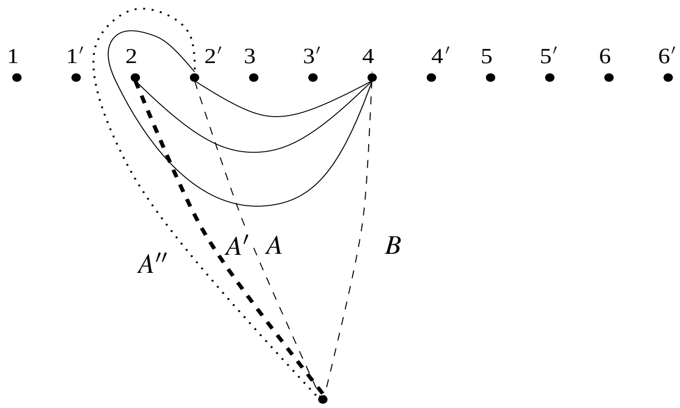


Fig. 7.

Following the same technique in Fig. 7 we get three relations according to three paths. These paths represent the three braids derived from three cusps. The relations are: $\langle \Gamma_{2'}, \Gamma_4 \rangle = 1$, $\langle \Gamma_2, \Gamma_4 \rangle = 1$, $\langle (\Gamma_{2'})^{\Gamma_2}, \Gamma_4 \rangle = 1$

Following the same technique in Fig. 8, we get two relations corresponding to the two paths. These paths represent the two braids derived from two nodes. The relations are: $[\Gamma_5, (\Gamma_2)^{\Gamma_{1'}\Gamma_1}] = 1$ and $[\Gamma_5, (\Gamma_{2'})^{\Gamma_{1'}\Gamma_1}] = 1$.

Since we followed the process on the point V_2, we continue and present the list of relations for V_2. We get an infinite number of relations by Corollary 10. We present partially the infinite list for V_2, following the list of paths and braids involved in H_{V_2} (Subsection 2.2). Recall that the generators involved in this cer-

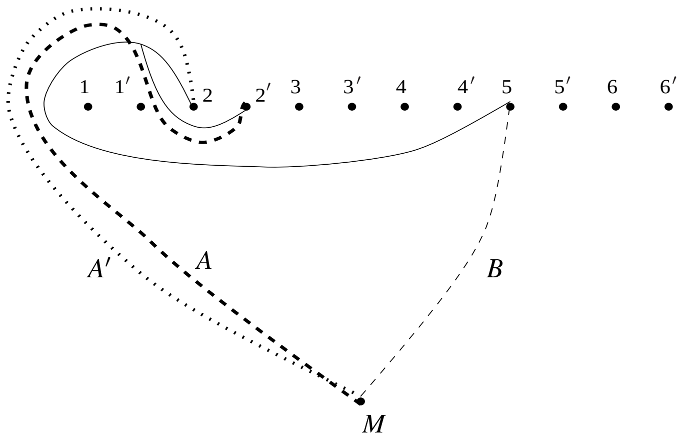


Fig. 8.

tain case are $\{\Gamma_j, \Gamma_{j'}\}_{i=1,3,7,9,14,17}$ (see [4, Fig. 6]) which are numerated locally as $\{\Gamma_j, \Gamma_{j'}\}_{i=1,2,3,4,5,6}$. Moreover, we add the relations $\Gamma_j^2 = \Gamma_{j'}^2 = 1$.

3.2. Relations H_{V_2}. The following relations are induced from the factors in H_{V_2} and their paths in the first table.

Relation[1] : $\langle \Gamma_3, \Gamma_4 \rangle = 1$
Relation[2] : $\langle \Gamma_3, \Gamma_{4'} \rangle = 1$
Relation[3] : $\langle \Gamma_{3'}, \Gamma_4 \rangle = 1$
Relation[4] : $\langle \Gamma_{3'}, \Gamma_{4'} \rangle = 1$
Relation[5] : $[\Gamma_1, \Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}] = 1$
Relation[6] : $[\Gamma_1, \Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}] = 1$
Relation[7] : $[\Gamma_{1'}, \Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}] = 1$
Relation[8] : $[\Gamma_{1'}, \Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}] = 1$
Relation[9] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5] = 1$
Relation[10] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}] = 1$
Relation[11] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}] = 1$
Relation[12] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5^{\Gamma_{5'}}] = 1$
Relation[13] : $[\Gamma_{4'}^{(\Gamma_4 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5] = 1$
Relation[14] : $[\Gamma_{4'}^{(\Gamma_4 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}^{\Gamma_5}] = 1$
Relation[15] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_6] = 1$
Relation[16] : $[\Gamma_4^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{6'}] = 1$
Relation[17] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_6] = 1$
Relation[18] : $[\Gamma_{4'}^{(\Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{6'}] = 1$
Relation[19] : $\langle \Gamma_2, \Gamma_4 \rangle = 1$
Relation[20] : $\langle \Gamma_2, \Gamma_{4'} \rangle = 1$
Relation[21] : $\langle \Gamma_{2'}, \Gamma_4 \rangle = 1$

Relation[22] : $\langle \Gamma_{2'}, \Gamma_{4'} \rangle = 1$
Relation[23] : $[\Gamma_1, \Gamma_4] = 1$
Relation[24] : $[\Gamma_1, \Gamma_{4'}] = 1$
Relation[25] : $[\Gamma_{1'}, \Gamma_4] = 1$
Relation[26] : $[\Gamma_{1'}, \Gamma_{4'}] = 1$
Relation[27] : $[\Gamma_4^{(\Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5^{\Gamma_{5'}}] = 1$
Relation[28] : $[\Gamma_4^{(\Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}] = 1$
Relation[29] : $[\Gamma_{4'}^{(\Gamma_4 \cdot \Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5] = 1$
Relation[30] : $[\Gamma_{4'}^{(\Gamma_4 \cdot \Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}] = 1$
Relation[31] : $[\Gamma_4^{(\Gamma_{4'} \cdot \Gamma_4 \cdot \Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_5] = 1$
Relation[32] : $[\Gamma_4^{(\Gamma_{4'} \cdot \Gamma_4 \cdot \Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{5'}^{\Gamma_5}] = 1$
Relation[33] : $[\Gamma_4^{(\Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_6] = 1$
Relation[34] : $[\Gamma_4^{(\Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{6'}] = 1$
Relation[35] : $[\Gamma_{4'}^{(\Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_6] = 1$
Relation[36] : $[\Gamma_{4'}^{(\Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)}, \Gamma_{6'}] = 1$
Relation[37] : $\Gamma_4^{(\Gamma_{3'} \cdot \Gamma_3 \cdot \Gamma_{2'} \cdot \Gamma_2)} = \Gamma_{4'}$
Relation[38] : $\langle \Gamma_5, \Gamma_6 \rangle = 1$
Relation[39] : $\langle \Gamma_5, \Gamma_{6'} \rangle = 1$
Relation[40] : $\langle \Gamma_{5'}, \Gamma_6 \rangle = 1$
Relation[41] : $\langle \Gamma_{5'}, \Gamma_{6'} \rangle = 1$
Relation[42] : $[\Gamma_3^{\Gamma_4}, \Gamma_5] = 1$
Relation[43] : $[\Gamma_3^{\Gamma_{4'}}, \Gamma_{5'}] = 1$
Relation[44] : $[\Gamma_{3'}^{\Gamma_4}, \Gamma_5] = 1$
Relation[45] : $[\Gamma_{3'}^{\Gamma_{4'}}, \Gamma_{5'}] = 1$
Relation[46] : $[\Gamma_2, \Gamma_5] = 1$
Relation[47] : $[\Gamma_2, \Gamma_{5'}] = 1$
Relation[48] : $[\Gamma_{2'}, \Gamma_5] = 1$
Relation[49] : $[\Gamma_{2'}, \Gamma_{5'}] = 1$
Relation[50] : $\langle \Gamma_1, \Gamma_5 \rangle = 1$
Relation[51] : $\langle \Gamma_1, \Gamma_{5'} \rangle = 1$
Relation[52] : $\langle \Gamma_{1'}, \Gamma_5 \rangle = 1$
Relation[53] : $\langle \Gamma_{1'}, \Gamma_{5'} \rangle = 1$
Relation[54] : $[\Gamma_3^{\Gamma_4}, \Gamma_{5'}^{(\Gamma_6 \cdot \Gamma_{6'} \cdot \Gamma_5)}] = 1$
Relation[55] : $[\Gamma_{3'}^{\Gamma_4}, \Gamma_{5'}^{(\Gamma_6 \cdot \Gamma_{6'} \cdot \Gamma_5)}] = 1$
Relation[56] : $[\Gamma_3^{(\Gamma_4 \cdot \Gamma_{4'} \cdot \Gamma_4)}, \Gamma_5^{(\Gamma_6 \cdot \Gamma_{6'} \cdot \Gamma_5 \cdot \Gamma_{5'} \cdot \Gamma_5)}] = 1$
Relation[57] : $[\Gamma_{3'}^{(\Gamma_4 \cdot \Gamma_{4'} \cdot \Gamma_4)}, \Gamma_5^{(\Gamma_6 \cdot \Gamma_{6'} \cdot \Gamma_5 \cdot \Gamma_{5'} \cdot \Gamma_5)}] = 1$
Relation[58] : $[\Gamma_2^{(\Gamma_{1'} \cdot \Gamma_1)}, \Gamma_5] = 1$
Relation[59] : $[\Gamma_2^{(\Gamma_{1'} \cdot \Gamma_1)}, \Gamma_{5'}] = 1$
Relation[60] : $[\Gamma_{2'}^{(\Gamma_{1'} \cdot \Gamma_1)}, \Gamma_5] = 1$
Relation[61] : $[\Gamma_{2'}^{(\Gamma_{1'} \cdot \Gamma_1)}, \Gamma_{5'}] = 1$

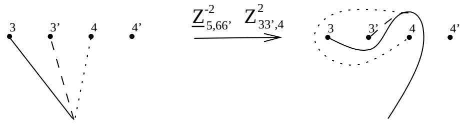
Relation[62] : $\Gamma_5^{(\Gamma_1 \cdot \Gamma_{1'} \cdot \Gamma_2 \cdot \Gamma_{2'} \cdot \Gamma_5)} = \Gamma_{5'}^{(\Gamma_6 \cdot \Gamma_{6'})}$
Relation[63] : $\Gamma_1 \cdot \Gamma_1 = 1$
Relation[64] : $\Gamma_{1'} \cdot \Gamma_{1'} = 1$
Relation[65] : $\Gamma_2 \cdot \Gamma_2 = 1$
Relation[66] : $\Gamma_{2'} \cdot \Gamma_{2'} = 1$
Relation[67] : $\Gamma_3 \cdot \Gamma_3 = 1$
Relation[68] : $\Gamma_{3'} \cdot \Gamma_{3'} = 1$
Relation[69] : $\Gamma_4 \cdot \Gamma_4 = 1$
Relation[70] : $\Gamma_{4'} \cdot \Gamma_{4'} = 1$
Relation[71] : $\Gamma_5 \cdot \Gamma_5 = 1$
Relation[72] : $\Gamma_{5'} \cdot \Gamma_{5'} = 1$
Relation[73] : $\Gamma_6 \cdot \Gamma_6 = 1$
Relation[74] : $\Gamma_{6'} \cdot \Gamma_{6'} = 1.$

3.3. Relations $\hat{F}_1(\hat{F}_1)^{\rho^{-1}}$. $\left(\hat{F}_1(\hat{F}_1)^{\rho^{-1}}\right)^{\underline{Z}^{-2}_{5,66'} Z^2_{33',4}}$ are conjugated by $\underline{Z}^{-2}_{5,66'} Z^2_{33',4}$. Therefore, the corresponding elements A, B in this case are conjugations of the corresponding Γ_j, $\Gamma_{j'}$ by $\underline{Z}^{-2}_{5,66'} Z^2_{33',4}$, $1 \leq j \leq 6$.
We denote the conjugated Γ_j and $\Gamma_{j'}$ by $\overline{\Gamma}_j$ and $\overline{\Gamma}_{j'}$ respectively.

$\overline{\Gamma}_j = \Gamma_j$ for $j = 1, 1', 2, 2', 4'$.

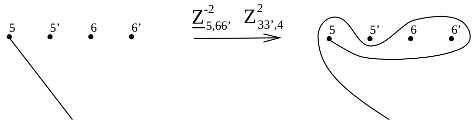
We can view the other ones in the following figures.

$\underline{\overline{\Gamma}_3, \overline{\Gamma}_{3'}, \overline{\Gamma}_4}$:



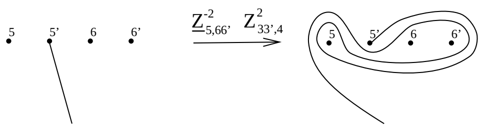
$\overline{\Gamma}_3 = \Gamma_3^{\Gamma_4}$
$\overline{\Gamma}_{3'} = \Gamma_{3'}^{\Gamma_4}$
$\overline{\Gamma}_4 = \Gamma_4^{(\Gamma_3 \cdot \Gamma_{3'} \cdot \Gamma_4)}$.

$\underline{\overline{\Gamma}_5}$:



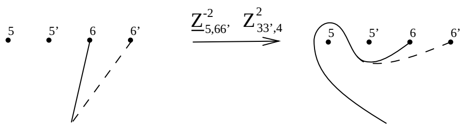
$\overline{\Gamma}_5 = \Gamma_5^{(\Gamma_{6'} \cdot \Gamma_6 \cdot \Gamma_5)}$.

$\underline{\overline{\Gamma}_{5'}}$:



$\overline{\Gamma}_{5'} = \Gamma_{5'}^{(\Gamma_6 \cdot \Gamma_{6'} \cdot \Gamma_5 \cdot \Gamma_{6'} \cdot \Gamma_6 \cdot \Gamma_5)}$.

$\underline{\bar{\Gamma}_6, \bar{\Gamma}_{6'}}$:



$\bar{\Gamma}_6 = \Gamma_6^{\Gamma_5}$
$\bar{\Gamma}_{6'} = \Gamma_{6'}^{\Gamma_5}$.

The following relations are induced from the factors in $\left(\hat{F}_1(\hat{F}_1)^{\rho^{-1}}\right)^{\underline{Z}^{-2}_{5,66'} Z^2_{33',4}}$ and their paths in the second table of H_{V_2}.

Relation[1] : $\langle \bar{\Gamma}_2, \bar{\Gamma}_3 \rangle = 1$
Relation[2] : $\langle \bar{\Gamma}_2, \bar{\Gamma}_{3'} \rangle = 1$
Relation[3] : $\langle \bar{\Gamma}_{2'}, \bar{\Gamma}_3 \rangle = 1$
Relation[4] : $\langle \bar{\Gamma}_{2'}, \bar{\Gamma}_{3'} \rangle = 1$
Relation[5] : $\langle \bar{\Gamma}_2, \bar{\Gamma}_{3'}^{\bar{\Gamma}_3} \rangle = 1$
Relation[6] : $\langle \bar{\Gamma}_{2'}, \bar{\Gamma}_{3'}^{\bar{\Gamma}_3} \rangle = 1$
Relation[7] : $[\bar{\Gamma}_{3'}, \bar{\Gamma}_6] = 1$
Relation[8] : $[\bar{\Gamma}_3^{\Gamma_{3'}}, \bar{\Gamma}_{6'}] = 1$
Relation[9] : $[\bar{\Gamma}_3, \bar{\Gamma}_{6'}^{\bar{\Gamma}_6}] = 1$
Relation[10] : $[\bar{\Gamma}_3^{(\bar{\Gamma}_2 \cdot \bar{\Gamma}_{2'} \cdot \bar{\Gamma}_3)}, \bar{\Gamma}_6] = 1$
Relation[11] : $[\bar{\Gamma}_{3'}^{(\bar{\Gamma}_2 \cdot \bar{\Gamma}_{2'} \cdot \bar{\Gamma}_{3'})}, \bar{\Gamma}_{6'}] = 1$
Relation[12] : $[\bar{\Gamma}_{3'}^{(\bar{\Gamma}_3 \cdot \bar{\Gamma}_2 \cdot \bar{\Gamma}_{2'} \cdot \bar{\Gamma}_3 \cdot \bar{\Gamma}_{3'} \cdot \bar{\Gamma}_3)}, \bar{\Gamma}_{6'}^{\bar{\Gamma}_6}] = 1$
Relation[13] : $\langle \bar{\Gamma}_2^{\bar{\Gamma}_3}, \bar{\Gamma}_6 \rangle = 1$
Relation[14] : $\langle \bar{\Gamma}_2^{\bar{\Gamma}_{3'}}, \bar{\Gamma}_{6'} \rangle = 1$
Relation[15] : $\langle \bar{\Gamma}_{2'}^{\bar{\Gamma}_3}, \bar{\Gamma}_6 \rangle = 1$
Relation[16] : $\langle \bar{\Gamma}_{2'}^{\bar{\Gamma}_{3'}}, \bar{\Gamma}_{6'} \rangle = 1$
Relation[17] : $\langle \bar{\Gamma}_2^{(\bar{\Gamma}_3 \cdot \bar{\Gamma}_{3'} \cdot \bar{\Gamma}_3)}, \bar{\Gamma}_{6'}^{\bar{\Gamma}_6} \rangle = 1$
Relation[18] : $\langle \bar{\Gamma}_{2'}^{(\bar{\Gamma}_3 \cdot \bar{\Gamma}_{3'} \cdot \bar{\Gamma}_3)}, \bar{\Gamma}_{6'}^{\bar{\Gamma}_6} \rangle = 1$
Relation[19] : $\bar{\Gamma}_1 = \bar{\Gamma}_{2'}^{(\bar{\Gamma}_3 \cdot \bar{\Gamma}_6)}$
Relation[20] : $\bar{\Gamma}_{1'} = \bar{\Gamma}_2^{(\bar{\Gamma}_{2'} \cdot \bar{\Gamma}_3 \cdot \bar{\Gamma}_6)}$
Relation[21] : $\bar{\Gamma}_1 = \bar{\Gamma}_{2'}^{(\bar{\Gamma}_3 \cdot \bar{\Gamma}_{3'} \cdot \bar{\Gamma}_3 \cdot \bar{\Gamma}_6 \cdot \bar{\Gamma}_{6'} \cdot \bar{\Gamma}_6)}$
Relation[22] : $\bar{\Gamma}_{1'} = \bar{\Gamma}_2^{(\bar{\Gamma}_{2'} \cdot \bar{\Gamma}_3 \cdot \bar{\Gamma}_{3'} \cdot \bar{\Gamma}_3 \cdot \bar{\Gamma}_6 \cdot \bar{\Gamma}_{6'} \cdot \bar{\Gamma}_6)}$

3.4. Relations C_i. In the same way as above, we derive relations from the braids C_i. The list appears in [4, Section 4.11] in a global numeration of the generators.

4. Results

In this paper we presented the list of braids and the braid monodromy factorization Δ^2_{54} corresponding to the branch curve S of $T \times T$. Recall that S compounds nine curves, each one of them is treated in [4]. The computations in detail appear in [4, Chapter 3].

We quoted the van Kampen Method and Theorems and we used them to obtain relations for the fundamental group $\pi_1(\mathbb{C}^2 - S, M)$. We presented some relations, the complete list appears in [4, Chapter 4]. Setting $\Gamma_j^2 = \Gamma_{j'}^2 = 1$, we can get a presentation for the group $\tilde{\pi}_1$.

Through this paper we concentrated on the 6-point V_2 and showed complete computations concerning this point. As quoted above, all other computations appear in [4, Chapters 3 and 4].

In [7] we compute the fundamental group $\pi_1((T \times T)_{Gal})$ of the Galois cover of $T \times T$ with respect to a generic projection to $\mathbb{CP}^2$. Recall that the fundamental group $\pi_1((T \times T)_{Gal}^{Aff})$ is the kernel of the surjection $\psi\colon \tilde{\pi}_1 \to S_{18}$.

The Galois cover is a surface of a general type. In [7] we verify the Bogomolov Conjecture. Bogomolov conjectured that if a surface of a general type has a positive index, then it has an infinite fundamental group.

Let X be a surface, $f\colon X \to \mathbb{CP}^2$ is a generic projection, $S \subset \mathbb{CP}^2$ its branch curve, $m = \deg S$, d = # nodes in S, ρ = # of cusps in S, μ = # of tangency points in S for a generic projection to $\mathbb{CP}^1$, $n = \deg f$. Then

$$C_1^2(X_{Gal}) = \frac{n!}{4}(m-6)^2,$$
$$C_2(X_{Gal}) = n!\left(\frac{m^2}{2} - \frac{3m}{2} + 3 - \frac{3d}{4} - \frac{4\rho}{3}\right)$$

and the index

$$\tau = \frac{1}{3}\big(C_1^2(X_{Gal}) - 2C_2(X_{Gal})\big),$$

see [16, pp. 603–604].

We compute the index of $(T \times T)_{Gal}$.

By the above computations, $n = 18$, $\mu = 54$, $\rho = 216$, $d = 1080$, $m = 54$. $C_1^2(X_{Gal}) = 576 \cdot 18!$ and $C_2(X_{Gal}) = 282 \cdot 18!$. Therefore $\tau(X_{Gal}) = 4 \cdot 18!$. The index is positive and the group we derive in [7] is infinite.

5. Notations

$(A)_B = B^{-1}AB = A^B$.

X an algebraic surface, $X \subseteq \mathbb{CP}^n$.

X_0 a degenerated object of a surface X, $X_0 \subset \mathbb{CP}^N$.

S an algebraic curve defined over $\mathbb{R}$, $S \subset \mathbb{C}^2$.

E (resp. D) be a closed disk on the x-axis (resp. y-axis) with the center on the real part of the x-axis (resp. y-axis), such that $\{\text{singularities of } \pi_1\} \subseteq E * (D - \partial D)$.
$\pi : S \to E$.
$K(x) = \pi^{-1}(x)$.
$N = \{x \in E : \#K(x) < n\}$.
u real number such that $x \ll u \quad \forall x \in N$.
$\mathbb{C}_u = \pi^{-1}(u)$.
$\rho_j = Z_{jj'}$.
φ = the braid monodromy of an algebraic curve S in M.
$B_p[D, K]$ = the braid group.
$\Delta_p^2 = (H_1 \cdots H_{p-1})^p$.
$\pi_1(\mathbb{C}^2 - S, M)$ = the fundamental group of a complement of a branch curve S.
$\tilde{\pi}_1 = \frac{\pi_1(\mathbb{C}^2 - S, M)}{\langle \Gamma_j^2, \Gamma_{j'}^2 \rangle}$.
T = complex torus.
$\underline{z}_{ij}$ (resp. $\overline{z}_{ij}$) = a path from q_i to q_j below (resp. above) the real line.
The corresponding halftwists are: $H(\underline{z}_{ij}) = \underline{Z}_{ij}$; $H(\overline{z}_{ij}) = \overline{Z}_{ij}$.

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