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ON MILNOR MOVES AND ALEXANDER POLYNOMIALS OF KNOTS

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1. Introduction

Recently, several local moves of knots and links were defined and studied actively in many papers, for example [2], [5], [7], and [8].

In this paper, we define a new local move on knot diagram called a Milnor move of order n or simply an *M_n-move*. Namely, let k be an oriented knot in an oriented 3-space R^3 and let B^3 be a 3-ball in R^3 such that $k \cap B^3$ is the tangle illustrated in Fig. 1. The transformation from Fig. 1(a) to 1(b) is called an M_n^+-move and that from Fig. 1(b) to 1(a) is called an M_n^--move. Furthermore an M_n-move means either an M_n^+-move or an M_n^--move. For two knots k, k' in R^3, k is said to be *M_n-equivalent to k'* or k and k' are said to be *M_n-equivalent* if k can be transformed into k' by a finite sequence of M_n-moves, [5].

In [6], Milnor introduced the Milnor link. Namely a link L is called the Milnor link if L is transformed into a trivial link by an M_2-move. Now we generalize this move to an M_n-move for any positive integer $n\,(\geq 2)$.

Almost local moves known up to the present change the knot cobordism, [1]. But we will see that an M_n-move does not change the knot cobordism for any integer $n\,(\geq 2)$, see Proposition.

In Section 2, we study a relation between the Alexander polynomials of M_n-equivalent knots and a property of M_n-equivalence of knots and prove Theorems 1 and 2.

A relation of Alexander polynomials of cobordant knots was known in [1]. The result we obtain in Theorem 1 is more concrete than that of [1] for cobordant knots which are M_n-equivalent. Theorems 1 and 2 give a classification of cobordant knots by an M_n-move.

For a knot k, $\Delta_k(t)$ means the Alexander polynomial of k.

Theorem 1. *For two knots k, k' and an integer $n \geq 2$, if k is M_n-equivalent to k', then*

$$\prod_{i=1}^{u}\{(1-t)^n-(-t)^{p_i}\}\{(1-t)^n-(-t)^{q_i}\}\Delta_k(t)$$

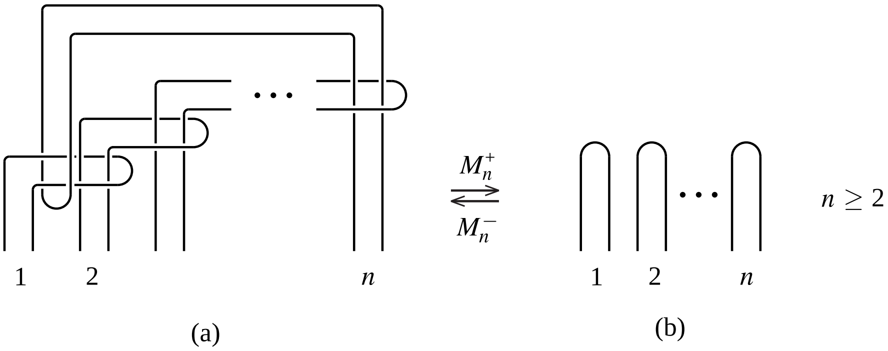

Fig. 1.

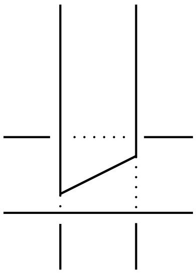

Fig. 2.

$$= \pm t^s \prod_{j=1}^{v} \{(1-t)^n - (-t)^{r_j}\}\{(1-t)^n - (-t)^{s_j}\}\Delta_{k'}(t)$$

for some integers s, u, v, p_i, q_i, r_j *and* s_j, $0 \leq p_i, q_i, r_j, s_j \leq n$, $p_i + q_i = r_j + s_j = n$.

Theorem 2. *For two knots* k, k' *and an integer* $n \geq 2$, *let* k *be* M_n*-equivalent to* k'. *Then* k *is not* M_m*-equivalent to* k' *for any integer* $m\,(\neq n) \geq 2$.

A knot k is a ribbon knot if k bounds a singular disk with only so-called ribbon singularities, Fig. 2. Moreover it is easily seen that k is a ribbon knot if and only if k ($\subset R^3[0]$) bounds a non-singular locally flat disk which does not have minimal points in the half space $R^4_+ = \{(x, y, z, t) \in R^4 \mid t \geq 0\}$ of R^4, where $R^3[a] = \{(x, y, z, t) \in R^4 \mid t = a\}$. (If k bounds a non-singular locally flat disk in R^4_+, k is called a slice knot.)

If k can be transformed into a trivial knot by a finite sequence of M_n^+-moves, we see that k is a ribbon knot, Proposition, and so we can use Theorem 1 to classify rib-

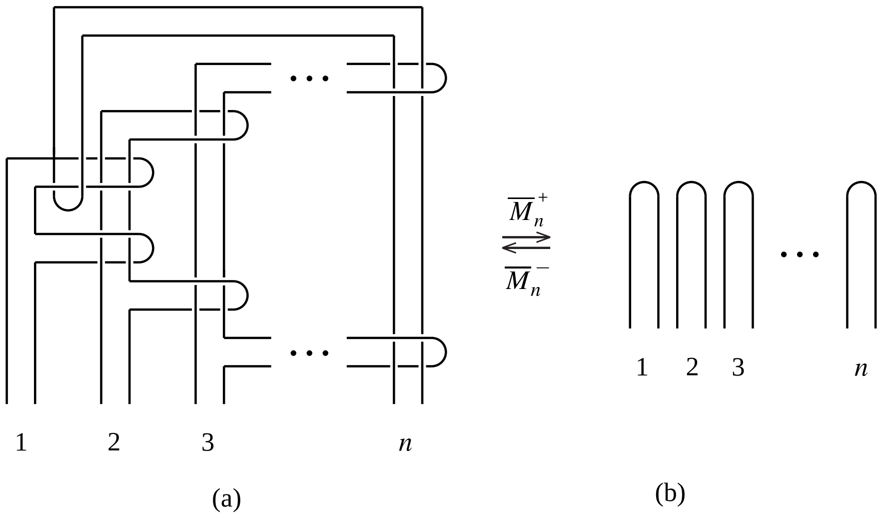

Fig. 3.

bon knots by M_n- moves. Indeed, we will classify almost all prime ribbon knots up to 10 crossing points by Theorem 1 in Section 3.

2. Properties of M_n-moves

In this section, we study some properties of M_n-moves and prove Theorems. We prepare Lemmas 1 and 2 to prove Theorem 1.

To calculate the Alexander polynomial of M_n-equivalent knots, let us define a local move, called $\overline{M}_n^{\pm}$ -moves. The tangle transformation from Fig. 3(a) to 3(b) is called an $\overline{M}_n^+$-move and that of Fig. 3(b) to 3(a) is called an $\overline{M}_n^-$-move.

Lemma 1. (1) *An M_n^+ (or M_n^-)-move can be realized by an $\overline{M}_n^+$ (resp. $\overline{M}_n^-$)-move.*
(2) *An $\overline{M}_n^+$ (or $\overline{M}_n^-$)-move can be realized by an M_n^+ (resp. M_n^-)-move.*

Proof. (1) By the deformations illustrated in Fig. 4, we obtain (1).
(2) We easily see (2) by the definitions of these moves. □

Lemma 2. *For two knots k, k' and an integer $n\,(\geq 2)$, if k can be transformed into k' by an M_n^+-move, then*

$$\Delta_k(t) = \pm t^r\{(1-t)^n - (-t)^p\}\{(1-t)^n - (-t)^q\}\Delta_{k'}(t)$$

for some integers p, q and r, $0 \leq p, q \leq n$, $p+q=n$.

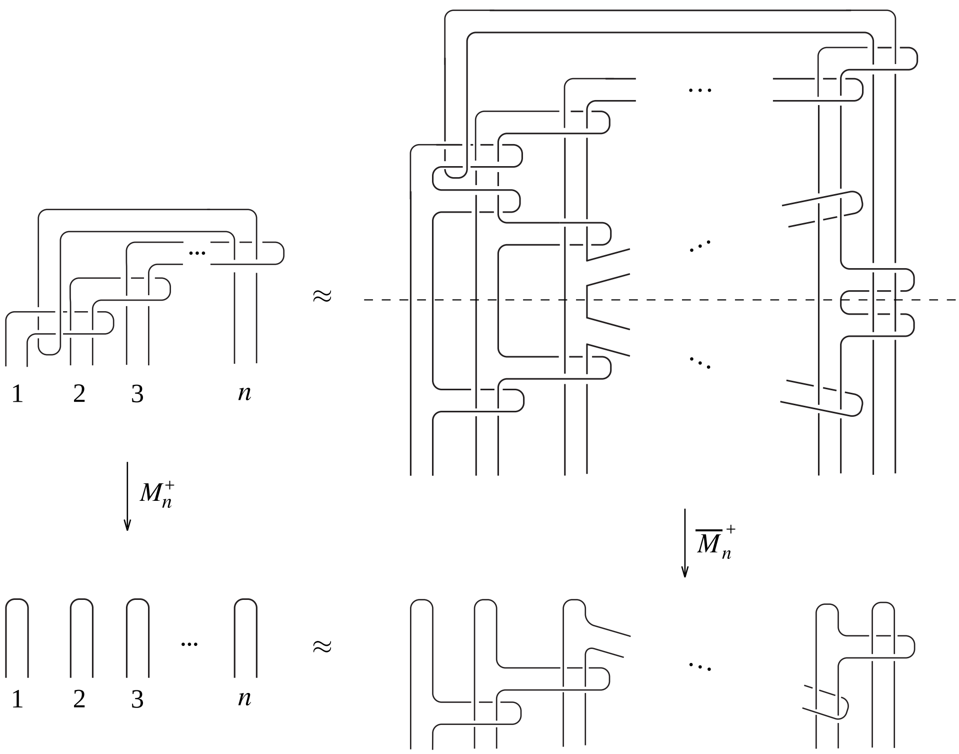

Fig. 4.

Proof. Suppose that k can be transformed into k' by an M_n^+-move, hence by an $\overline{M}_n^+$-move by Lemma 1. Namely k can be ambient isotopic to the band sum of k' and an n-component trivial link $\mathcal{L}_n$, by n bands, say $B_1, \ldots, B_n$, and let us span n disks $D_1, \ldots, D_n$ with singularities, say d_1, d_{21}, $d_{22}, \ldots, d_{n1}$, d_{n2} of ribbon type to $\mathcal{L}_n$, where $d_1 = D_1 \cap D_2$, $d_{i1} \cup d_{i2} = D_i \cap D_{i+1}$ for $2 \leq i \leq n-1$ and $d_{n1} \cup d_{n2} = D_n \cap B_1$, Fig. 5(a).

Performing an orientation preserving cut along d_1 and attach a tube T_i along a subdisk of D_{i+1} or B_1 for $2 \leq i \leq n$, Fig. 5(b). Hence we obtain an orientable surface $F_1 \cup \cdots \cup F_n$, where F_1 is obtained from $D_1 \cup B_1$ by an orientation preserving cut along d_1 and $F_i = (D_i - N(d_{i1} \cup d_{i2} : D_i)) \cup T_i \cup B_i$ for $2 \leq i \leq n$, where $N(x : X)$ means the regular neighborhood of x in X.

Let F' be an orientable surface of k'. If the singularity of $F' \cap F_i$ is not empty, it consists of arcs of ribbon type of $F' \cap B_i$. Performing the orientation preserving cut along these arcs for each i, we obtain an orientable surface F of k.

To calculate $\Delta_k(t)$ of k, we take a set of basis of the first homology $H_1(F)$ of F including a_i, b_i illustrated in Fig. 6. Let M be a Seifert matrix of k and hence $\Delta_k(t)$ is the following, where a_i^+, b_j^+ mean the lift of a_i, b_j respectively over the positive side of F_i.

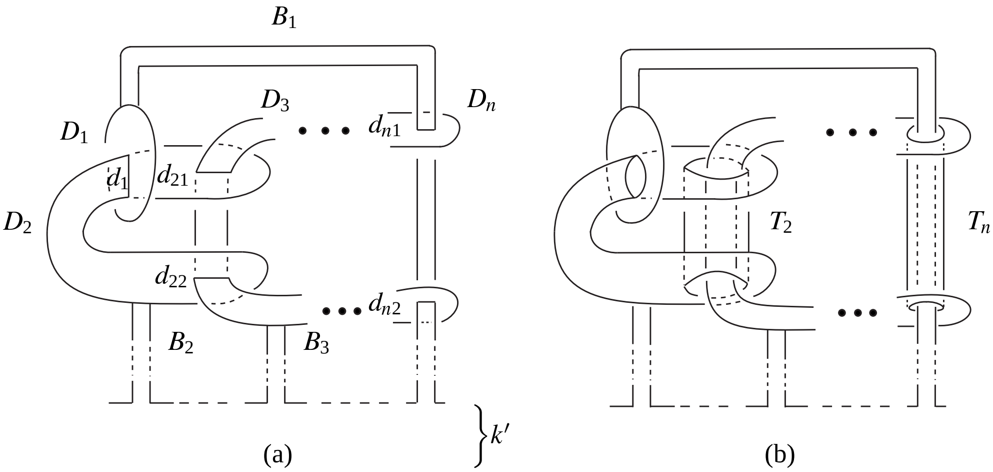

Fig. 5.

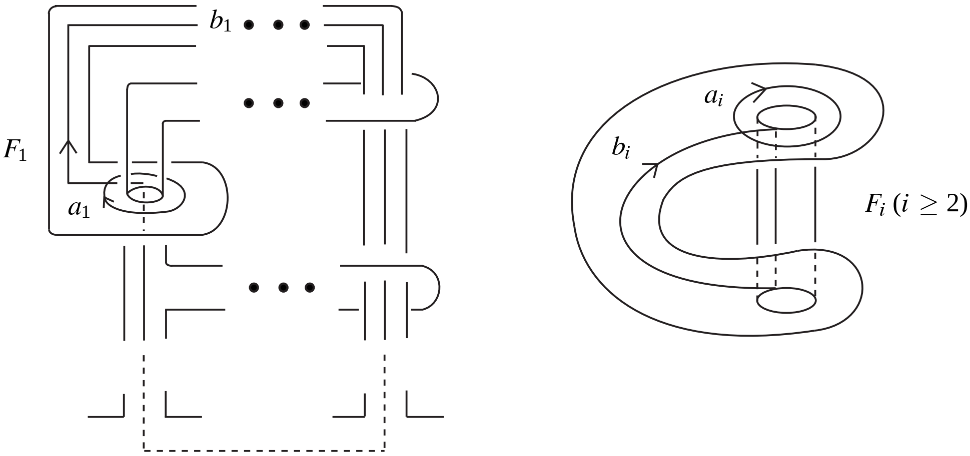

Fig. 6.

$$\Delta_k(t) = |M - tM'|$$

$$= \begin{array}{cc} & \begin{array}{ccccccccc} a_1^+ & \cdots & a_{n-1}^+ & a_n^+ & b_1^+ & b_2^+ & \cdots & b_n^+ & \end{array} \\ \begin{array}{c} a_1 \\ \vdots \\ a_{n-1} \\ a_n \\ b_1 \\ b_2 \\ \vdots \\ b_n \\ \\ \end{array} & \left| \begin{array}{cccc|cccc|c} & & & & \epsilon_1 t^{\delta_1} & t-1 & & 0 & \\ & & 0 & & & \ddots & \ddots & & 0 \\ & & & & & 0 & \ddots & t-1 & \\ & & & & t-1 & & & \epsilon_n t^{\delta_n} & \\ \hline -\epsilon_1 t^{1-\delta_1} & & & t-1 & & & & & \\ t-1 & \ddots & 0 & & & & & & \\ & \ddots & \ddots & & & * & & & * \\ 0 & & t-1 & -\epsilon_n t^{1-\delta_n} & & & & & \\ \hline & 0 & & & & * & & & \Delta_{k'}(t) \end{array} \right| \end{array},$$

where $\delta_i = 0$, $\epsilon_i = 1$ or $\delta_i = 1$, $\epsilon_i = -1$. Let us denote $p = \delta_1 + \cdots + \delta_n$ and $q = n - p$. Then $\epsilon_1 \cdots \epsilon_n = (-1)^p$ and $(-1)^n \epsilon_1 \cdots \epsilon_n = (-1)^q$. Therefore

$$\begin{aligned} \Delta_k(t) &= \{(-1)^{n-1}(t-1)^n + (-t)^p\}\{(-1)^{n-1}(t-1)^n + (-t)^q\}\Delta_{k'}(t) \\ &= \{(1-t)^n - (-t)^p\}\{(1-t)^n - (-t)^q\}\Delta_{k'}(t). \end{aligned} \qquad \square$$

Let k, k' be those of Lemma 2. Then k' can be transformed into k by an M_n^--move. Hence we easily obtain Theorem 1 by Lemmas 1 and 2.

Now, we apply Lemma 2 for $n = 2$, 3 and 4.

Corollary 1. *Suppose that a knot K can be transformed into a trivial knot by a finite sequence of M_n^+-moves.*

(1) *If* $n = 2$, $\Delta_K(t) = \pm t^r \prod_{i,j} (t-2)^{m_i}(2t-1)^{m_i}(t^2 - t + 1)^{2n_j}$.

(2) *If* $n = 3$, $\Delta_K(t) = \pm t^r \prod_{i,j} (t^2 - 3t + 3)^{m_i}(3t^2 - 3t + 1)^{m_i}$
$\times (t^3 - 3t^2 + 2t - 1)^{n_j}(t^3 - 2t^2 + 3t - 1)^{n_j}$.

(3) *If* $n = 4$, $\Delta_K(t) = \pm t^r \prod_{i,j,k} (t^3 - 4t^2 + 6t - 4)^{m_i}(4t^3 - 6t^2 + 4t - 1)^{m_i}$
$\times (t^4 - 4t^3 + 6t^2 - 3t + 1)^{n_j}(t^4 - 3t^3 + 6t^2 - 4t + 1)^{n_j}$
$\times (t^4 - 4t^3 + 5t^2 - 4t + 1)^{2l_k}$.

Proof. We apply to Lemma 2 in the following cases respectively. If $n = 2$, we consider the case that $p_i = 0$, $q_i = 2$ and $p_i = q_i = 1$. If $n = 3$, we do the cases that $p_i = 0$, $q_i = 3$ and $p_i = 1$, $q_i = 2$. If $n = 4$, we do the cases that $p_i = 0$, $q_i = 4$ and $p_i = 1$, $q_i = 3$ and $p_i = q_i = 2$. $\square$

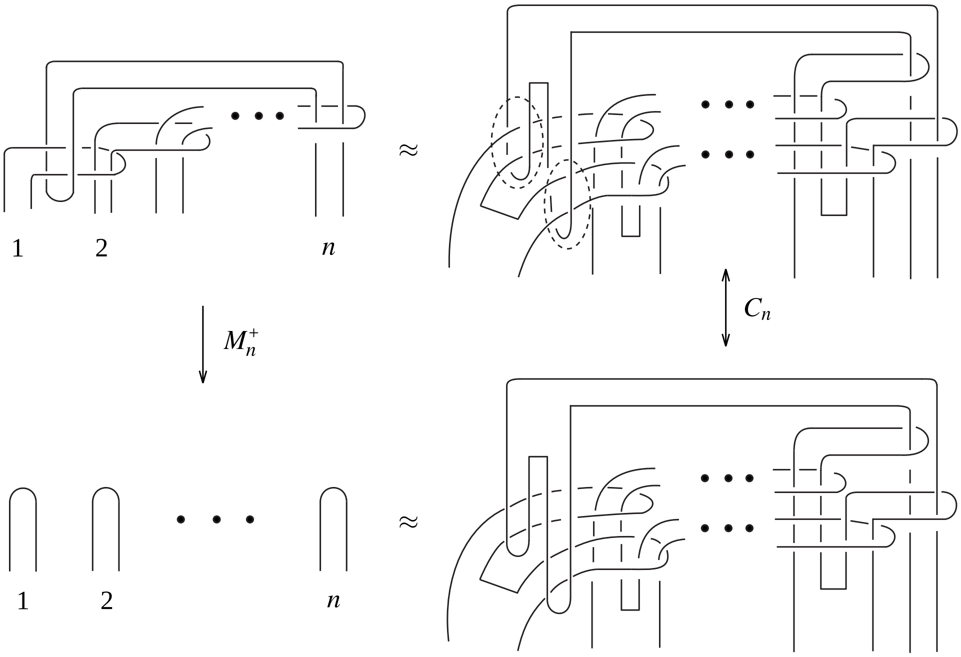

Fig. 7.

Let $\nabla_K(z)$ be the Conway polynomial of K. It is well-known that $\nabla_K(t - t^{-1}) = \Delta_K(t^2)$. Therefore we easily obtain the following.

Corollary 2. (1) *If K can be transformed into a trivial knot by a finite sequence of M_2^+-moves,* $\nabla_K(z) = \prod_{i,j}(1 - 2z^2)^{m_i}(1 + z^2)^{2n_j}$.
(2) *If K can be transformed into a trivial knot by a finite sequence of M_3^+-moves,* $\nabla_K(z) = \prod_{i,j}(1 + 3z^4)^{m_i}(1 - z^4 - z^6)^{n_j}$.

K. Habiro introduced a local move, called the C_n-move, [2], [7]. We see that an M_n-move can be realized by a finite sequence of C_n-moves as illustated in Fig. 7, which is also obtained by the result of [2]. But the converse is false by Example 1.

EXAMPLE 1. For any integer $n \geq 2$, there is a knot k_n which is C_n-equivalent to a trivial knot $\mathcal{O}$ (namely k_n can be transformed into $\mathcal{O}$ by a finite sequence of C_n-moves) but not M_n-equivalent to $\mathcal{O}$. For example, let k_n be the knot illustrated in Fig. 8. Then we easily see that k_n is C_n-equivalent to $\mathcal{O}$. Suppose that k_n is M_n-equivalent to $\mathcal{O}$. Then we obtain that $\Delta_{k_n}(-1) = \pm(2^n - 1)^{2m}$ for an integer m by putting $t = -1$ in Theorem 1. On the other hand, we obtain that $\Delta_{k_n}(t) = (t - 1)^{2(n-1)} \pm t^{n-1}$ by calculating the determinant of Seifert matrix of k_n. Hence

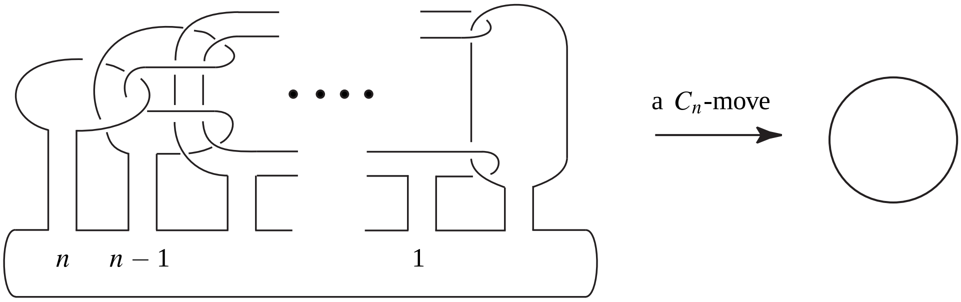

Fig. 8.

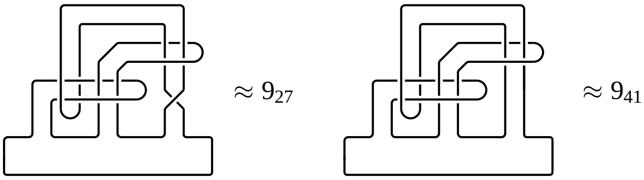

Fig. 9.

$\Delta_{k_n}(-1) = 2^{2(n-1)} \pm (-1)^{n-1} \neq \pm(2^n - 1)^{2m}$, which is a contradiction.

EXAMPLE 2. By the projections of ribbon knots in [4], we easily see that 6_1, 8_{20}, 9_{46} and 10_{140} are M_2-equivalent to a trivial knot $\mathcal{O}$. Since the knots in Fig. 9 are ambient isotopic to 9_{27} and 9_{41} respectively, 9_{27} and 9_{41} are M_3-equivalent to $\mathcal{O}$.

Next let us prove Theorem 2.

Proof of Theorem 2. Suppose that there is an integer $m\,(\neq n) \geq 2$ such that k is M_m-equivalent to k'. Then we obtain that

$$\begin{aligned}&\prod_{i=1}^{u}\{(1-t)^n - (-t)^{p_i}\}\{(1-t)^n - (-t)^{q_i}\}\Delta_k(t)\\ &= \pm t^s \prod_{j=1}^{v}\{(1-t)^n - (-t)^{r_j}\}\{(1-t)^n - (-t)^{s_j}\}\Delta_{k'}(t)\end{aligned}$$

and

$$\begin{aligned}&\prod_{i=1}^{U}\{(1-t)^m - (-t)^{P_i}\}\{(1-t)^m - (-t)^{Q_i}\}\Delta_k(t)\\ &= \pm t^S \prod_{j=1}^{V}\{(1-t)^m - (-t)^{R_j}\}\{(1-t)^m - (-t)^{S_j}\}\Delta_{k'}(t)\end{aligned}$$

for some integers s, u, v, p_i, q_i, r_j and s_j, $0 \leq p_i, q_i, r_j, s_j \leq n$, $p_i + q_i = r_j + s_j = n$ and S, U, V, P_i, Q_i, R_j and S_j, $0 \leq P_i, Q_i, R_j, S_j \leq m$, $P_i + Q_i = R_j + S_j = m$ by Theorem 1. By putting $t = -1$, we obtain that $(2^n - 1)^{2u}\alpha = \pm(2^n - 1)^{2v}\beta$, $(2^m - 1)^{2U}\alpha = \pm(2^m - 1)^{2V}\beta$, where $\alpha = \Delta_k(-1)$ and $\beta = \Delta_{k'}(-1)$. Therefore we obtain that $(2^n - 1)^p = (2^m - 1)^q$ for some integers p, q.

But we may show that it is a contradiction in the following. We suppose that there exist m, n, p, q with $n > m \geq 2$ such that $(2^n - 1)^p = (2^m - 1)^q$. Let $p = as$ and $q = bt$, where $a, b \in \{2^i\}_{i=0}^{\infty}$ and integers s, t are odd. After replacing (p, q) by (q, p), we can assume that $a \geq b$ and $c = a/b \in \{2^i\}_{i=0}^{\infty}$. Then we have $(2^n - 1)^{cs} = (2^m - 1)^t$. Since s, t are odd and $2^n > 2^m \geq 4$, we have $(-1)^c \equiv (-1)^{cs} \equiv (-1)^t \equiv -1 \pmod 4$. Thus $c = 1$, so $(2^n - 1)^s = (2^m - 1)^t$. Let $A = 2^m - 1$. Then we have

$$A^t = (2^m - 1)^t = (2^n - 1)^s \equiv (-1)^s \equiv -1 \pmod{2^n}. \tag{1}$$

Squaring the above, we have

$$A^{2t} \equiv 1 \pmod{2^n}. \tag{2}$$

Now, since $(A, 2^n) = 1$, by Euler's Theorem (cf. [3, p. 33]) we have

$$A^{\phi(2^n)} \equiv 1 \pmod{2^n}, \tag{3}$$

where $\phi(2^n)$ is Euler's phi function (the number of positive integers prime to 2^n and $\leq 2^n$). Since $\phi(2^n) = 2^{n-1}$ and $(2t, 2^{n-1}) = 2$, (2) and (3) imply $A^2 \equiv 1 \pmod{2^n}$. Since $n \geq 3$, this equation has 4 solutions $A \equiv \pm 1,\ 2^{n-1} \pm 1 \pmod{2^n}$. But, by (1) it has only $A \equiv -1 \pmod{2^n}$, so $2^m \equiv 0 \pmod{2^n}$. Hence $m \geq n$. This is a contradiction. □

3. A classification of ribbon knots by M_n-moves

For two knots $k(\subset R^3[a])$ and $k'(\subset R^3[b])$ for $a < b$, if there is a non-singular locally flat annulus $\mathcal{A}$ in $R^3[a, b]$ with $\mathcal{A} \cap R^3[a] = k$ and $\mathcal{A} \cap R^3[b] = -k'$, we say that k is cobordant to k', [1]. Hence if k is cobordant to a trivial knot $\mathcal{O}$, k is a slice knot and moreover if $\mathcal{A}$ does not have minimal points, k is a ribbon knot.

Proposition. *For two knots k, k' and an integer $n\,(\geq 2)$, if k is M_n-equivalent to k', then k is cobordant to k'.*

Proof. Since k is M_n-equivalent to k', there are knots $k_0(= k), k_1, \ldots, k_p(= k')$ such that k_i can be transformed into k_{i+1} by an M_n^+-move or an M_n^--move. Suppose that k_i is contained in $R^3[2i]$ for $i = 0, 1, \ldots, p$.

If we perform a hyperbolic transformation, Fig. 10, to k_i (or k_{i+1}) in $R^3[2i + 1]$ and obtain k_{i+1} (resp. k_i) and a trivial knot split from k_{i+1} (resp. k_i).

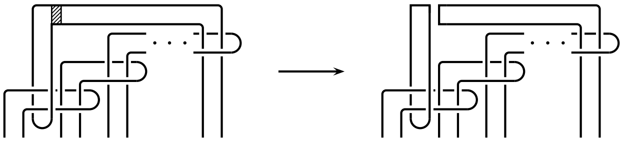

Fig. 10.

Performing the above discussion to each i, we obtain a non-singular locally flat annulus $\mathcal{A}$ in $R^3[0, 2p]$ with $\partial\mathcal{A} = k \cup (-k')$, namely k is cobordant to k'. □

Hence if k can be transformed into a trivial knot by a finite sequence of M_n (or M_n^+)-moves, k is a slice (resp. a ribbon) knot. Therefore if k is not a slice knot, k is not M_n-equivalent to a trivial knot $\mathcal{O}$.

In this section, we consider the following by using Theorem 1: Are the prime ribbon knots up to 10 crossing points M_n-equivalent to $\mathcal{O}$ for some integer n (≥ 2)?

By Example 2, we already see that 6_1, 8_{20}, 9_{46} and 10_{140} are M_2-equivalent to $\mathcal{O}$ and that 9_{27} and 9_{41} are M_3-equivalent to $\mathcal{O}$.

ribbon knot	Alexander polynomial	M_2	M_3	M_n ($n \geq 4$)
6_1	$2t^2 - 5t + 2$	Y	N	N
8_8	$2t^4 - 6t^3 + 9t^2 - 6t + 2$	N	N	N
8_9	$t^6 - 3t^5 + 5t^4 - 7t^3 + 5t^2 - 3t + 1$	N	N	N
8_{20}	$(t^2 - t + 1)^2$	Y	N	N
9_{27}	$t^6 - 5t^5 + 11t^4 - 15t^3 + 11t^2 - 5t + 1$	N	Y	N
9_{41}	$3t^4 - 12t^3 + 19t^2 - 12t + 3$	N	Y	N
9_{46}	$2t^2 - 5t + 2$	Y	N	N
10_3	$6t^2 - 13t + 6$	N	N	N
10_{22}	$2t^6 - 6t^5 + 10t^4 - 13t^3 + 10t^2 - 6t + 2$	N	N	N
10_{35}	$2t^4 - 12t^3 + 21t^2 - 12t + 2$	N	N	N
10_{42}	$t^6 - 7t^5 + 19t^4 - 27t^3 + 19t^2 - 7t + 1$	N	N	N
10_{48}	$t^8 - 3t^7 + 6t^6 - 9t^5 + 11t^4 - 9t^3 + 6t^2 - 3t + 1$	N	N	N
10_{75}	$t^6 - 7t^5 + 19t^4 - 27t^3 + 19t^2 - 7t + 1$	N	N	N
10_{87}	$(t^2 - t + 1)^2(-2t^2 + 5t - 2)$	?	N	N
10_{99}	$(t^2 - t + 1)^4$	?	N	N
10_{123}	$(t^4 - 3t^3 + 3t^2 - 3t + 1)^2$	N	N	N
10_{129}	$2t^4 - 6t^3 + 9t^2 - 6t + 2$	N	N	N
10_{137}	$(t^2 - 3t + 1)^2$	N	N	N
10_{140}	$(t^2 - t + 1)^2$	Y	N	N
10_{153}	$t^6 - t^5 - t^4 + 3t^3 - t^2 - t + 1$	N	N	N
10_{155}	$t^6 - 3t^5 + 5t^4 - 7t^3 + 5t^2 - 3t + 1$	N	N	N

Here Y and N mean "yes" and "no" respectively.

Question. Are 10_{87} and 10_{99} M_2-equivalent to $\mathcal{O}$?

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