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Osaka J. Math.
40 (2003), 801–833

ON THE ADDITIVITY OF THE CLASP NUMBER OF KNOTS

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(Received February 12, 2002)

1. Introduction

It is of great interest to know whether a knot invariant is additive under connected sum. The genus $g(K)$ of a knot K is additive [13], that is, $g(K_1 \# K_2) = g(K_1) + g(K_2)$, where $K_1 \# K_2$ denotes a connected sum of two knots K_1 and K_2. For the braid index $b(K)$ of a knot K, Birman and Menasco [1] showed that $b(K) - 1$ is additive, that is, $b(K_1 \# K_2) - 1 = (b(K_1) - 1) + (b(K_2) - 1)$. For the tunnel number $t(K)$ of a knot K, it is known that $t(K)$ is not additive under connected sum. See, for example, [8], [9], [5]. In this paper, we study the additivity of the clasp number of knots which is defined in the following.

Let K be a knot in S^3. We denote by f an immersion $f\colon D \to S^3$ of a disc D into S^3 such that $f|_{\partial D}\colon \partial D \to K$ is a homeomorphism onto K. Let $\widetilde{\Sigma}$ denote the singular set $\{x \in f(D) \mid |f^{-1}(x)| \geq 2\}$ of the immersion f, and let Σ denote $f^{-1}(\widetilde{\Sigma})$ on D. The following lemma is a special case of Lemma 1 in [14]. See also Lemma 1 in [15].

Lemma 1.1. *We may choose an immersion f so that each connected component of Σ is an embedded arc on D joining a point in ∂D and a point in $\operatorname{int} D$.*

Note that this immersion f satisfies $\{x \in f(D) \mid |f^{-1}(x)| \geq 3\} = \emptyset$. An immersed disc $B = f(D)$ with these properties is called a *clasp disc* of K. Fig. 1.1 illustrates a clasp disc of a trefoil knot. Let $cp_B(K)$ denote the number of connected components of $\widetilde{\Sigma}$ in B. The minimal number of $cp_B(K)$ among all clasp discs B of K is called the *clasp number* of K, denoted by $cp(K)$. Shibuya defined also the clasp number of a link in S^3. See Definition 3 in [14]. We refer to Appendix for the clasp number of prime knots of eight or fewer crossings except 8_{18}. The following proposition is a special case of Theorem 1 in [14].

Proposition 1.2. *Let K be a knot in S^3. Then we have inequalities $cp(K) \geq g(K)$ and $cp(K) \geq u(K)$, where $u(K)$ denotes the unknotting number of K.*

The author is partially supported by JSPS Research Fellowships for Young Scientists.

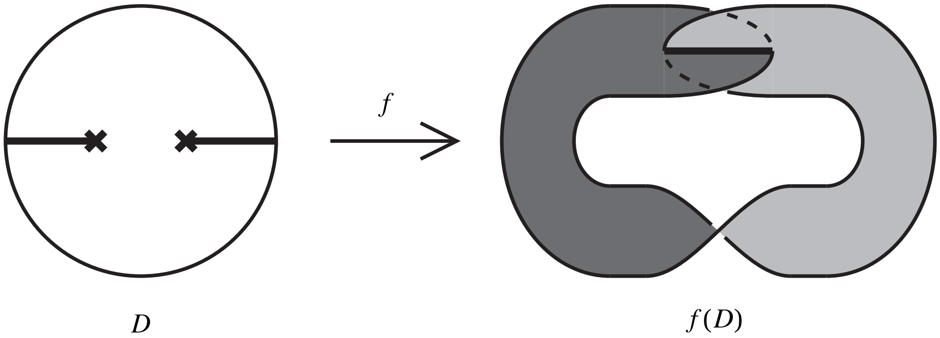


Fig. 1.1.

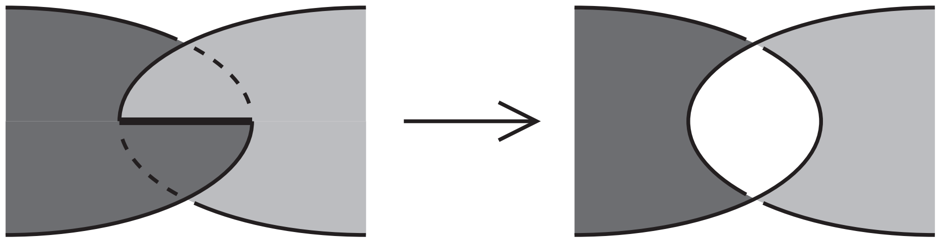


Fig. 1.2.

Fig. 1.2 illustrates a sketch of an operation to show the inequality $cp(K) \geq g(K)$. In [14], Shibuya called the operation, illustrated in Fig. 1.2, an orientation preserving cut along a clasp arc. This operation has been also called a smoothing.

Using Proposition 1.2, Morimoto [7] determined the clasp number of torus knots. Note that the genus of a torus knot of type (p, q) is $(|p| - 1)(|q| - 1)/2$. See, for example, Theorem 7.5.2 in [10]. Since the unknotting number of a torus knot of type (p, q) is equal to $(|p| - 1)(|q| - 1)/2$ (see [3], [4]), we obtain the following theorem.

Theorem 1.3. *Let K be a torus knot of type (p, q). Then the clasp number of K is $(|p| - 1)(|q| - 1)/2$, that is, $cp(K) = g(K) = u(K)$.*

Concerning the additivity of $cp(K)$, Morimoto made the following conjecture in [6].

Conjecture 1.4. $cp(K_1 \# K_2) = cp(K_1) + cp(K_2)$.

He obtained a partial solution to this conjecture in the same paper.

Theorem 1.5 ([6]). *If $cp(K_1 \# K_2) \leq 2$, then $cp(K_1 \# K_2) = cp(K_1) + cp(K_2)$.*

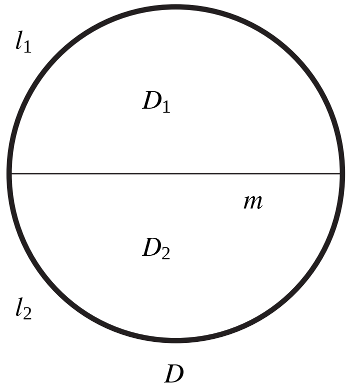


Fig. 2.1.

In this paper, we prove the following theorem.

Theorem 1.6. *Let K_1, K_2 be non-trivial knots. If $cp(K_1\#K_2) = 3$, then $cp(K_1) = 1$ and $cp(K_2) = 2$.*

Together with Theorem 1.5, we obtain the following corollary.

Corollary 1.7. *If $cp(K_1\#K_2) \leq 3$, then $cp(K_1\#K_2) = cp(K_1) + cp(K_2)$.*

2. Preliminary lemmas

Let $B = f(D)$ be a clasp disc of K with $cp_B(K) = cp(K)$. Let $K = K_1\#K_2$ denote a knot which is a connected sum of two non-trivial knots K_1 and K_2. Then there exists a 2-sphere S which realizes a non-trivial decomposition of $K = K_1\#K_2$. We may isotope S so that S intersects B and $\widetilde{\Sigma}$ transversely. Let T denote the set $f^{-1}(S \cap B)$ on D. Then T consists of a properly embedded arc m in D and some simple closed curves embedded in $int\,D$. Let D_1 and D_2 denote discs in D such that $D_1 \cap D_2 = m$ and $D_1 \cup D_2 = D$, and let l_i ($i = 1$ and 2) denote the arc $D_i \cap \partial D$. See Fig. 2.1. Let Q_i be the 3-ball which is bounded by S in S^3 and which contains the arc $f(l_i)$. Let k be a simple arc on S which connects the two points of $f(\partial D) \cap S = K \cap S$. We may regard the knot K_i as the union of two arcs k and $f(l_i)$. In the following, Z_i ($Z = D$, K, l, $\ldots$, etc.) denotes Z_1 or Z_2.

Loop components of T separate D_i to many regions. Let $\bar{D_i}$ denote the region in D_i separated by loop components of T such that l_i is a subarc of $\partial\bar{D_i}$. If there is no loop component of T in D_i, then $\bar{D_i}$ is D_i itself. Let g be the restriction of f to $\bar{D_i}$. Let $\bar{\Sigma_i}$ denote the set $\{x \in g(\bar{D_i}) \mid |g^{-1}(x)| \geq 2\}$, and let Σ_i denote $g^{-1}(\bar{\Sigma_i})$ on $\bar{D_i}$. By the definition of a clasp disc, a connected component of Σ_i on $\bar{D_i}$ belongs to one of arcs of the following four types (see Fig. 2.2);

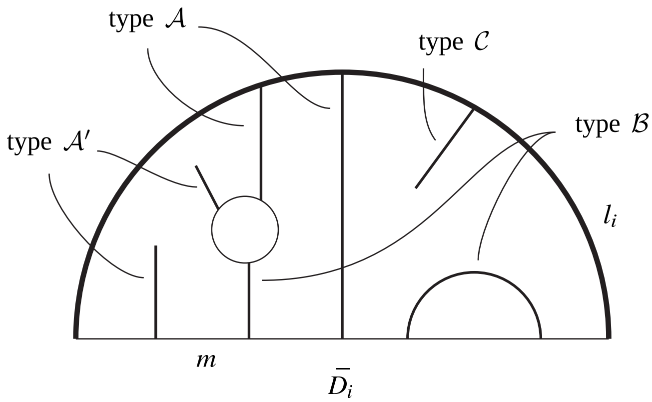


Fig. 2.2.

type $\mathcal{A}$: an arc which connects a point in l_i and a point in $cl(\partial\overline{D}_i - l_i)$,
type $\mathcal{A}'$: an arc which connects a point in $int\,\overline{D}_i$ and a point in $cl(\partial\overline{D}_i - l_i)$,
type $\mathcal{B}$: an arc which connects two distinct points in $cl(\partial\overline{D}_i - l_i)$,
type $\mathcal{C}$: an arc which connects a point in $int\,\overline{D}_i$ and a point in l_i.

Note that an arc of type $\mathcal{A}$ is identified by g with an arc of type $\mathcal{A}'$, that an arc of type $\mathcal{B}$ with another arc of type $\mathcal{B}$, and that an arc of type $\mathcal{C}$ with another arc of type $\mathcal{C}$. Let X_i denote the union of endpoints of arcs of types $\mathcal{A}'$ and $\mathcal{C}$ in $int\,\overline{D}_i$. The following lemma is essentially the same as Lemma 1 (2) in [6]. We refer to [6] for a proof.

Lemma 2.1. *Let α be an arc of type $\mathcal{A}$ on $\overline{D}_i$. Suppose that α and a subarc of m together with a subarc of l_i cobound a disc δ with $(int\,\delta)\cap\Sigma_i = \emptyset$ on $\overline{D}_i$. Then there are a surface D_i^* and an immersion $g^*\colon D_i^* \to Q_i$ satisfying the following properties;*
(i) *The surface D_i^* is homeomorphic to $\overline{D}_i$,*
(ii) *Every connected component of $(g^*)^{-1}(\{x \in g^*(D_i^*) \mid |(g^*)^{-1}(x)| \geq 2\}) = \Sigma_i^*$ belongs to an arc of type $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ or $\mathcal{C}$, where these arcs of four types are defined on D_i^* in the same way as they are on $\overline{D}_i$,*
(iii) *The numbers of arcs of types $\mathcal{B}$ and $\mathcal{C}$ in Σ_i^* are equal to those of types $\mathcal{B}$ and $\mathcal{C}$, respectively, in Σ_i,*
(iv) *The numbers of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ in Σ_i^* are strictly less than those of types $\mathcal{A}$ and $\mathcal{A}'$, respectively, in Σ_i, and*
(v) *There is a subarc l_i^* of ∂D_i^* such that $g^*(l_i^*) = g(l_i)$ and that $g^*(\partial D_i^* - l_i^*)$ is contained in S.*

Lemma 2.2. *Let β_1 and β_2 be arcs of type $\mathcal{B}$ on $\overline{D}_i$ with $g(\beta_1) = g(\beta_2)$. Suppose that β_1 and β_2 together with two subarcs of $\partial\overline{D}_i - l_i$ cobound a disc δ in $\overline{D}_i$. Then there are a surface D_i^* and an immersion $g^*\colon D_i^* \to Q_i$ satisfying the following* *properties*;

(i) *The surface* D_i^* *is homeomorphic to* $\overline{D}_i$,

(ii) *Every connected component of* $(g^*)^{-1}(\{x \in g^*(D_i^*) \mid |(g^*)^{-1}(x)| \geq 2\}) = \Sigma_i^*$ *belongs to an arc of type* $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ *or* $\mathcal{C}$, *where these arcs of four types are defined on* D_i^* *in the same way as they are on* $\overline{D}_i$,

(iii) *The numbers of arcs of types* $\mathcal{A}$, $\mathcal{A}'$ *and* $\mathcal{C}$ *in* Σ_i^* *are equal to those of types* $\mathcal{A}$, $\mathcal{A}'$ *and* $\mathcal{C}$, *respectively, in* Σ_i,

(iv) *The number of arcs of type* $\mathcal{B}$ *in* Σ_i^* *is strictly less than that of type* $\mathcal{B}$ *in* Σ_i, *and*

(v) *There is a subarc* l_i^* *of* ∂D_i^* *such that* $g^*(l_i^*) = g(l_i)$ *and that* $g^*(\partial D_i^* - l_i^*)$ *is contained in* S.

Proof. Let V denote a regular neighborhood $N(g(\beta_1); Q_i)$ of the arc $g(\beta_1) = g(\beta_2)$ in Q_i. We may choose V so that $g(\overline{D}_i) \cap V$ consists of two discs $g(N(\beta_1; \overline{D}_i))$ and $g(N(\beta_2; \overline{D}_i))$. The disc $g(N(\beta_1; \overline{D}_i))$ intersects $g(N(\beta_2; \overline{D}_i))$ transversely along the arc $g(\beta_1) = g(\beta_2)$. We regard V as the set $\{(x, y, z) \mid x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}$ so that two discs of $V \cap S$ correspond to $\{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ and $\{(x, y, 1) \mid x^2 + y^2 \leq 1\}$. We may assume that $g(N(\beta_1; \overline{D}_i))$ corresponds to $\{(x, 0, z) \mid -1 \leq x \leq 1,\ 0 \leq z \leq 1\}$, and that $g(N(\beta_2; \overline{D}_i))$ corresponds to $\{(0, y, z) \mid -1 \leq y \leq 1,\ 0 \leq z \leq 1\}$. We may also assume that $g(\delta) \cap V$ corresponds to the union of $\{(x, 0, z) \mid 0 \leq x \leq 1,\ 0 \leq z \leq 1\}$ and $\{(0, y, z) \mid 0 \leq y \leq 1,\ 0 \leq z \leq 1\}$. Let $\widetilde{d}_1$ be the disc $\{(x, y, z) \mid x^2 + y^2 = 1,\ x \geq 0,\ y \leq 0,\ 0 \leq z \leq 1\}$, and $\widetilde{d}_2$ be the disc $\{(x, y, z) \mid x^2 + y^2 = 1,\ x \leq 0,\ y \geq 0,\ 0 \leq z \leq 1\}$.

Now we define an immersion g^* of a surface D_i^* into Q_i. Let $g^*(D_i^*)$ be the immersed surface which is the union of $g(\overline{D}_i - N(\beta_1 \cup \beta_2; \overline{D}_i))$, $\widetilde{d}_1$ and $\widetilde{d}_2$. By this construction, $g^*(D_i^*)$ satisfies the properties (ii)–(v).

The surface D_i^* is the union of $\overline{D}_i - N(\beta_1 \cup \beta_2; \overline{D}_i)$, d_1 and d_2, where d_j ($j = 1$ and 2) is a disc corresponding to $\widetilde{d}_j$. Since $\widetilde{d}_1$ may be regarded as a band which connects two arcs $\{(1, 0, z) \mid 0 \leq z \leq 1\}$ and $\{(0, -1, z) \mid 0 \leq z \leq 1\}$, the disc d_1 may be regarded as a band which connects the subarc $c_{1,1}$ of $\partial N(\beta_1; \overline{D}_i)$ and the subarc $c_{1,2}$ of $\partial N(\beta_2; \overline{D}_i)$, where $g(c_{1,1}) = \{(1, 0, z) \mid 0 \leq z \leq 1\}$ and $g(c_{1,2}) = \{(0, -1, z) \mid 0 \leq z \leq 1\}$. Similarly, the disc d_2 may be regarded as a band which connects the subarc $c_{2,1}$ of $\partial N(\beta_1; \overline{D}_i)$ and the subarc $c_{2,2}$ of $\partial N(\beta_2; \overline{D}_i)$, where $g(c_{2,1}) = \{(-1, 0, z) \mid 0 \leq z \leq 1\}$ and $g(c_{2,2}) = \{(0, 1, z) \mid 0 \leq z \leq 1\}$. We notice that $g(\delta)$ is either an immersed annulus or an immersed Möbius band in Q_i, because $g(\beta_1) = g(\beta_2)$. This construction of D_i^* shows that D_i^* is homeomorphic to $\overline{D}_i$. $\square$

Fig. 2.3 (1) illustrates a sketch of the operation described in the proof of Lemma 2.2. Similar arguments as in the proof of Lemma 2.2 show the following two lemmas. See Fig. 2.3 (2) and (3) for sketches of operations to prove Lemmas 2.3 and 2.4, respectively.

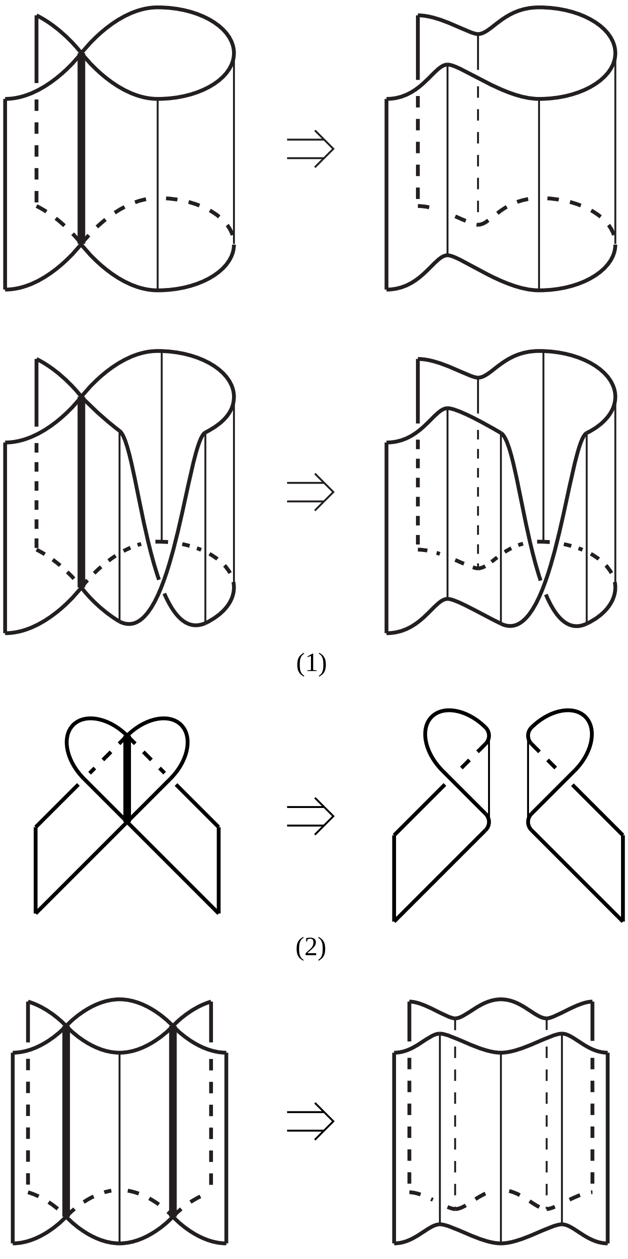


Fig. 2.3.

Lemma 2.3. *Let β_1 and β_2 be arcs of type $\mathcal{B}$ on $\overline{D}_i$ with $g(\beta_1) = g(\beta_2)$. Suppose that β_j ($j = 1$ and 2) and a subarc of $\partial\overline{D}_i$ cobound a disc δ_j in $\overline{D}_i$ such that $\delta_1 \cap \delta_2 = \emptyset$. Then there are a surface D_i^* and an immersion $g^*\colon D_i^* \to Q_i$ satisfying the following properties*;
(i) *The surface D_i^* is homeomorphic to $\overline{D}_i$,*
(ii) *Every connected component of $(g^*)^{-1}(\{x \in g^*(D_i^*) \mid |(g^*)^{-1}(x)| \geq 2\}) = \Sigma_i^*$ belongs to an arc of type $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ or $\mathcal{C}$, where these arcs of four types are defined on D_i^* in the same way as they are on $\overline{D}_i$,*
(iii) *The numbers of arcs of types $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{C}$ in Σ_i^* are equal to those of types $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{C}$, respectively, in Σ_i,*
(iv) *The number of arcs of type $\mathcal{B}$ in Σ_i^* is strictly less than that of type $\mathcal{B}$ in Σ_i, and*
(v) *There is a subarc l_i^* of ∂D_i^* such that $g^*(l_i^*) = g(l_i)$ and that $g^*(\partial D_i^* - l_i^*)$ is contained in S.*

Lemma 2.4. *Let β_1, β_2, γ_1 and γ_2 be arcs of type $\mathcal{B}$ on $\overline{D}_i$ with $g(\beta_j) = g(\gamma_j)$ for $j = 1$ and 2. Suppose that β_1, β_2 and two subarcs of $\partial\overline{D}_i$ cobound a disc d_β in $\overline{D}_i$, and that γ_1, γ_2 and two subarcs of $\partial\overline{D}_i - l_i$ cobound a disc d_γ in $\overline{D}_i$ such that $d_\beta \cap d_\gamma = \emptyset$. Suppose also that $g(d_\beta) \cup g(d_\gamma)$ forms an immersed annulus in Q_i. Then there are a surface D_i^* and an immersion $g^*\colon D_i^* \to Q_i$ satisfying the following properties*;
(i) *The surface D_i^* is homeomorphic to $\overline{D}_i$,*
(ii) *Every connected component of $(g^*)^{-1}(\{x \in g^*(D_i^*) \mid |(g^*)^{-1}(x)| \geq 2\}) = \Sigma_i^*$ belongs to an arc of type $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{B}$ or $\mathcal{C}$, where these arcs of four types are defined on D_i^* in the same way as they are on $\overline{D}_i$,*
(iii) *The numbers of arcs of types $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{C}$ in Σ_i^* are equal to those of types $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{C}$, respectively, in Σ_i,*
(iv) *The number of arcs of type $\mathcal{B}$ in Σ_i^* is strictly less than that of type $\mathcal{B}$ in Σ_i, and*
(v) *There is a subarc l_i^* of ∂D_i^* such that $g^*(l_i^*) = g(l_i)$ and that $g^*(\partial D_i^* - l_i^*)$ is contained in S.*

For a positive integer p and an immersion $g^p\colon \overline{D}_i^p \to Q_i$, l_i^p denotes, in the following, the subarc of $\partial\overline{D}_i^p$ with $g^p(l_i^p) = g(l_i)$, and Σ_i^p denotes the set $(g^p)^{-1}$ $(\{x \in g^p(D_i^p) \mid |(g^p)^{-1}(x)| \geq 2\})$ on D_i^p.

3. Proof of Theorem 1.6

In this section, we give a proof of Theorem 1.6 assuming propositions we prove in §§4 and 5. Suppose $cp(K) = 3$, so that Σ consists of six arcs $\sigma_1, \ldots, \sigma_6$ on D. Let x_j ($j = 1, \ldots, 6$) be the point $\partial\sigma_j \cap \operatorname{int} D$, and X be the union of the points $\{x_1, \ldots, x_6\}$. The following proposition is the same as Lemma 1 (3) in [6]. We refer to [6] for a proof.

Proposition 3.1. *Let α be a loop component of T, and δ_α be the disc bounded by α in D. Then there exists a 2-sphere S such that S realizes a non-trivial decomposition of $K = K_1 \# K_2$, and that $|\delta_\alpha \cap X| \geq 2$ for every loop component α of T.*

The following four propositions are proved in §4.

Proposition 4.1. *Suppose that $\bar{D_i} = D_i$ is a disc, and that the number of points of X_i on $\bar{D_i}$ is at most one. Then K_i is the trivial knot.*

Proposition 4.2. *Suppose that $\bar{D_i} = D_i$ is a disc, and that the number of points of X_i on $\bar{D_i}$ is two. Then the clasp number of K_i is at most one.*

Proposition 4.5. *Suppose that $\bar{D_i} = D_i$ is a disc, and that the number of points of X_i on $\bar{D_i}$ is three. Then the clasp number of K_i is at most one.*

Proposition 4.6. *Suppose that $\bar{D_i} = D_i$ is a disc, and that the number of points of X_i on $\bar{D_i}$ is four. Then the clasp number of K_i is at most two.*

The following four propositions are proved in §5.

Proposition 5.1. *Suppose that $\bar{D_i}$ is an annulus, and that the number of points of X_i on $\bar{D_i}$ is 0. Then K_i is the trivial knot.*

Proposition 5.3. *Suppose that $\bar{D_i}$ is an annulus, and that the number of points of X_i on $\bar{D_i}$ is one. Then the clasp number of K_i is at most one.*

Proposition 5.6. *Suppose that $\bar{D_i}$ is an annulus, and that the number of points of X_i on $\bar{D_i}$ is two. Then the clasp number of K_i is at most one.*

Proposition 5.14. *Suppose that $\bar{D_i}$ is a twice-punctured disc, and that the number of points of X_i on $\bar{D_i}$ is 0. Then the clasp number of K_i is at most one.*

By Propositions 3.1 and 4.1, we may suppose that $|D_1 \cap X| \geq 2$ and $|D_2 \cap X| \geq 2$. Without loss of generality, we may suppose that $(|D_1 \cap X|, |D_2 \cap X|) = (2, 4)$ or $(3, 3)$.

First suppose $|D_1 \cap X| = |D_2 \cap X| = 3$. Propositions 3.1, 4.5, 5.1, and 5.3 show that the clasp numbers of K_1 and K_2 are at most one. By the definition of the clasp number, we see that $cp(K_1 \# K_2) \leq cp(K_1) + cp(K_2)$. Hence $cp(K_1 \# K_2) \leq 2$. This contradicts our supposition.

Next suppose $|D_1 \cap X| = 2$ and $|D_2 \cap X| = 4$. Propositions 3.1, 4.2 and 5.1 show that the clasp number of K_1 is at most one. Propositions 3.1, 4.6, 5.1, 5.3, 5.6 and 5.14 show that the clasp number of K_2 is at most two. Therefore we have

$cp(K_1) = 1$ and $cp(K_2) = 2$.

This completes the proof of Theorem 1.6. □

4. The case where D_i contains no loop component of T

In this section, we deal with an immersed surface $g(\bar{D}_i)$ when there is no loop component of T in D_i. Therefore $\bar{D}_i = D_i$ is a disc.

Proposition 4.1. *Suppose that the number of points of X_i on D_i is at most one. Then K_i is the trivial knot.*

Proposition 4.1 is essentially the same as Claim 1 in [6].

Proof. First suppose $|X_i| = 0$. Then Σ_i consists only of arcs of type $\mathcal{B}$. By Lemma 2.3, we obtain an embedding g^1 of a disc D_i^1 into Q_i. This embedded disc $g^1(D_i^1)$ shows that K_i is the trivial knot.

Next suppose $|X_i| = 1$. Then Σ_i consists of one arc of type $\mathcal{A}$, one arc of type $\mathcal{A}'$ and some arcs of type $\mathcal{B}$. By Lemma 2.3, we obtain an immersion g^1 of a disc D_i^1 into Q_i such that Σ_i^1 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Then we obtain, by Lemma 2.1, an embedding g^2 of a disc D_i^2 into Q_i. This embedded disc $g^2(D_i^2)$ shows that K_i is the trivial knot. □

Proposition 4.2. *Suppose that the number of points of X_i on D_i is two. Then the clasp number of K_i is at most one.*

Proof. Since $|X_i| = 2$, Σ_i consists of either (1) two arcs of type $\mathcal{C}$ and some arcs of type $\mathcal{B}$, or (2) two arcs of type $\mathcal{A}$, two arcs of type $\mathcal{A}'$ and some arcs of type $\mathcal{B}$. In both cases, we obtain, by Lemma 2.3, an immersion g^1 of a disc D_i^1 into Q_i such that there is no arc of type $\mathcal{B}$ in Σ_i^1.

First suppose that Σ_i^1 consists of two arcs of type $\mathcal{C}$. Then the immersed disc $g^1(D_i^1)$ shows that the clasp number of K_i is at most one.

Next suppose that Σ_i^1 consists of two arcs of type $\mathcal{A}$ and two arcs of type $\mathcal{A}'$. Let α_1, α_2 be arcs of type $\mathcal{A}$, and α_1', α_2' be arcs of type $\mathcal{A}'$ such that $g^1(\alpha_j) = g^1(\alpha_j')$ for $j = 1$ and 2. Let m^1 be the arc $cl(\partial D_i^1 - l_i^1)$. When we proceed on m^1 from one endpoint of m^1, we may assume, by Lemma 2.1 and Proposition 4.1, that the first point of $\Sigma_i^1 \cap m^1$ we encounter is an endpoint of an arc of type $\mathcal{A}'$. Hence we may assume that the order of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ whose endpoints we encounter, when we proceed on m^1 from one endpoint of m^1, is either α_1', α_1, α_2, α_2', or α_1', α_2, α_1, α_2' in this order. If the order is α_1', α_1, α_2, α_2', then a configuration of the singular arc $g^1(m^1)$ on S is that of Fig. 4.1 (1) or (2), up to symmetry and isotopy on S. If the order is α_1', α_2, α_1, α_2', then a configuration of the singular arc $g^1(m^1)$ on S is that of Fig. 4.1 (3), up to symmetry and isotopy on S.

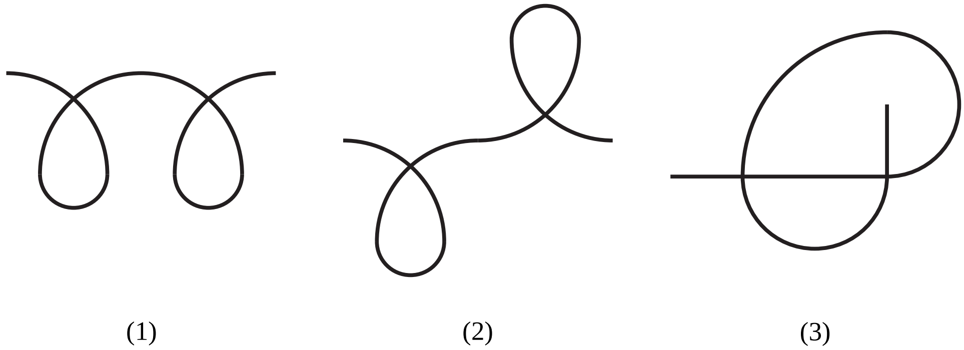


Fig. 4.1.

Lemma 4.3. *Suppose that a configuration of* $g^1(m^1)$ *on* S *is that of* Fig. 4.1 (1) *or* (2). *Then* K_i *is the trivial knot.*

Proof. Suppose that the configuration is that of Fig. 4.1 (1). Let k be a simple arc on S such that $k \cap g^1(m^1) = \partial k = g^1(\partial m^1)$. The immersed disc $g^1(D_i^1)$ implies that the singular arc $g^1(m^1)$ is a projection of the arc $g^1(l_i^1)$ to S fixing its boundary $g^1(\partial l_i^1)$. Therefore the union $k \cup g^1(m^1)$ may be regarded as a projection of K_i to S. This shows that the crossing number of K_i is at most two, so K_i is the trivial knot.

Similar arguments as above prove the case in the configuration of Fig. 4.1 (2). □

Lemma 4.4. *Suppose that a configuration of* $g^1(m^1)$ *on* S *is that of* Fig. 4.1 (3). *Then the clasp number of* K_i *is at most one.*

Proof. Let k be a simple arc on S which connects two points of $g^1(\partial m^1)$ and which intersects $g^1(int\, m^1)$ transversely in one point. The immersed disc $g^1(D_i^1)$ implies that the union $k \cup g^1(m^1)$ may be regarded as a projection of K_i to S. Therefore the crossing number of K_i is at most three, and the clasp number of K_i is at most one. See Appendix for the clasp number of prime knots of eight or fewer crossings except 8_{18}. □

This completes the proof of Proposition 4.2. □

Proposition 4.5. *Suppose that the number of points of* X_i *on* D_i *is three. Then the clasp number of* K_i *is at most one.*

Proof. Since $|X_i| = 3$, Σ_i consists of either (1) three arcs of type $\mathcal{A}$, three arcs of type $\mathcal{A}'$ and some arcs of type $\mathcal{B}$, or (2) one arc of type $\mathcal{A}$, one arc of type $\mathcal{A}'$, two arcs of type $\mathcal{C}$ and some arcs of type $\mathcal{B}$. In both cases, we obtain, by Lemma 2.3, an immersion g^1 of a disc D_i^1 into Q_i such that there is no arc of type $\mathcal{B}$ in Σ_i^1.

First suppose that Σ_i^1 consists of three arcs of type $\mathcal{A}$ and three arcs of type $\mathcal{A}'$. Similar arguments as in the proof of Proposition 4.2 show that the crossing number of K_i is at most four, and the clasp number of K_i is at most one.

Next suppose that Σ_i^1 consists of one arc of type $\mathcal{A}$, one arc of type $\mathcal{A}'$ and two arcs of type $\mathcal{C}$. Let α and α' denote these arcs of types $\mathcal{A}$ and $\mathcal{A}'$, respectively. Let V denote a regular neighborhood $N(g^1(\alpha); Q_i)$ of the arc $g^1(\alpha)$ in Q_i. We may choose V so that $g^1(D_i^1) \cap V$ consists of two discs $g^1(N(\alpha; D_i^1))$ and $g^1(N(\alpha'; D_i^1))$. The disc $g^1(N(\alpha; D_i^1))$ intersects $g^1(N(\alpha'; D_i^1))$ transversely along the arc $g^1(\alpha) = g^1(\alpha')$. We regard V as the set $\{(x, y, z) \mid x^2 + y^2 \leq 1,\ 0 \leq z \leq 2\}$ so that the disc $V \cap S$ corresponds to the set $\{(x, y, 0) \mid x^2 + y^2 \leq 1\}$. We suppose that $g^1(N(\alpha; D_i^1))$ corresponds to the set $\{(0, y, z) \mid -1 \leq y \leq 1,\ 0 \leq z \leq 1\}$, and that $g^1(N(\alpha'; D_i^1))$ corresponds to the set $\{(x, 0, z) \mid -1 \leq x \leq 1,\ 0 \leq z \leq 2\}$. See Fig. 4.2. We may suppose that an image of the outward-normal to D_i^1 in $N(\alpha; D_i^1)$ agrees with the direction of increasing x, and that an image of the outward-normal to D_i^1 in $N(\alpha'; D_i^1)$ agrees with the direction of increasing y. Let $\widetilde{d_1}$ be the disc $\{(x, y, z) \mid x^2 + y^2 = 1,\ x \geq 0,\ y \geq 0,\ 0 \leq z \leq 1\}$, and $\widetilde{d_2}$ be the disc $\{(x, y, z) \mid x^2 + y^2 = 1,\ x \leq 0,\ y \leq 0,\ 0 \leq z \leq 1\}$ in V. Let $\widetilde{d_3}$ denote the disc embedded in V which is the union of discs $\{(x, 0, z) \mid 2 - z \leq x \leq 1,\ 1 \leq z \leq 2\}$, $\{(x, y, z) \mid x^2 + y^2 = (2 - z)^2,\ x \geq 0,\ y \geq 0,\ 1 \leq z \leq 2\}$, $\{(0, y, z) \mid z - 2 \leq y \leq 2 - z,\ 1 \leq z \leq 2\}$, $\{(x, y, z) \mid x^2 + y^2 = (2 - z)^2,\ x \leq 0,\ y \leq 0,\ 1 \leq z \leq 2\}$ and $\{(x, 0, z) \mid -1 \leq x \leq z - 2,\ 1 \leq z \leq 2\}$. We note here that the arc $g^1(l_i^1) \cap V$ which corresponds to the set $\{(0, y, 1) \mid -1 \leq y \leq 1\}$ is disjoint from $\mathit{int}\,\widetilde{d_3}$, that $\widetilde{d_3} \cap \partial V$ is an arc consisting of the three subarcs $\widetilde{d_1} \cap \{(x, y, 1) \mid x^2 + y^2 = 1\}$, $\widetilde{d_2} \cap \{(x, y, 1) \mid x^2 + y^2 = 1\}$ and $g^1(\partial N(\alpha'; D_i^1)) \cap \{(x, y, z) \mid x^2 + y^2 \leq 1,\ 1 \leq z \leq 2\}$, and that $\partial\widetilde{d_3}$ consists of the two arcs $g^1(l_i^1) \cap V$ and $\widetilde{d_3} \cap \partial V$.

Now we define an immersion g^2 of a surface D_i^2 into Q_i. Let $g^2(D_i^2)$ be the immersed surface which is the union of $g^1(D_i^1 - (N(\alpha; D_i^1 - l_i^1) \cup N(\alpha'; D_i^1)))$, $\widetilde{d_1}$, $\widetilde{d_2}$ and $\widetilde{d_3}$. We say that $g^2(D_i^2)$ is obtained from $g^1(D_i^1)$ by a *CP surgery* along $g^1(\alpha)$. A CP surgery may be regarded as a detailed explanation of a smoothing operation, illustrated in Fig. 1.2, in a regular neighborhood of an endpoint of the clasp arc. By this construction, we see that Σ_i^2 consists of two arcs of type $\mathcal{C}$, and that there is a subarc l_i^2 of ∂D_i^2 with $g^2(l_i^2) = g^1(l_i^1) = g(l_i)$. Now we investigate the surface D_i^2 in detail. The surface D_i^2 is the union of $D_i^1 - (N(\alpha; D_i^1 - l_i^1) \cup N(\alpha'; D_i^1))$, d_1, d_2 and d_3, where d_j ($j = 1$, 2 and 3) is a disc corresponding to $\widetilde{d_j}$. Let $\widetilde{c_1}$ be the arc $\{(0, 1, z) \mid 0 \leq z \leq 1\}$ in V, and $\widetilde{c_2}$ be the arc $\{(0, -1, z) \mid 0 \leq z \leq 1\}$ in V. Note that one endpoint of $\widetilde{c_p}$ ($p = 1$, 2) is contained in S. Let c_p denote the arc on D_i^1 with $g^1(c_p) = \widetilde{c_p}$. Let $\widetilde{\gamma_{1,1}}$, $\widetilde{\gamma_{1,2}}$, $\widetilde{\gamma_{2,1}}$ and $\widetilde{\gamma_{2,2}}$ be the arcs $\{(1, 0, z) \mid 0 \leq z \leq 1\}$, $\{(1, 0, z) \mid 1 \leq z \leq 2\}$, $\{(-1, 0, z) \mid 0 \leq z \leq 1\}$ and $\{(-1, 0, z) \mid 1 \leq z \leq 2\}$ in V, respectively. Note that one endpoint of $\widetilde{\gamma_{q,1}}$ ($q = 1, 2$) is contained in S. Let $\gamma_{q,r}$ ($q = 1$, 2; $r = 1$, 2) denote the arc on D_i^1 with $g^1(\gamma_{q,r}) = \widetilde{\gamma_{q,r}}$. We may suppose, without loss

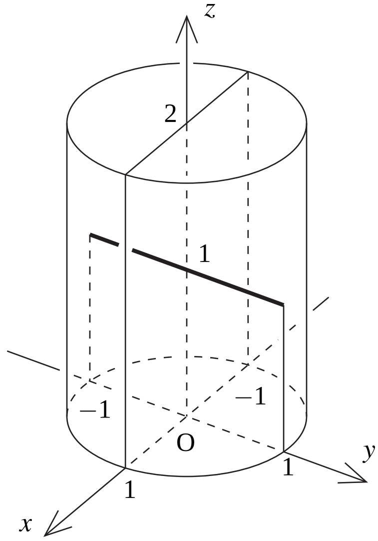


Fig. 4.2.

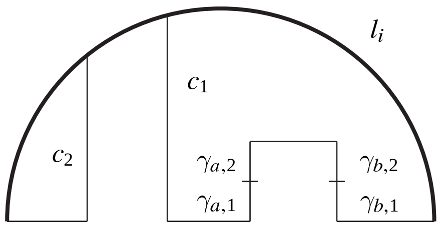


Fig. 4.3.

of generality, that Fig. 4.3 shows the location of the arcs c_1 and c_2 on $\partial N(\alpha; D_i^1)$, and the location of the arcs $\gamma_{a,1}$, $\gamma_{a,2}$, $\gamma_{b,1}$ and $\gamma_{b,2}$ on $\partial N(\alpha'; D_i^1)$, where $(a, b) = (1, 2)$ or $(2, 1)$. Considering images of the outward-normal to D_i^1 in $N(\alpha; D_i^1)$ and $N(\alpha'; D_i^1)$, we see that $(a, b) = (1, 2)$. Since the disc $\widetilde{d_j}$ $(j = 1, 2)$ may be regarded as a band which connects $\widetilde{c_j}$ and $\widetilde{\gamma_{j,1}}$, the disc d_j may be regarded as a band which connects c_j and $\gamma_{j,1}$. This construction of D_i^2 shows that the surface D_i^2 is homeomorphic to an annulus.

Let n^2 denote the component of ∂D_i^2 such that l_i^2 is not contained in n^2. Let m^2 be the arc $cl(\partial D_i^2 - (n^2 \cup l_i^2))$. The simple closed curve $g^2(n^2)$ bounds a disc δ on S such that $g^2(m^2)$ is not contained in δ. Isotope $g^2(N(n^2; D_i^2)) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an immersion g^3 of a disc D_i^3 into Q_i such that Σ_i^3 consists of two arcs of type $\mathcal{C}$. This immersed disc $g^3(D_i^3)$ shows that the clasp number of K_i is at most one. $\square$

Proposition 4.6. *Suppose that the number of points of X_i on D_i is four. Then the clasp number of K_i is at most two.*

Proof. Since $|X_i| = 4$, Σ_i consists of either (1) four arcs of type $\mathcal{A}$, four arcs of type $\mathcal{A}'$ and some arcs of type $\mathcal{B}$, or (2) four arcs of type $\mathcal{C}$ and some arcs of type $\mathcal{B}$, or (3) two arcs of type $\mathcal{A}$, two arcs of type $\mathcal{A}'$, two arcs of type $\mathcal{C}$ and some arcs of type $\mathcal{B}$. In these three cases, we obtain, by Lemma 2.3, an immersion g^1 of a disc D_i^1 into Q_i such that there is no arc of type $\mathcal{B}$ in Σ_i^1.

First suppose that Σ_i^1 consists of four arcs of type $\mathcal{A}$ and four arcs of type $\mathcal{A}'$. Similar arguments as in the proof of Proposition 4.2 show that the crossing number of K_i is at most six, and the clasp number of K_i is at most two.

Next suppose that Σ_i^1 consists of four arcs of type $\mathcal{C}$. Then the immersed disc $g^1(D_i^1)$ shows that the clasp number of K_i is at most two.

Finally suppose that Σ_i^1 consists of two arcs of type $\mathcal{A}$, two arcs of type $\mathcal{A}'$ and two arcs of type $\mathcal{C}$. Let α_1, α_2 be arcs of type $\mathcal{A}$, and α_1', α_2' be arcs of type $\mathcal{A}'$ such that $g^1(\alpha_j) = g^1(\alpha_j')$ for $j = 1$ and 2. Let m^1 denote the arc $cl(\partial D_i^1 - l_i^1)$.

Now we consider configurations of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1. When we proceed on m^1 from one endpoint of m^1, we may assume, without loss of generality, that the first point of $\Sigma_i^1 \cap m^1$ we encounter is an endpoint of either α_1 or α_1'. First suppose that the first point of $\Sigma_i^1 \cap m^1$ is an endpoint of α_1. If the second point of $\Sigma_i^1 \cap m^1$ is an endpoint of α_1', then a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1 is that of Fig. 4.4 (1) or (2). In the configurations of Fig. 4.4, we omit arcs of type $\mathcal{C}$. If the second point of $\Sigma_i^1 \cap m^1$ is an endpoint of α_2, then the configuration is that of Fig. 4.4 (3) or (4). If the second point is an endpoint of α_2', then the configuration is that of Fig. 4.4 (5) or (6). Next suppose that the first point of $\Sigma_i^1 \cap m^1$ is an endpoint of α_1'. If the second point is an endpoint of α_1, then the configuration is that of Fig. 4.4 (1) or (7), up to exchange of the suffix. If the second point is an endpoint of α_2, then the configuration is that of Fig. 4.4 (6) or (8). If the second point is an endpoint of α_2', then the configuration is that of Fig. 4.4 (3) or (4), up to exchange of the suffix.

Lemma 4.7. *Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1 is that of* Fig. 4.4 (1), (2), (4), (6) or (7)*. Then the clasp number of K_i is at most one.*

Proof. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1 is that of Fig. 4.4 (1). Performing a CP surgery to $g^1(D_i^1)$ along the arc $g^1(\alpha_1) = g^1(\alpha_1')$, we obtain an immersion g^2 of an annulus D_i^2 into Q_i such that Σ_i^2 consists of one arc of type $\mathcal{A}$, one arc of type $\mathcal{A}'$ and two arcs of type $\mathcal{C}$. Let n^2 denote the component of ∂D_i^2 such that l_i^2 is not contained in n^2. Let m^2 be the arc $cl(\partial D_i^2 - (n^2 \cup l_i^2))$. The simple closed curve $g^2(n^2)$ bounds a disc δ on S such that $g^2(m^2)$ is not contained in δ. Isotope $g^2(N(n^2; D_i^2)) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an immersion g^3 of a disc into Q_i such that Σ_i^3 consists of one arc of type $\mathcal{A}$, one arc of type $\mathcal{A}'$ and two

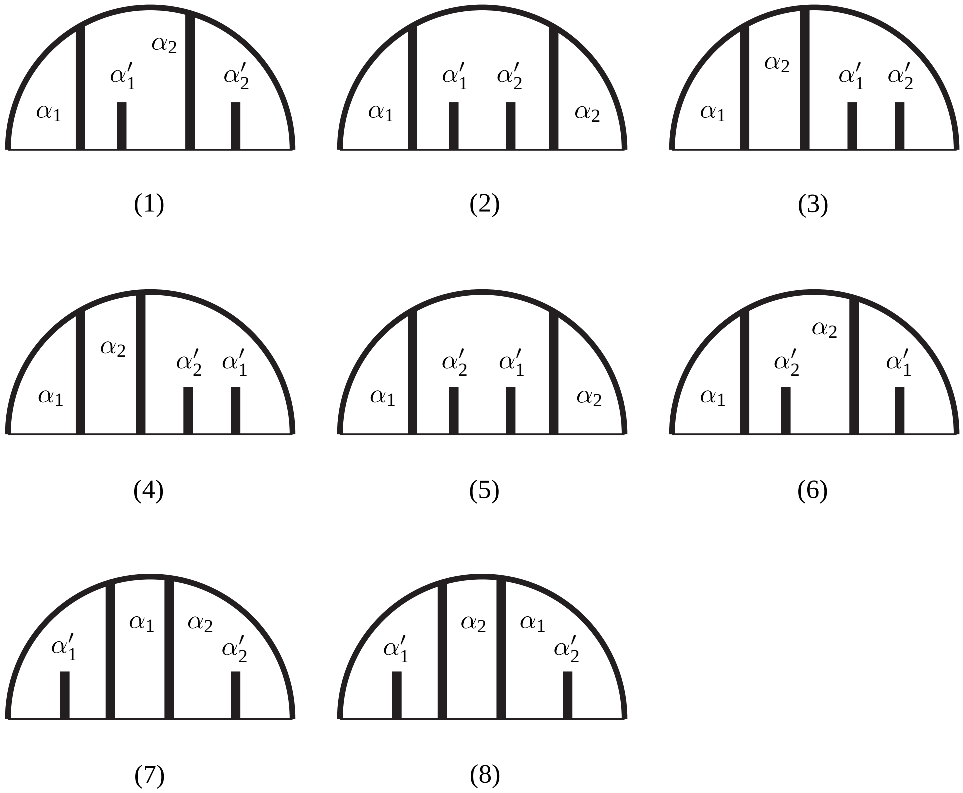


Fig. 4.4.

arcs of type $\mathcal{C}$. Proposition 4.5 shows that the clasp number of K_i is at most one.

Similar arguments as above prove the cases in the configurations of Fig. 4.4 (2), (4), (6) and (7). □

Lemma 4.8. *Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1 is that of* Fig. 4.4 (3) or (8). *Then the clasp number of K_i is at most two.*

Proof. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1 is that of Fig. 4.4 (3). Performing a CP surgery to $g^1(D_i^1)$ along the arc $g^1(\alpha_1)$, we obtain an immersion g^2 of an annulus D_i^2 into Q_i such that Σ_i^2 consists of one arc of type $\mathcal{A}$, one arc of type $\mathcal{A}'$ and two arcs of type $\mathcal{C}$. Let α_2 and α'_2 denote these arcs of types $\mathcal{A}$ and $\mathcal{A}'$ in Σ_i^2, respectively. Let n^2 denote the component of ∂D_i^2 such that l_i^2 is not contained in n^2, and let m^2 denote the arc $cl(\partial D_i^2 - (n^2 \cup l_i^2))$. The simple closed curve $g^2(n^2)$ bounds a disc δ on S which contains one endpoint of the simple arc $g^2(m^2)$. Isotope $g^2(N(n^2; D_i^2)) \cup \delta$ slightly into *int* Q_i. Then we obtain an immersion g^3

of a disc D_i^3 into Q_i. This isotopy changes the union of the arcs $g^2(\alpha_2) = g^2(\alpha_2')$ and $g^2(m^2) \cap \delta$ to a singular arc γ of $g^3(D_i^3)$ such that $(g^3)^{-1}(\gamma)$ consists of two arcs of type $\mathcal{C}$ in D_i^3. Hence Σ_i^3 consists of four arcs of type $\mathcal{C}$. This immersed disc $g^3(D_i^3)$ shows that the clasp number of K_i is at most two.

Similar arguments as above prove the case in the configuration of Fig. 4.4 (8). □

Lemma 4.9. *Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1 is that of* Fig. 4.4 (5). *Then the clasp number of K_i is at most two.*

Proof. Let γ_1 be the subarc of m^1 with $\partial\gamma_1 = (\alpha_1 \cap m^1) \cup (\alpha_2' \cap m^1)$, and γ_2 be the subarc of m^1 with $\partial\gamma_2 = (\alpha_2 \cap m^1) \cup (\alpha_1' \cap m^1)$. Note that the singular arc $g^1(m^1)$ on S has the same configuration as that of Fig. 4.1 (3), up to symmetry and isotopy on S. The two arcs $g^1(\gamma_1)$ and $g^1(\gamma_2)$ cobound a disc δ on S such that $g^1(\partial m^1)$ is not contained in δ. Isotope $g^1(N(\gamma_1; D_i^1))$ along δ. Then we obtain an immersion g^2 of a disc D_i^2 into Q_i such that $g^2(m^2)$ is an embedded arc on S, where $m^2 = cl(\partial D_i^2 - l_i^2)$. This isotopy changes the union of the arcs $g^1(\alpha_1) = g^1(\alpha_1')$, $g^1(\alpha_2) = g^1(\alpha_2')$ and $g^1(\gamma_2)$ to a singular arc γ of $g^2(D_i^2)$ such that $(g^2)^{-1}(\gamma)$ consists of two arcs of type $\mathcal{C}$ in D_i^2. Therefore Σ_i^2 consists of four arcs of type $\mathcal{C}$. This immersed disc $g^2(D_i^2)$ shows that the clasp number of K_i is at most two. □

This completes the proof of Proposition 4.6 □

5. The case where D_i contains loop components of T

In this section, we deal with an immersed surface $g(\overline{D}_i)$ when there are loop components of T in D_i. Recall that $\overline{D}_i$ is the region separated by loop components of T in D_i such that l_i is a subarc of $\partial\overline{D}_i$.

Proposition 5.1. *Suppose that $\overline{D}_i$ is an annulus, and that the number of points of X_i on $\overline{D}_i$ is* 0. *Then K_i is the trivial knot.*

Proof. Since $X_i = \emptyset$, Σ_i consists only of arcs of type $\mathcal{B}$. Let n denote the component of $\partial\overline{D}_i$ such that l_i is not contained in n.

First suppose $\Sigma_i = \emptyset$. Then the simple closed curve $g(n)$ bounds a disc δ on S such that $g(m)$ is not contained in δ. Isotope $g(N(n; \overline{D}_i)) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an embedding of a disc into Q_i. This embedded disc shows that K_i is the trivial knot.

Next suppose $\Sigma_i \neq \emptyset$. A properly embedded arc b on $\overline{D}_i$ is said to be *of type b_1* if the two points of ∂b are contained in m, and if b and a subarc b' of $\partial\overline{D}_i$ cobound a disc on $\overline{D}_i$ such that l_i is contained in b'. See Fig. 5.1. A properly embedded arc b on $\overline{D}_i$ is *of type b_2* if b together with a subarc of m cobounds a disc on $\overline{D}_i$. A

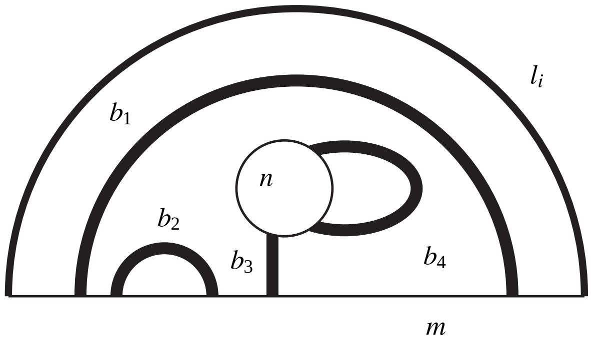


Fig. 5.1.

properly embedded arc b on $\bar{D}_i$ is *of type* b_3 if b connects a point on m and a point on n. A properly embedded arc b on $\bar{D}_i$ is *of type* b_4 if b together with a subarc of n cobounds a disc on $\bar{D}_i$. We note that an arc of type $\mathcal{B}$ on $\bar{D}_i$ is of type b_1, b_2, b_3 or b_4.

Now we consider configurations of a pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g. If one of the arcs of type $\mathcal{B}$ in the pair is of type b_1, then we may assume, by Lemmas 2.2 and 2.3, that a configuration of the pair is, up to symmetry of $\bar{D}_i$, that of Fig. 5.2 (1) or (2). If one of the arcs of type $\mathcal{B}$ is of type b_2, then we may assume, by Lemmas 2.2 and 2.3, that a configuration of the pair is, up to symmetry of $\bar{D}_i$, that of Fig. 5.2 (1) or (3). If one of the arcs of type $\mathcal{B}$ is of type b_3, then we may assume, by Lemma 2.2, that a configuration of the pair is, up to symmetry of $\bar{D}_i$, that of Fig. 5.2 (2), (3) or (4). If one of the arcs of type $\mathcal{B}$ is of type b_4, then we may assume, by Lemmas 2.2 and 2.3, that a configuration of the pair is that of Fig. 5.2 (4).

Lemma 5.2. *Suppose that a configuration of the pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g is that of* Fig. 5.2 (1), (2), (3) *or* (4). *Then there exists an immersion g^2 of an annulus D_i^2 into Q_i satisfying the following properties*;
(i) *Every component of Σ_i^2 is an arc of type $\mathcal{B}$,*
(ii) *The number of arcs of type $\mathcal{B}$ in Σ_i^2 is strictly less than that of type $\mathcal{B}$ in Σ_i, and*
(iii) *There is a subarc l_i^2 of ∂D_i^2 such that $g^2(l_i^2) = g(l_i)$, and that $g^2(\partial D_i^2 - l_i^2)$ is contained in S.*

Proof. Suppose that a configuration of the pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g is that of Fig. 5.2 (1). Let β_1 and β_2 be the two arcs of type $\mathcal{B}$. Let V denote a regular neighborhood $N(g(\beta_1); Q_i)$. We may choose V so that $g(\bar{D}_i) \cap V$ consists of two discs $g(N(\beta_1; \bar{D}_i))$ and $g(N(\beta_2; \bar{D}_i))$. The disc $g(N(\beta_1; \bar{D}_i))$ intersects $g(N(\beta_2; \bar{D}_i))$ transversely along the arc $g(\beta_1) = g(\beta_2)$. We regard V as the set

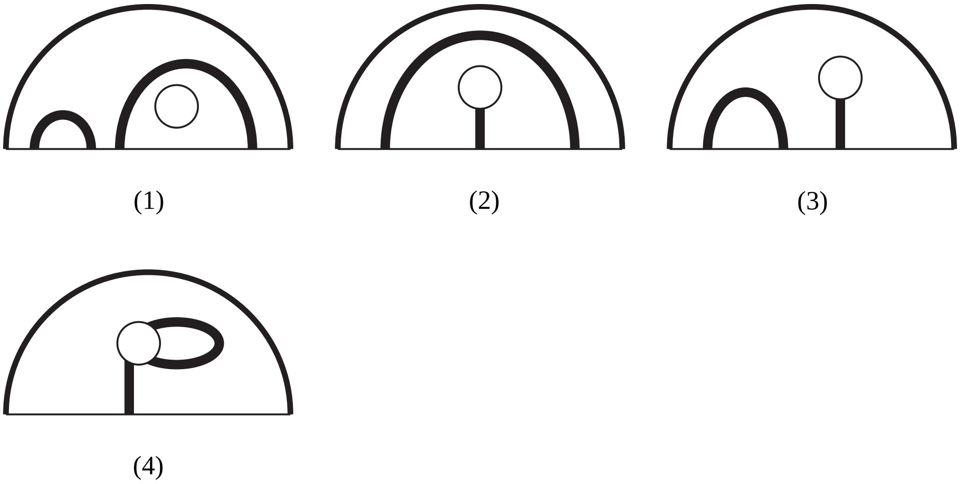


Fig. 5.2.

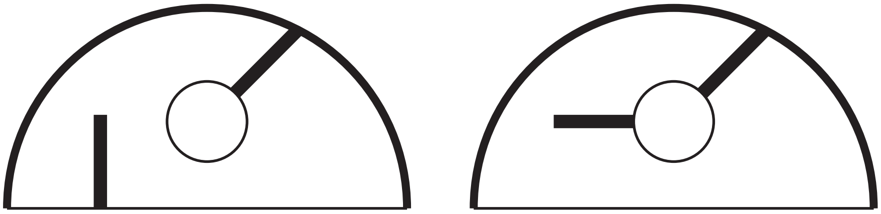
$\{(x, y, z) \mid x^2 + y^2 \leq 1,\ 0 \leq z \leq 1\}$ so that two discs of $V \cap S$ correspond to the sets $\{(x, y, 0) \mid x^2 + y^2 \leq 1\}$ and $\{(x, y, 1) \mid x^2 + y^2 \leq 1\}$. Without loss of generality, we may assume that $g(N(\beta_1; \bar{D}_i))$ corresponds to $\{(x, 0, z) \mid -1 \leq x \leq 1,\ 0 \leq z \leq 1\}$, and that $g(N(\beta_2; \bar{D}_i))$ corresponds to $\{(0, y, z) \mid -1 \leq y \leq 1,\ 0 \leq z \leq 1\}$. We may also assume that an image of the outward-normal to $\bar{D}_i$ in $N(\beta_1; \bar{D}_i)$ agrees with the direction of increasing y, and that an image of the outward-normal to $\bar{D}_i$ in $N(\beta_2; \bar{D}_i)$ agrees with the direction of increasing x. Let $\tilde{d}_1$ be the disc $\{(x, y, z) \mid x^2 + y^2 = 1, x \geq 0,\ y \geq 0,\ 0 \leq z \leq 1\}$, and $\tilde{d}_2$ be the disc $\{(x, y, z) \mid x^2 + y^2 = 1,\ x \leq 0,\ y \leq 0, 0 \leq z \leq 1\}$.

Now we define an immersion g^1 of a surface $\bar{D}_i^1$ into Q_i. Let $g^1(\bar{D}_i^1)$ be the immersed surface which is the union of $g(\bar{D}_i - N(\beta_1 \cup \beta_2; \bar{D}_i))$, $\tilde{d}_1$ and $\tilde{d}_2$. We say that $g^1(\bar{D}_i^1)$ is obtained from $g(\bar{D}_i)$ by an *oriented double curve surgery* along the arc $g(\beta_1) = g(\beta_2)$. This operation was called an *orientation preserving cut* along the arc $g(\beta_1) = g(\beta_2)$. See, for example, [11, p. 4]. By this construction, $g^1(\bar{D}_i^1)$ satisfies the properties (i)–(iii). The surface $\bar{D}_i^1$ is the union of $\bar{D}_i - N(\beta_1 \cup \beta_2; \bar{D}_i)$, d_1 and d_2, where d_j $(j = 1, 2)$ is a disc corresponding to $\tilde{d}_j$. Similar arguments as in the proof of Lemma 2.2 show that $\bar{D}_i^1$ is homeomorphic to either an annulus or two annuli.

Let $\bar{D}_i^2$ denote the connected component of $\bar{D}_i^1$ such that l_i^1 is a subarc of $\partial \bar{D}_i^2$. Let g^2 be the restriction of g^1 to $\bar{D}_i^2$. Thus we obtain an immersion g^2 of an annulus $\bar{D}_i^2$ into Q_i satisfying the properties (i)–(iii).

Similar arguments as above prove the cases in the configurations of Fig. 5.2 (2), (3) and (4). $\square$

By Lemma 5.2, we obtain an embedding g^3 of an annulus $\bar{D}_i^3$ into Q_i with



(1) (2)

Fig. 5.3.

$g^3(l_i^3) = g(l_i)$. Therefore we have $\Sigma_i^3 = \emptyset$, and K_i is the trivial knot. This completes the proof of Proposition 5.1. □

Proposition 5.3. *Suppose that $\overline{D_i}$ is an annulus, and that the number of points of X_i on $\overline{D_i}$ is one. Then the clasp number of K_i is at most one.*

Proof. Since $|X_i| = 1$, Σ_i consists of one arc of type $\mathcal{A}$, one arc of type $\mathcal{A}'$ and some arcs of type $\mathcal{B}$. By similar arguments as in the proof of Proposition 5.1, we obtain an immersion g^1 of an annulus $\overline{D_i^1}$ into Q_i such that there is no arc of type $\mathcal{B}$ in Σ_i^1, and that there is a subarc l_i^1 of $\partial \overline{D_i^1}$ with $g^1(l_i^1) = g(l_i)$. If $\Sigma_i^1 = \emptyset$, then Proposition 5.1 shows that K_i is the trivial knot. So we may assume that Σ_i^1 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Let n^1 denote the component of $\partial \overline{D_i^1}$ such that l_i^1 is not contained in n^1. Let m^1 be the arc $cl(\partial \overline{D_i^1} - (n^1 \cup l_i^1))$.

First suppose that there are no endpoints of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on n^1. Then the simple closed curve $g^1(n^1)$ bounds a disc δ on S such that $g^1(m^1)$ is not contained in δ. Isotope $g^1(N(n^1; \overline{D_i^1})) \cup \delta$ slightly into *int* Q_i. Then we obtain an immersion g^2 of a disc $\overline{D_i^2}$ into Q_i such that Σ_i^2 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Proposition 4.1 shows that K_i is the trivial knot.

Next suppose that there are endpoints of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on n^1. If there is an endpoint of only the arc of type $\mathcal{A}$ on n^1, then a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D_i^1}$ is that of Fig. 5.3 (1). If there is an endpoint of only the arc of type $\mathcal{A}'$ on n^1, then the arc of type $\mathcal{A}$ satisfies the supposition of Lemma 2.1, and we obtain an embedding of an annulus into Q_i. Proposition 5.1 shows that K_i is the trivial knot. If there are endpoints of the arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on n^1, then a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D_i^1}$ is that of Fig. 5.3 (2).

Similar arguments as in the proof of Lemma 4.8 prove the following lemma.

Lemma 5.4. *Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D_i^1}$ is that of* Fig. 5.3 (1). *Then the clasp number of K_i is at most one.*

Lemma 5.5. *Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D}_i^1$ is that of* Fig. 5.3 (2). *Then K_i is the trivial knot.*

Proof. Performing a CP surgery to $g^1(\overline{D}_i^1)$ along the singular arc, we obtain an embedding g^2 of a twice-punctured disc $\overline{D}_i^2$ into Q_i. Let n_1^2 and n_2^2 denote the components of $\partial\overline{D}_i^2$ such that l_i^2 is not a subarc of n_1^2 or n_2^2. Let m^2 be the arc $cl(\partial D_i^2 - (n_1^2 \cup n_2^2 \cup l_i^2))$. At least one of the simple closed curves $g^2(n_1^2)$ and $g^2(n_2^2)$, say $g^2(n_1^2)$, bounds a disc δ on S such that $g^2(n_2^2)$ and $g^2(m^2)$ are not contained in δ. Isotope $g^2(N(n_1^2; D_i^2)) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an embedding of an annulus into Q_i. Proposition 5.1 shows that K_i is the trivial knot. □

This completes the proof of Proposition 5.3. □

Proposition 5.6. *Suppose that $\overline{D}_i$ is an annulus, and that the number of points of X_i on $\overline{D}_i$ is two. Then the clasp number of K_i is at most one.*

Proof. Since $|X_i| = 2$, Σ_i consists of either (1) two arcs of type $\mathcal{A}$, two arcs of type $\mathcal{A}'$ and some arcs of type $\mathcal{B}$, or (2) two arcs of type $\mathcal{C}$ and some arcs of type $\mathcal{B}$. In both cases, we obtain, by similar arguments as in the proof of Proposition 5.1, an immersion g^1 of an annulus $\overline{D}_i^1$ into Q_i such that there is no arc of type $\mathcal{B}$ in Σ_i^1, and that there is a subarc l_i^1 of ∂D_i^1 with $g^1(l_i^1) = g(l_i)$. We may assume, by Propositions 5.1 and 5.3, that Σ_i^1 consists of two arcs of type $\mathcal{A}$ and two arcs of type $\mathcal{A}'$ in the case of (1), and that Σ_i^1 consists of two arcs of type $\mathcal{C}$ in the case of (2). Let n^1 denote the component of $\partial\overline{D}_i^1$ such that l_i^1 is not contained in n^1. Let m^1 be the arc $cl(\partial D_i^1 - (n^1 \cup l_i^1))$. □

Lemma 5.7. *Suppose that Σ_i^1 consists of two arcs of type $\mathcal{C}$. Then the clasp number of K_i is at most one.*

Proof. The simple closed curve $g^1(n^1)$ bounds a disc δ on S such that $g^1(m^1)$ is not contained in δ. Isotope $g^1(N(n^1; D_i^1)) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an immersion g^2 of a disc $\overline{D}_i^2$ into Q_i such that Σ_i^2 consists of two arcs of type $\mathcal{C}$. This immersed disc $g^2(D_i^2)$ shows that the clasp number of K_i is at most one. □

Lemma 5.8. *Suppose that Σ_i^1 consists of two arcs of type $\mathcal{A}$ and two arcs of type $\mathcal{A}'$. Then the clasp number of K_i is at most one.*

Proof. We consider configurations of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D}_i^1$.

CLAIM 5.9. If there are no endpoints of arcs of type $\mathcal{A}$ or $\mathcal{A}'$ on n^1, then the clasp number of K_i is at most one.

Proof. Suppose that there are no endpoints of arcs of type $\mathcal{A}$ or $\mathcal{A}'$ on n^1. Then the simple closed curve $g^1(n^1)$ bounds a disc δ on S such that $g^1(m^1)$ is not contained in δ. Isotope $g^1(N(n^1; D_i^1))\cup\delta$ slightly into *int* Q_i. Then we obtain an immersion g^2 of a disc into Q_i such that Σ_i^2 consists of two arcs of type $\mathcal{A}$ and two arcs of type $\mathcal{A}'$. Proposition 4.2 shows that the clasp number of K_i is at most one. □

We may assume, by Claim 5.9, that there is at least one endpoint of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on n^1. If there is an endpoint of only one arc of type $\mathcal{A}$ on n^1, then we may assume, by Lemma 2.1 and Proposition 5.3, that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on D_i^1 is, up to symmetry of D_i^1, that of Fig. 5.4 (1) or (2). If there is an endpoint of only one arc of type $\mathcal{A}'$ on n^1, then we may assume, by Lemma 2.1 and Proposition 5.3, that the configuration is, up to symmetry of D_i^1, that of Fig. 5.4 (3). If there are endpoints of only one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$ on n^1, then we may assume, by Lemma 2.1 and Proposition 5.3, that the configuration is, up to symmetry of D_i^1, that of Fig. 5.4 (4). If there are endpoints of only two arcs of type $\mathcal{A}$ on n^1, then the configuration is that of Fig. 5.4 (5). If there are endpoints of only two arcs of type $\mathcal{A}'$ on n^1, then at least one of the two arcs of type $\mathcal{A}$ satisfies the supposition of Lemma 2.1, and we obtain an immersion g^2 of an annulus into Q_i such that Σ_i^2 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Proposition 5.3 shows that the clasp number of K_i is at most one. If there are endpoints of only one arc of type $\mathcal{A}$ and two arcs of type $\mathcal{A}'$ on n^1, then the same arguments as above show that the clasp number of K_i is at most one. If there are endpoints of only two arcs of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$ on n^1, then the configuration is that of Fig. 5.4 (6) or (7). If there are endpoints of two arcs of type $\mathcal{A}$ and two arcs of type $\mathcal{A}'$ on n^1, then the configuration is that of Fig. 5.4 (8), (9) or (10).

CLAIM 5.10. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\bar{D}_i^1$ is that of Fig. 5.4 (1), (2), (3), (6) or (7). Then the clasp number of K_i is at most one.

Proof. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\bar{D}_i^1$ is that of Fig. 5.4 (1). Let α_1', α_2' denote arcs of type $\mathcal{A}'$, and α_j ($j = 1$ or 2) denote the arc of type $\mathcal{A}$ as illustrated in the figure. We assume that $g^1(\alpha_p) = g^1(\alpha_p')$ for $p = 1$ and 2.

First suppose $j = 2$. The simple closed curve $g^1(n^1)$ intersects the immersed arc $g^1(m^1)$ transversely in one point on S. Since $g^1(\alpha_1) = g^1(\alpha_1')$, we can construct a simple closed curve on S which intersects $g^1(n^1)$ transversely in one point. This shows that $g^1(n^1)$ is a non-separating simple closed curve on a 2-sphere, a contradiction.

Next suppose $j = 1$. Performing a CP surgery to $g^1(D_i^1)$ along the arc $g^1(\alpha_2) = g^1(\alpha_2')$, we obtain an immersion g^2 of a twice-punctured disc D_i^2 into Q_i such that Σ_i^2 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Let n^2 be the connected component of ∂D_i^2 such that there is no endpoint of the arc of type $\mathcal{A}$ or $\mathcal{A}'$ on n^2. Then the simple closed curve $g^2(n^2)$ bounds a disc δ on S such that $g^2(\partial D_i^2 - n^2)\cap S$

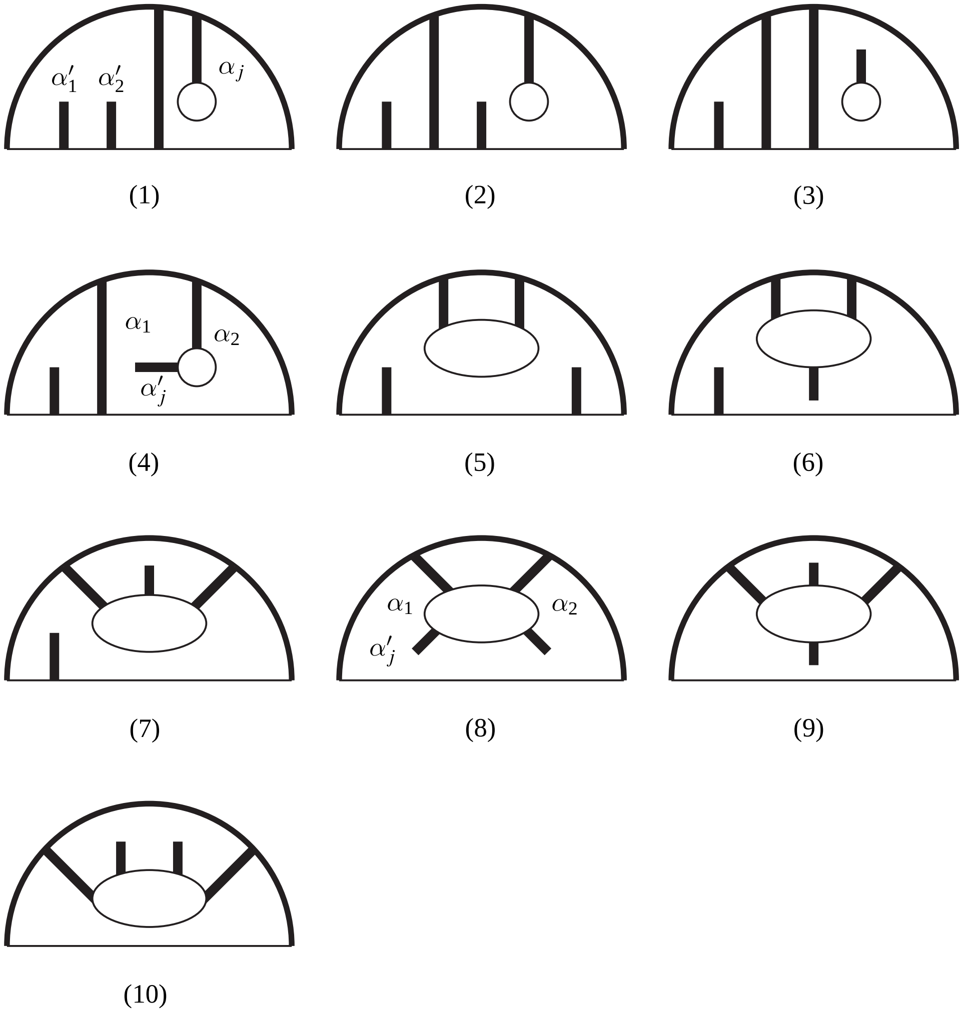


Fig. 5.4.

is not contained in δ. Isotope $g^2(N(n^2;\overline{D_i^2}))\cup\delta$ slightly into *int* Q_i. Then we obtain an immersion g^3 of an annulus into Q_i such that Σ_i^3 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Proposition 5.3 shows that the clasp number of K_i is at most one.

Similar arguments as above prove the cases in the configurations of Fig. 5.4 (2), (3), (6) and (7). □

CLAIM 5.11. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D_i^1}$ is that of Fig. 5.4 (4). Then the clasp number of K_i is at most one.

Proof. Let α_1, α_2 denote arcs of type $\mathcal{A}$, and α'_j (j = 1 or 2) denote the arc of type $\mathcal{A}'$ as illustrated in the figure. We assume that $g^1(\alpha_p) = g^1(\alpha'_p)$ for p = 1 and 2.

First suppose j = 1. Then the simple closed curve $g^1(n^1)$ bounds a disc δ on S which contains no endpoints of the simple arc $g^1(m^1)$. Isotope $g^1(N(n^1; \overline{D}_i^1)) \cup \delta$ slightly into $\mathit{int}\, Q_i$. Then we obtain an immersion g^2 of a disc D_i^2 into Q_i. This isotopy changes the union of the arcs $g^1(\alpha_1) = g^1(\alpha'_1)$, $g^1(\alpha_2) = g^1(\alpha'_2)$ and $g^1(m^1) \cap \delta$ to a singular arc γ of $g^2(D_i^2)$ such that $(g^2)^{-1}(\gamma)$ consists of two arcs of type $\mathcal{C}$ in D_i^2. Hence Σ_i^2 consists of two arcs of type $\mathcal{C}$. This immersed disc $g^2(D_i^2)$ shows that the clasp number of K_i is at most one.

Next suppose j = 2. Perform a CP surgery to $g^1(\overline{D}_i^1)$ along the arc $g^1(\alpha_2) = g^1(\alpha'_2)$. Then we obtain an immersion g^2 of a twice-punctured disc into Q_i. By similar arguments as in the proof of Lemma 5.5, we obtain an immersion g^3 of a disc into Q_i such that Σ_i^3 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Proposition 4.1 shows that K_i is the trivial knot. □

CLAIM 5.12. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D}_i^1$ is that of Fig. 5.4 (5). Then K_i is the trivial knot.

Proof. Let α_1 and α_2 be arcs of type $\mathcal{A}$ on $\overline{D}_i^1$. The simple closed curve $g^1(n^1)$ bounds a disc δ on S which contains no endpoints of the simple arc $g^1(m^1)$. Isotope $g^1(N(n^1; \overline{D}_i^1)) \cup \delta$ slightly into $\mathit{int}\, Q_i$. Then we obtain an immersion g^2 of a disc D_i^2 into Q_i. This isotopy changes the union of the arcs $g^1(\alpha_1)$, $g^1(\alpha_2)$ and $g^1(m^1) \cap \delta$ to a singular arc γ of $g^2(D_i^2)$ such that $(g^2)^{-1}(\gamma)$ consists of two arcs γ_1 and γ_2 embedded in D_i^2, where $\partial\gamma_1$ is contained in l_i^2 and γ_2 is contained in $\mathit{int}\, D_i^2$. Note that $\gamma_1 \cap \gamma_2 = \emptyset$ on D_i^2, and that γ_1 and a subarc of l_i^2 cobound a disc d_γ in D_i^2 such that γ_2 is not contained in d_γ. Isotope $g^2(N(\gamma_2; D_i^2))$ along $g^2(d_\gamma)$. Then we obtain an embedding g^3 of a disc D_i^3 into Q_i. This embedded disc $g^3(D_i^3)$ shows that K_i is the trivial knot. □

CLAIM 5.13. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D}_i^1$ is that of Fig. 5.4 (8), (9) or (10). Then K_i is the trivial knot.

Proof. Suppose that a configuration of arcs of types $\mathcal{A}$ and $\mathcal{A}'$ on $\overline{D}_i^1$ is that of Fig. 5.4 (8). Let α_1 and α_2 denote arcs of type $\mathcal{A}$, and α'_j (j = 1 or 2) denote the arc of type $\mathcal{A}'$ as illustrated in the figure. We assume that $g^1(\alpha_p) = g^1(\alpha'_p)$ for p = 1 and 2.

First suppose j = 2. Then there is no configuration of an immersed closed curve $g^1(n^1)$ on a 2-sphere.

Next suppose j = 1. Performing CP surgeries to $g^1(\overline{D}_i^1)$ along the arcs $g^1(\alpha_1)$ and $g^1(\alpha_2)$, we obtain an embedding g^2 of a twice-punctured annulus D_i^2 into Q_i. Similar arguments as in the proof of Lemma 5.5 shows that K_i is the trivial knot.

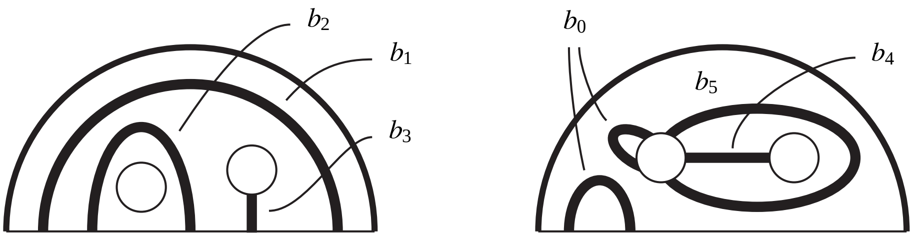


Fig. 5.5.

The same arguments as above prove the cases in the configurations of Fig. 5.4 (9) and (10). □

This completes the proofs of Lemma 5.8 and Proposition 5.6. □

Proposition 5.14. *Suppose that $\bar{D}_i$ is a twice-punctured disc, and that the number of points of X_i on $\bar{D}_i$ is 0. Then the clasp number of K_i is at most one.*

Proof. Since $|X_i| = 0$, Σ_i consists only of arcs of type $\mathcal{B}$. Let n_1, n_2 denote connected components of $\partial\bar{D}_i - (m \cup l_i)$.

A properly embedded arc b on $\bar{D}_i$ is said to be *of type* b_0 if b and a subarc of $cl(\partial\bar{D}_i - l_i)$ cobound a disc on $\bar{D}_i$. See Fig. 5.5. A properly embedded arc b on $\bar{D}_i$ is *of type* b_1 if the two points of ∂b are contained in m, and if b together with a subarc b' of $\partial\bar{D}_i$ cobounds a disc on $\bar{D}_i$ such that l_i is contained in b'. A properly embedded arc b on $\bar{D}_i$ is *of type* b_2 if the two points of ∂b are contained in m, and if b separates $\bar{D}_i$ to two annuli d_1 and d_2 so that n_j is a component of ∂d_j for $j = 1$ and 2. A properly embedded arc b on $\bar{D}_i$ is *of type* b_3 if b connects a point on m and a point on n_j for $j = 1$ or 2. A properly embedded arc b on $\bar{D}_i$ is *of type* b_4 if b connects a point on n_1 and a point on n_2. A properly embedded arc b on $\bar{D}_i$ is *of type* b_5 if the two points of ∂b are contained in n_j, and if b and a subarc of n_j together with n_k cobound an annulus on $\bar{D}_i$ for $(j, k) = (1, 2)$ or $(2, 1)$. We note that an arc of type $\mathcal{B}$ on $\bar{D}_i$ is of type b_0, b_1, b_2, b_3, b_4 or b_5.

The following lemma is essentially the same as Lemma 1 (1) in [6]. We refer to [6] for a proof.

Lemma 5.15. *Let β be an arc of type $\mathcal{B}$ which is of type b_0 on $\bar{D}_i$. Then there are an orientable surface D_i^1 and an immersion $g^1 \colon D_i^1 \to Q_i$ satisfying the following properties*;

(i) *The Euler characteristics of $\bar{D}_i^1$ is equal to or greater than that of $\bar{D}_i$,*
(ii) *Every component of Σ_i^1 is an arc of type $\mathcal{B}$,*
(iii) *The number of arcs of type $\mathcal{B}$ in Σ_i^1 is strictly less than that in Σ_i, and*
(iv) *There is a subarc l_i^1 of ∂D_i^1 such that $g^1(l_i^1) = g(l_i)$, and that $g^1(\partial D_i^1 - l_i^1)$ is*

contained in S.

We may assume, by Lemma 5.15 and Propositions 4.1 and 5.1, that there is no arc of type $\mathcal{B}$ which is of type b_0 on a twice-punctured disc $\bar{D}_i$.

Let β be an arc of type $\mathcal{B}$ which is of type b_1 on $\bar{D}_i$, and d_β be the disc on $\bar{D}_i$ which is cobounded by β and a subarc of $\partial \bar{D}_i$. We may assume, by Lemma 2.2, that the restriction of g to d_β is an embedding. We can isotope the string $g(l_i)$ of the 1-string tangle $(Q_i, g(l_i))$ along the embedded disc $g(d_\beta)$ in Q_i to the string $g(\beta)$ of the 1-string tangle $(Q_i, g(\beta))$. It follows that it has no effect on the knot type of K_i to replace $\bar{D}_i$ with $cl(\bar{D}_i - N(d_\beta; \bar{D}_i))$. So we suppose, in the following, that there is no arc of type $\mathcal{B}$ which is of type b_1 on $\bar{D}_i$.

Now we consider configurations of a pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g. If one of the arcs of type $\mathcal{B}$ is of type b_2, then we may assume, by Lemma 2.2, that a configuration of the pair is, up to symmetry of $\bar{D}_i$, that of Fig. 5.6 (1), (2) or (3). If one of the arcs of type $\mathcal{B}$ is of type b_3, then we may assume, by Lemma 2.2, that a configuration of the pair is, up to symmetry of $\bar{D}_i$, that of Fig. 5.6 (2)–(6) or (7). If one of the arcs of type $\mathcal{B}$ is of type b_4, then we may assume, by Lemma 2.2, that a configuration of the pair is, up to symmetry of $\bar{D}_i$, that of Fig. 5.6 (6) or (8). If one of the arcs of type $\mathcal{B}$ is of type b_5, then we may assume, by Lemma 2.2, that a configuration of the pair is, up to symmetry of $\bar{D}_i$, that of Fig. 5.6 (7) or (8).

Lemma 5.16. *Suppose that a configuration of a pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g is that of* Fig. 5.6 (1), (2), (3), (5), (7) *or* (8). *Then there is an immersion $g^2 \colon \bar{D}_i^2 \to Q_i$ with the following properties*;
(i) *The surface $\bar{D}_i^2$ is homeomorphic to either an annulus or a twice-punctured disc,*
(ii) *Every component of Σ_i^2 is an arc of type $\mathcal{B}$,*
(iii) *The number of arcs of type $\mathcal{B}$ in Σ_i^2 is strictly less than that in Σ_i, and*
(iv) *There is a subarc l_i^2 of $\partial \bar{D}_i^2$ such that $g^2(l_i^2) = g(l_i)$ and that $g^2(\partial \bar{D}_i^2 - l_i^2)$ is contained in S.*

Proof. Suppose that a configuration of a pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g is that of Fig. 5.6 (1). Performing an oriented double curve surgery to $g(\bar{D}_i)$ along the singular arc, we obtain an immersion $g^1 \colon \bar{D}_i^1 \to Q_i$ which satisfies the conditions (ii), (iii) and (iv). Similar arguments as in the proof of Lemma 2.2 show that the surface $\bar{D}_i^1$ is homeomorphic to either a union of an annulus and a twice-punctured disc, or a twice-punctured disc. Let $\bar{D}_i^2$ denote the connected component of $\bar{D}_i^1$ such that l_i^1 is a subarc of $\partial \bar{D}_i^2$. Let g^2 be the restriction of g^1 to $\bar{D}_i^2$. Then the immersion $g^2 \colon \bar{D}_i^2 \to Q_i$ satisfies the conditions (i)–(iv).

The same arguments as above prove the cases in the configurations of Fig. 5.6 (2), (3), (5), (7) and (8). □

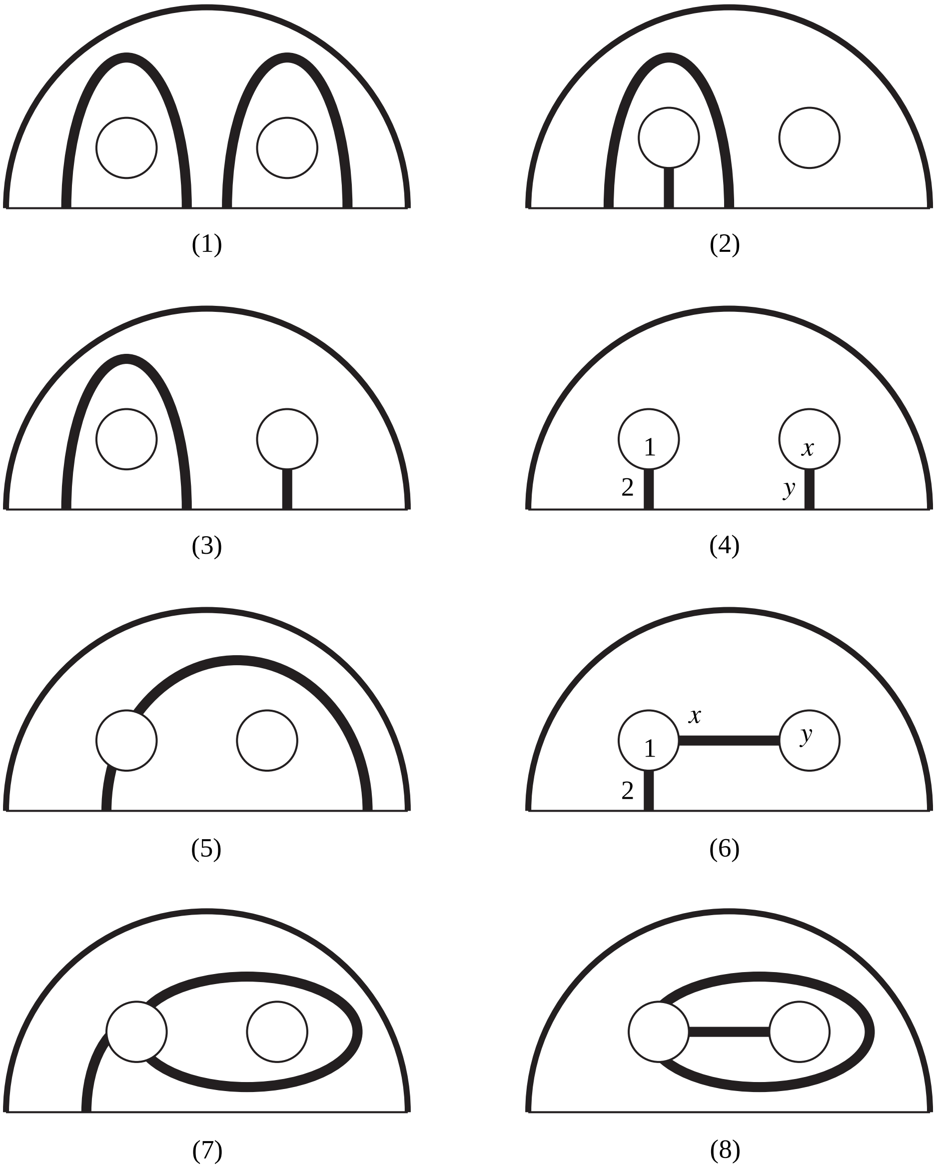


Fig. 5.6.

The same arguments as in the proof of Lemma 5.16 prove the following two lemmas.

Lemma 5.17. *Suppose that a configuration of a pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g is that of* Fig. 5.6 (4). *Let* 1, 2, x, y *denote endpoints of arcs of type $\mathcal{B}$ as illustrated in the figure. If $\underline{g}(1) = g(x)$ and $g(2) = g(y)$, then there is an immersion g^2 of a twice-punctured disc D_i^2 into Q_i such that every component of Σ_i^2 is an arc of type $\mathcal{B}$, that the number of arcs of type $\mathcal{B}$ in Σ_i^2 is strictly less than that in Σ_i, and that there is a subarc l_i^2 of ∂D_i^2 with $g^2(l_i^2) = g(l_i)$.*

Lemma 5.18. *Suppose that a configuration of a pair of arcs of type $\mathcal{B}$ on $\bar{D}_i$ which are identified by g is that of* Fig. 5.6 (6). *Let* 1, 2, x, y *denote endpoints of arcs of type $\mathcal{B}$ as illustrated in the figure. If $\underline{g}(1) = g(x)$ and $g(2) = g(y)$, then there is an immersion g^2 of a twice-punctured disc D_i^2 into Q_i such that every component of Σ_i^2 is an arc of type $\mathcal{B}$, that the number of arcs of type $\mathcal{B}$ in Σ_i^2 is strictly less than that in Σ_i, and that there is a subarc l_i^2 of ∂D_i^2 with $g^2(l_i^2) = g(l_i)$.*

By Lemmas 5.16, 5.17 and 5.18, we may suppose either that $\bar{D}_i^2$ is an annulus such that Σ_i^2 consists only of arcs of type $\mathcal{B}$, or that D_i^2 is a twice-punctured disc such that a configuration of every pair of arcs of type $\mathcal{B}$ on D_i^2 which are identified by g^2 is that of Fig. 5.6 (4) or (6) with $g^2(1) = g^2(y)$ and $g^2(2) = g^2(x)$. If $\bar{D}_i^2$ is an annulus, then Proposition 5.1 shows that K_i is the trivial knot.

In the rest of the proof of Proposition 5.14, we suppose that $\bar{D}_i^2$ is a twice-punctured disc, and that every pair of arcs of type $\mathcal{B}$ on D_i^2 which are identified by g^2 is either the pair as in the configuration of Fig. 5.6 (4) with $g^2(1) = g^2(y)$ and $g^2(2) = g^2(x)$, or the pair as in the configuration of Fig. 5.6 (6) with $g^2(1) = g^2(y)$ and $g^2(2) = g^2(x)$.

We may assume, by Lemma 2.4, that Σ_i^2 contains at most one pair of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (4), and at most two pairs of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (6). Now we consider configurations of arcs of type $\mathcal{B}$ on D_i^2. If Σ_i^2 consists of only one pair of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (4), then a configuration of arcs of type $\mathcal{B}$ on D_i^2 is that of Fig. 5.7 (1). In the configurations of Fig. 5.7, we suppose that endpoints of arcs of type $\mathcal{B}$ which have the same labels are identified by g^2. If Σ_i^2 consists of only one pair of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (6), then the configuration is that of Fig. 5.7 (2). If Σ_i^2 consists of only one pair of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (4) and one pair of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (6), then the configuration is that of Fig. 5.7 (3), (4) or (5), up to symmetry of D_i^2. If Σ_i^2 consists of only two pairs of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (6), then the configuration is that of Fig. 5.7 (6), up to symmetry of D_i^2. If Σ_i^2 consists of one pair of arcs of type $\mathcal{B}$ as in the configuration of Fig. 5.6 (4) and two pairs of arcs of type $\mathcal{B}$ as

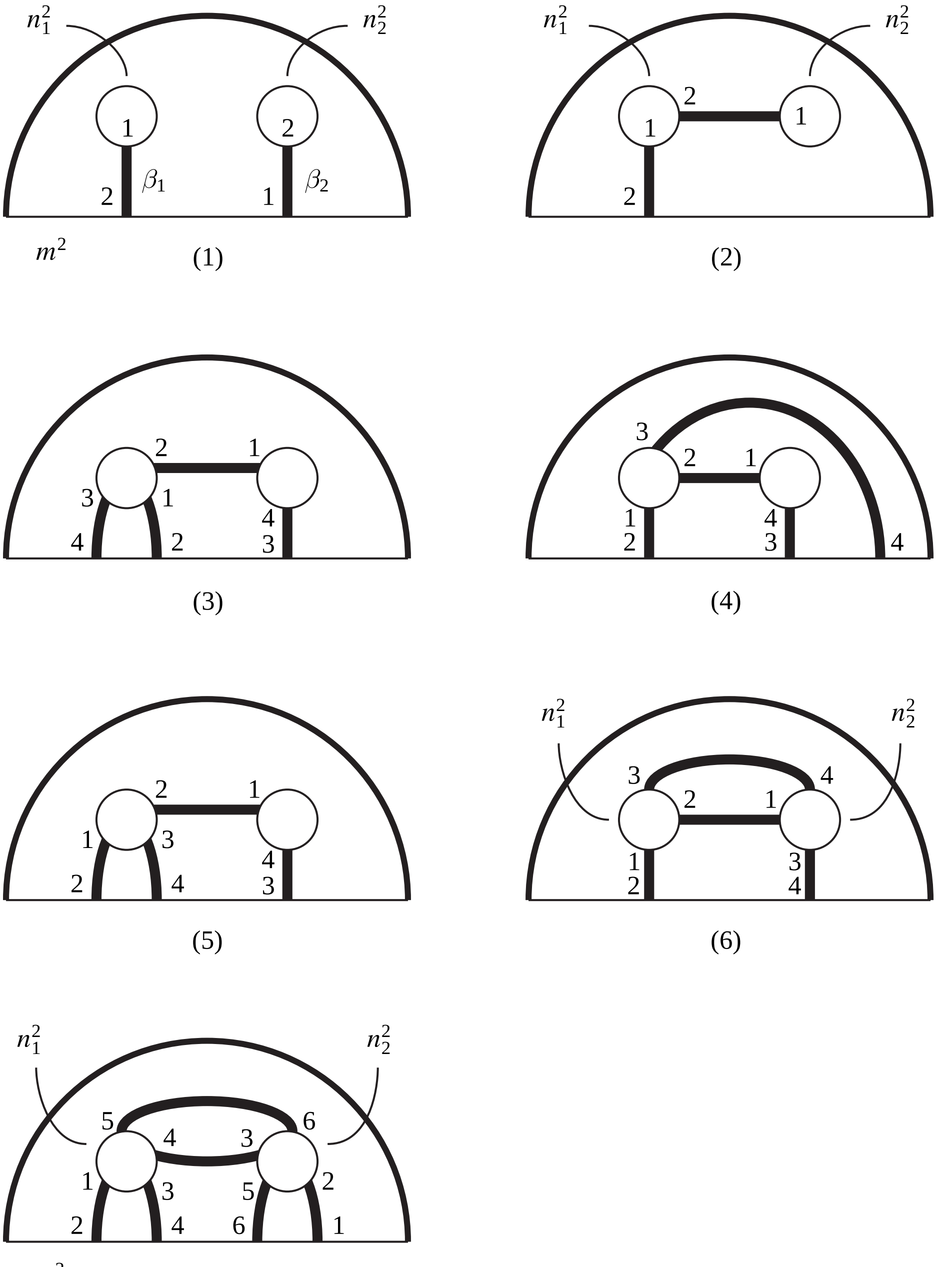


Fig. 5.7.

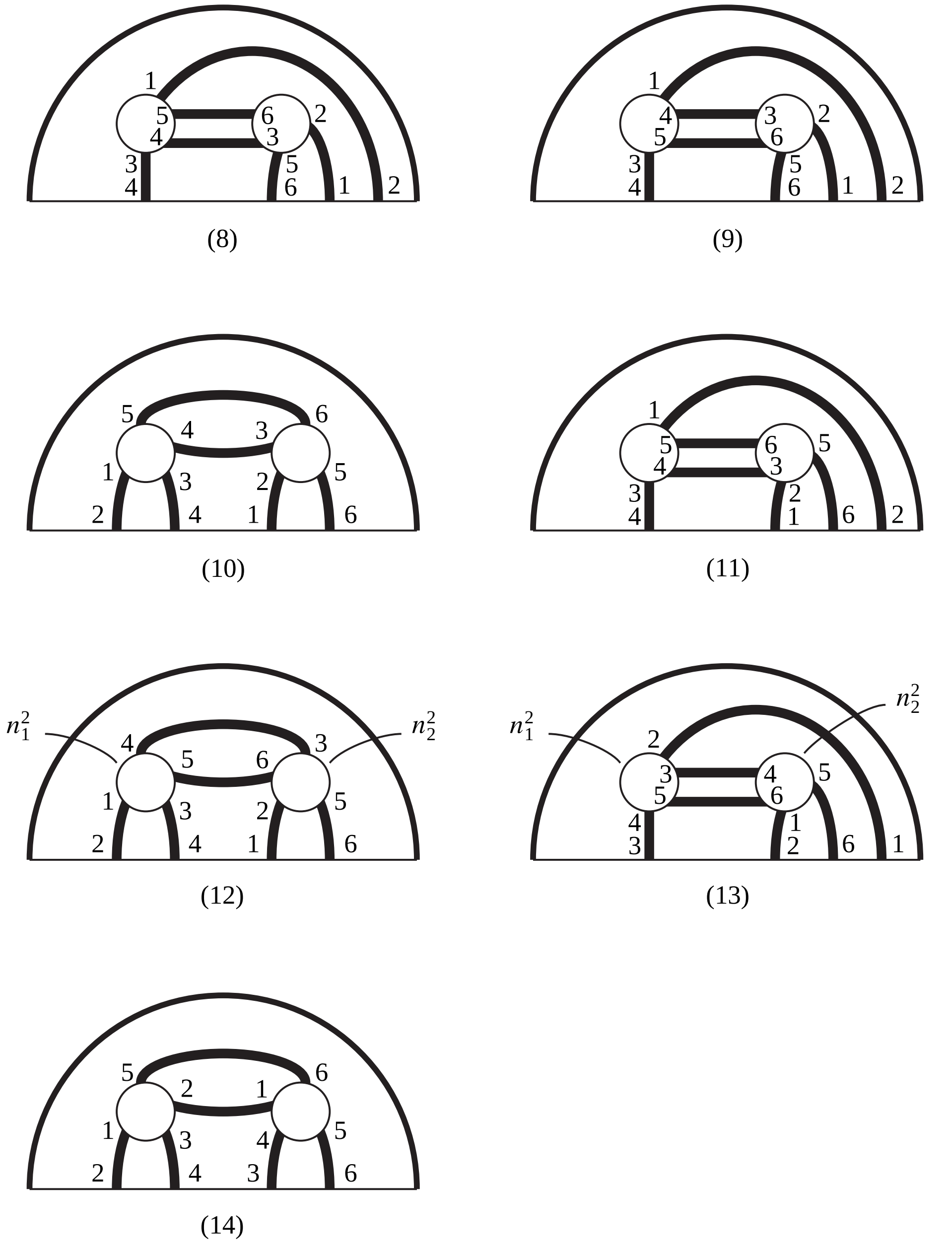


Fig. 5.7. (continued)

in the configuration of Fig. 5.6 (6), then the configuration is that of Fig. 5.7 (7)–(13) or (14), up to symmetry of D_i^2.

Lemma 5.19. *A configuration of arcs of type $\mathcal{B}$ on $\overline{D_i^2}$ is not that of* Fig. 5.7 (2)–(5), (7)–(11) *or* (14).

Proof. First we deal with the configuration of Fig. 5.7 (2). Let n_1^2, n_2^2 denote the components of ∂D_i^2 as illustrated in the figure. Note that both $g^2(n_1^2)$ and $g^2(n_2^2)$ are simple closed curves on S. The simple closed curve $g^2(n_2^2)$ intersects $g^2(n_1^2)$ transversely in one point at the image of the point labeled 1. This implies that each of $g^2(n_1^2)$ and $g^2(n_2^2)$ is a non-separating simple closed curve on a 2-sphere, a contradiction.

Similar arguments as above prove that a configuration of arcs of type $\mathcal{B}$ on $\overline{D_i^2}$ is not that of Fig. 5.7 (3), (4) or (5).

Next we deal with the configuration of Fig. 5.7 (7). Let n_1^2 and n_2^2 denote the components of ∂D_i^2, and let m^2 denote the arc $cl(\partial D_i^2 - (n_1^2 \cup n_2^2 \cup l_i^2))$ as illustrated in the figure. We denote by γ_m the subarc of m^2 such that $\partial\gamma_m$ consists of the two points labeled 2 and 6, and we denote by γ_n the subarc of n_2^2 such that $\partial\gamma_n$ consists of the two points labeled 2 and 6, and that $int\,\gamma_n$ is disjoint from the points labeled 3 and 5. Then the union $g^2(\gamma_m) \cup g^2(\gamma_n)$ forms a simple closed curve on S. The simple closed curve $g^2(n_1^2)$ intersects $g^2(\gamma_m) \cup g^2(\gamma_n)$ on S transversely in one point at the image of the point labeled 4. This implies that both $g^2(n_1^2)$ and $g^2(\gamma_m) \cup g^2(\gamma_n)$ are non-separating simple closed curves on a 2-sphere, a contradiction.

Similar arguments as above prove that a configuration of arcs of type $\mathcal{B}$ on $\overline{D_i^2}$ is not that of Fig. 5.7 (8)–(11) or (14). □

Lemma 5.20. *Suppose that a configuration of arcs of type $\mathcal{B}$ on $\overline{D_i^2}$ is that of* Fig. 5.7 (1). *Then the clasp number of K_i is at most one.*

Proof. Let n_1^2, n_2^2 denote the components of $\partial \overline{D_i^2}$, and β_1, β_2 denote arcs of type $\mathcal{B}$ as illustrated in the figure. Let m^2 be the arc $cl(\partial D_i^2 - (n_1^2 \cup n_2^2 \cup l_i^2))$. The image $g^2(n_1^2 \cup n_2^2 \cup m^2)$, which is unique up to isotopy and symmetry on S, is illustrated in Fig. 5.8 (1). The simple closed curve $g^2(n_1^2)$ bounds a disc δ on S such that $g^2(n_2^2)$ is not contained in δ. Isotope $g^2(N(n_1^2; D_i^2)) \cup \delta$ slightly into $int\,Q_i$. Then we obtain an immersion g^3 of an annulus D_i^3 into Q_i. This isotopy changes the union of the arcs $g^2(\beta_1) = g^2(\beta_2)$ and $g^2(m^2) \cap \delta$ to a singular arc γ of $g^3(D_i^3)$ such that $(g^3)^{-1}(\gamma)$ consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$ on D_i^3. Hence Σ_i^3 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Proposition 5.3 shows that the clasp number of K_i is at most one. □

Lemma 5.21. *Suppose that a configuraton of arcs of type $\mathcal{B}$ on $\overline{D_i^2}$ is that of* Fig. 5.7 (6). *Then the clasp number of K_i is at most one.*

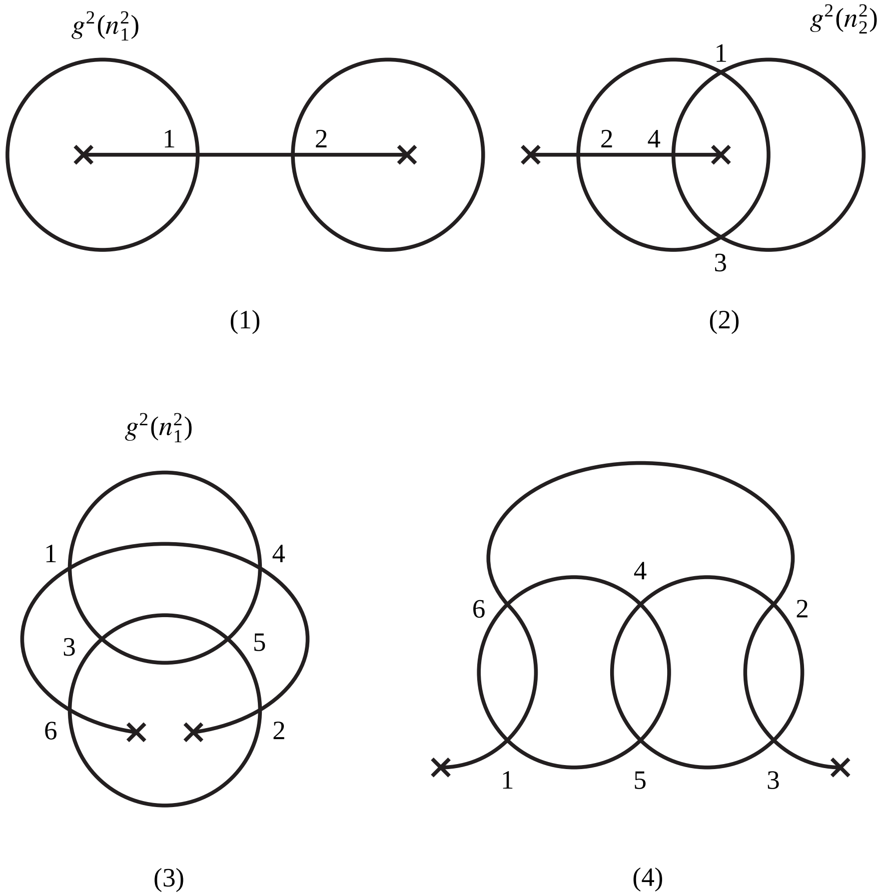


Fig. 5.8.

Proof. Let n_1^2, n_2^2 denote the components of $\partial\overline{D}_i^2$ as illustrated in the figure. Let m^2 be the arc $cl(\partial D_i^2 - (n_1^2 \cup n_2^2 \cup l_i^2))$. The image $g^2(n_1^2 \cup n_2^2 \cup m^2)$, which is unique up to isotopy and symmetry on S, is illustrated in Fig. 5.8 (2). The simple closed curve $g^2(n_2^2)$ bounds a disc δ on S such that the image of the point labeled 2 is not contained in δ. Isotope $g^2(N(n_2^2; \overline{D}_i^2)) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an immersion g^3 of an annulus into Q_i. Similar arguments as in the proof of Lemma 5.20 show that Σ_i^3 consists of one arc of type $\mathcal{A}$ and one arc of type $\mathcal{A}'$. Proposition 5.3 shows that the clasp number of K_i is at most one. $\square$

Lemma 5.22. *Suppose that a configuration of arcs of type $\mathcal{B}$ on $\overline{D}_i^2$ is that of* Fig. 5.7 (12). *Then K_i is the trivial knot.*

Proof. Let n_1^2, n_2^2 denote the components of $\partial \overline{D_i^2}$ as illustrated in the figure. Let m^2 denote the arc $cl(\partial \overline{D_i^2} - (n_1^2 \cup n_2^2 \cup l_i^2))$. The image $g^2(n_1^2 \cup n_2^2 \cup m^2)$, which is unique up to isotopy and symmetry on S, is illustrated in Fig. 5.8 (3). The simple closed curve $g^2(n_1^2)$ bounds a disc δ on S which contains no endpoints of the simple arc $g^2(m^2)$. Isotope $g^2(N(n_1^2; \overline{D_i^2})) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an immersion g^3 of an annulus into Q_i. Similar arguments as in the proof of Lemma 5.20 show that Σ_i^3 consists of two arcs of type $\mathcal{B}$. Proposition 5.1 shows that K_i is the trivial knot. □

Lemma 5.23. *Suppose that a configuration of arcs of type $\mathcal{B}$ on $\overline{D_i^2}$ is that of* Fig. 5.7 (13). *Then K_i is the trivial knot.*

Proof. Let n_1^2, n_2^2 denote the components of $\partial \overline{D_i^2}$ as illustrated in the figure. Let m^2 denote the arc $cl(\partial \overline{D_i^2} - (n_1^2 \cup n_2^2 \cup l_i^2))$. The image $g^2(n_1^2 \cup n_2^2 \cup m^2)$, which is unique up to isotopy and symmetry on S, is illustrated in Fig. 5.8 (4). The simple closed curve $g^2(n_1^2)$ bounds a disc δ on S which contains no endpoints of the simple arc $g^2(m^2)$. Isotope $g^2(N(n_1^2; \overline{D_i^2})) \cup \delta$ slightly into $int\, Q_i$. Then we obtain an immersion g^3 of an annulus into Q_i. Similar arguments as in the proof of Lemma 5.20 show that Σ_i^3 consists of two arcs of type $\mathcal{B}$. Proposition 5.1 shows that K_i is the trivial knot. □

By Lemmas 5.19–5.23, we may suppose that $\overline{D_i^2}$ is a twice-punctured disc and $\Sigma_i^2 = \emptyset$. The same arguments as in the proof of Lemma 5.5 show that K_i is the trivial knot.

This completes the proof of Proposition 5.14. □

Appendix

Kadokami obtained the following table in his Doctoral Dissertation [2]. This table gives us the clasp number of prime knots of eight or fewer crossings except 8_{18}. We refer to Rolfsen's table [12] for the nomenclature of knots. The clasp number of 8_{18} is not known yet.

knot	$cp(K)$
3_1	1
4_1	1
5_1	2
5_2	1
6_1	1
6_2	2
6_3	2
7_1	3
7_2	1
7_3	2
7_4	2
7_5	2
7_6	2
7_7	2
8_1	1
8_2	3
8_3	2

knot	$cp(K)$
8_4	2
8_5	3
8_6	2
8_7	3
8_8	2
8_9	3
8_{10}	3
8_{11}	2
8_{12}	2
8_{13}	2
8_{14}	2
8_{15}	2
8_{16}	3
8_{17}	3
8_{19}	3
8_{20}	2
8_{21}	2

ACKNOWLEDGEMENTS. The author would like to express his sincere gratitude to Professor Kanji Morimoto, Professor Chuichiro Hayashi and Dr. Teruhisa Kadokami for thier helpful comments.

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