Takai, M.
Osaka J. Math.
40 (2003), 751–778

DEGENERATING FAMILIES OF BRANCHED COVERINGS OF DISCS AND FUNDAMENTAL GROUPS OF 3-DIMENSIONAL MANIFOLDS

MAKI TAKAI

(Received May 18, 2001)

1. Introduction

The topological type of a degenerating family of finite branched coverings of discs can be determined by the pair (Φ, σ), the permutation monodromy Φ and the braid monodromy σ, which satisfy the equality $\Phi \circ \sigma = \Phi$.

By the theorem of Hilden [9]-Montesinos [11], every 3-dimensional compact oriented manifold can be expressed as a covering of degree 3 of the 3-sphere S^3 branching at a knot, whose monodromy at each branch point is a transposition. We regard S^3 as the boundary of a complex 2-dimensional polydisc. We also regard the knot as a braid. Taking cones, we get a topological degenerating family of branched coverings of discs. Thus every 3-dimensional compact oriented manifold can be constructed from the pair (Φ, σ), where Φ is a representation of the free group F_n of n generators onto the 3rd symmetric group S_3 such that the image by Φ of every generator is a transposition and σ is a braid of n strings with $\Phi \circ \sigma = \Phi$. Hence it is possible to compute the fundamental group of every 3-dimensional compact oriented manifold in this way, combining the theorem of Zariski-van Kampen (see Dimca [5]) and the method of Reidemeister-Schreier (see Rolfsen [14]).

There exist three canonincal forms of such Φ, that is, three canonical forms of monodromy representations Φ for coverings of discs of degree 3 with n (n is fixed) branch points such that the monodromy at each branch point is a transposition. Note that finite branched coverings of discs are compact Riemann surface deleted some discs from them. We consider branched coverings of degree 3, so we have compact Riemann surfaces deleted 1 (Case 3) or 2 (Case 2) or 3 (Case 1) discs from them. Each one has a canonical form of the monodromy. The braid σ such that $\Phi \circ \sigma = \Phi$ forms a subgroup of B_n of finite index. We call it the isotropy subgroup and denote it by $\mathrm{I}(\Phi)$. Birman and Wajnryb compute the generators of $\mathrm{I}(\Phi)$ for Case 2 and 3 in [3].

In this paper, we compute the generators of $\mathrm{I}(\Phi)$ for Case 1, and fundamental groups of some examples of 3-dimensional compact oriented manifolds using our method.

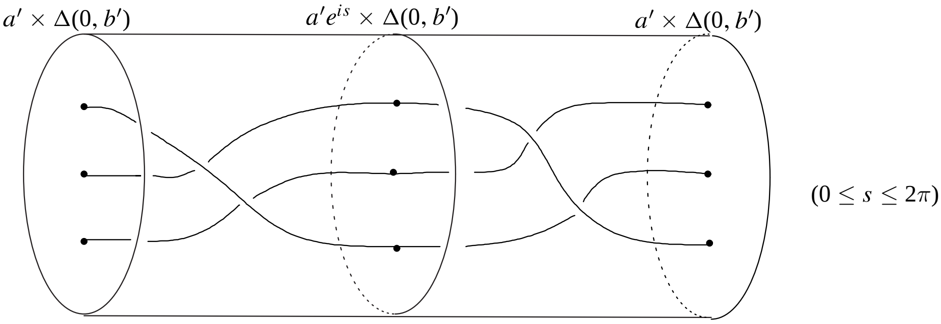

Fig. 1.

2. Connection between branched coverings of discs and 3-dimensional manifolds

By the theorem of Hilden-Montesinos (Hilden [9], Montesinos [11]), for every 3-dimensional compact oriented manifold Y, there exists a topological branched covering

$$h\colon Y \longrightarrow S^3$$

of the 3-sphere S^3 of degree 3 branching along a knot B_h, whose monodromy around the knot is given only by transpositions.

We regard the knot B_h as a braid, for every knot (and link) is isotopic in S^3 to a braid. We may identify S^3 with $\partial(\overline{\Delta(0, a')} \times \overline{\Delta(0, b')})$, where $\Delta(0, a')$ is the disc in the complex plane $\mathbb{C}$ with the center 0 and the radius a'. We may assume that B_h is contained in $\partial\overline{\Delta(0, a')} \times \Delta(0, b')$ as in Fig. 1.

Let B be the cone over B_h connecting every point of B_h with the origin of $\mathbb{C}^2$. Put $0 < a' < a$ and $0 < b' < b$. Let

$$f\colon X \longrightarrow \Delta(0, a) \times \Delta(0, b)$$

be the topological finite branched covering branching at B with the same monodromy as h. (Such a branched covering exists by Fox completion (Fox [7]). In fact X is a cone over Y.) Since X is a topological cone over Y,

$$\pi_1(X - \{x\}, p) \simeq \pi_1(Y, p), \quad (x = f^{-1}((0, 0))).$$

Put

$$\begin{aligned} X_t &= f^{-1}(t \times \Delta(0, b)), \\ f_t &= f|_{X_t}\colon X_t \longrightarrow t \times \Delta(0, b). \end{aligned}$$

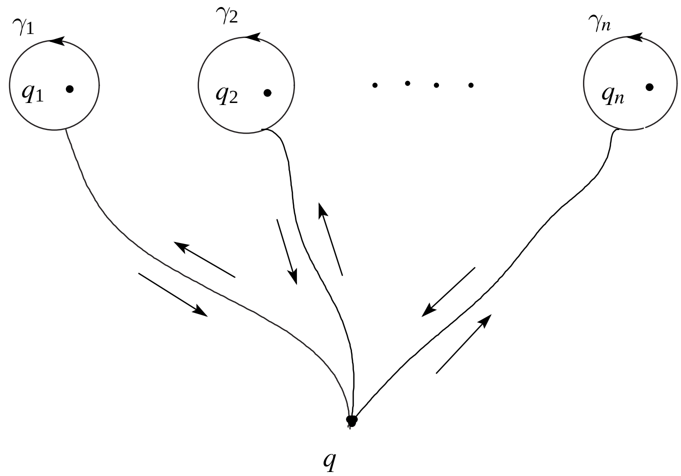

Fig. 2.

Then every $f_t (t \neq 0)$ is a finite branched covering of the disc $t \times \Delta(0, b)$, and f can be regarded as a topological degenerating family of finite branched coverings of discs: $f = \{f_t\}$. Its topological type is determined by the pair

$$(\Phi_t, \theta(\delta)), \quad (\delta\colon s \longmapsto a' e^{is}, (0 \leq s \leq 2\pi))$$

of the monodromy Φ_t of f_t (for a fixed $t \neq 0$) and the braid monodromy $\theta(\delta)$ of f. But they must satisfy the following equality (Namba [12]):

$$\Phi_t \circ \theta(\delta) = \Phi_t,$$

where $\theta(\delta)$ is regarded as an automorphism of $\pi_1(t \times \Delta(0, b) - B_f, q)$ (see Section 3).

Conversely, let

$$\Phi\colon \pi_1(\Delta(0, b) - \{n \text{ points}\}, q) \longrightarrow S_d$$

be a representation whose image is a transitive subgroup of the d-th symmetric group S_d. Let σ be a braid which satisfies

$$\Phi \circ \sigma = \Phi.$$

We denote the n points by $\{q_1, \ldots, q_n\}$ and let $\gamma_1, \ldots, \gamma_n$ be the lassos as in Fig. 2.

Then

$$\pi_1(\Delta(0, b) - \{q_1, \ldots, q_n\}, q) = \langle \gamma_1, \ldots, \gamma_n \rangle$$

is a free group. Put

$$A_j = \Phi(\gamma_j) \quad (j = 1, \ 2, \ldots, n).$$

We regard the braid σ as a link which is contained in $\partial\overline{\Delta(0,a')} \times \Delta(0,b')$ as in Fig. 1. By the condition $\Phi \circ \sigma = \Phi$, we can construct a topological branched covering

$$h\colon Y \longrightarrow \partial(\overline{\Delta(0,a')} \times \overline{\Delta(0,b')})$$

branching at the link σ whose monodromy is Φ. More precisely, we can construct a topological branched covering Y' of $\partial\overline{\Delta(0,a')} \times \Delta(0,b')$ branching at the link σ whose monodromy is Φ. We then attach solid tori to Y' at the part corresponding to the mutually prime cyclic decomposition of the permutation

$$A_\infty = \Phi(\gamma_n \cdots \gamma_1)^{-1} = (A_n \cdots A_1)^{-1}$$

over $\partial\overline{\Delta(0,a')} \times \partial\overline{\Delta(0,b')}$. Then we get a 3-dimensional compact oriented manifold Y and a topological finite branched covering

$$h\colon Y \longrightarrow \partial(\overline{\Delta(0,a')} \times \overline{\Delta(0,b')})$$

of the 3-sphere branching at the link σ whose monodromy is Φ.

We then construct the topological cone X of Y as above and construct a topological finite branched covering

$$f\colon X \longrightarrow \Delta(0,a) \times \Delta(0,b)$$

such that

$$\Phi_f = \Phi, \quad \theta(\delta) = \sigma.$$

This is regarded as a topological degenerating family of finite branched coverings of discs.

Thus to construct topological degenerating families of finite branched coverings of discs (hence to construct 3-dimensional compact oriented manifolds) is reduced to find out the pair (Φ, σ) as above such that $\Phi \circ \sigma = \Phi$.

3. Monodromy of a branched covering of degree 3 of the disc and its canonical forms

Let X and Y be Riemann surfaces and $f\colon X \longrightarrow Y$ a finite branched covering, that is, a surjective proper finite holomorphic mapping. A point p of X is called a *ramification point* of f if f is not biholomorphic around p. Its image $q = f(p)$ is called a *branch point* of f. The set of all ramification points (resp. branch points) is denoted by R_f (resp. B_f) and is called the *ramification locus* (resp. the *branch locus*). Then

$$f\colon X - f^{-1}(B_f) \longrightarrow Y - B_f$$

is an unbranched covering, whose mapping degree is called the *degree* of f and is denoted by $\deg f$. (X, f) (or simply f) is called a *finite branched covering* of Y.

DEFINITION 1. Two finite branched coverings

$$f : X \longrightarrow Y, \qquad f' : X' \longrightarrow Y$$

are said to be isomorphic if there is a biholomorphic mapping ψ which makes the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ Y & \xrightarrow{id} & Y \end{array}$$

DEFINITION 2. Two finite branched coverings

$$f : X \longrightarrow Y, \qquad f' : X' \longrightarrow Y$$

are said to be equivalent (resp. topologically equivalent) if there are biholomorphic mappings (resp. orientation preserving homeomorphisms) ψ and φ which make the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ Y & \xrightarrow{\varphi} & Y \end{array}$$

Let B_n be the Artin braid group of n strings. Then B_n is expressed as follows:

$$\begin{aligned} B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \; & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ & \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for} \quad |i - j| \geq 2 \rangle. \end{aligned}$$

Let $\{q_1, \ldots, q_n\}$ be a set of n distinct points in $\mathbb{C}$. The fundamental group $\pi_1(\mathbb{C} - \{q_1, \ldots, q_n\}, q)$ is the free group

$$\pi_1(\mathbb{C} - \{q_1, \ldots, q_n\}, q) = \langle \gamma_1, \ldots, \gamma_n \rangle$$

generated by the lassos $\gamma_1, \ldots, \gamma_n$ as in Fig. 2.

The braid group B_n acts on this group as follows:

$$\begin{aligned} \sigma_i(\gamma_i) &= \gamma_i^{-1} \gamma_{i+1} \gamma_i \\ \sigma_i(\gamma_{i+1}) &= \gamma_i \end{aligned}$$

$$\sigma_i(\gamma_j) = \gamma_j \qquad (j \neq i,\ i+1).$$

Note that this action is faithful (Birman [2]). A similar assertion holds if we replace $\mathbb{C}$ by a disc $\Delta(0,b)$.

The following theorem is well known:

Theorem 1. *Put $B = \{q_1, \dots, q_n\} \subset \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. For any homomorphism $\Phi\colon \pi_1(\mathbb{P}^1 - B, q) \longrightarrow S_d$ whose image $\operatorname{Im}\Phi$ is transitive, there exists a unique (up to isomorphisms) finite branched covering $f\colon X \longrightarrow \mathbb{P}^1$ such that*

$$B_f \subset B, \qquad \Phi_f = \Phi.$$

For the proof of Theorem 1, see Forster [6]. There is a higher dimensional analogy of the theorem (Grauert-Remmert [8]). The following theorem also seems to be well known:

Theorem 2. *For two finite branched coverings $f\colon X \longrightarrow \mathbb{P}^1$, $f'\colon X' \longrightarrow \mathbb{P}^1$ such that $B_f = B_{f'} = \{q_1, \dots, q_n\} \subset \mathbb{C}$, they are topologically equivalent if and only if there is a braid σ in B_n such that $\sigma^*(\Phi_f) = \Phi_f \circ \sigma = \Phi_{f'}$. Here the equality is that as representation classes. Moreover $\mathbb{P}^1$ can be replaced by $\mathbb{C}$ or a disc in $\mathbb{C}$.*

For the proof of Theorem 2, see Namba [12] or Namba-Takai [13].

Every branched covering

$$f\colon X \longrightarrow \Delta(0,b)$$

of degree d can be extended to a branched covering

$$\hat{f}\colon \hat{X} \longrightarrow \mathbb{P}^1$$

of degree d in the following canonical manner: Put

$$B_f = \{q_1, \dots, q_n\}, \quad A_j = \Phi_f(\gamma_j) \quad (j = 1, \dots, n),$$

where γ_j is a lasso as in Fig. 2. Let γ_∞ be the lasso around the point ∞ as in Fig. 3.

Then

$$\pi_1(\mathbb{P}^1 - \{q_1, \dots, q_n, \infty\}, q) = \langle \gamma_1, \dots, \gamma_n, \gamma_\infty \mid \gamma_\infty\gamma_n \cdots \gamma_1 = 1 \rangle.$$

Put

$$A_\infty = (A_n \cdots A_1)^{-1}.$$

We define a homomorphism

$$\Phi\colon \pi_1(\mathbb{P}^1 - \{q_1, \dots, q_n, \infty\}, q) \longrightarrow S_d$$

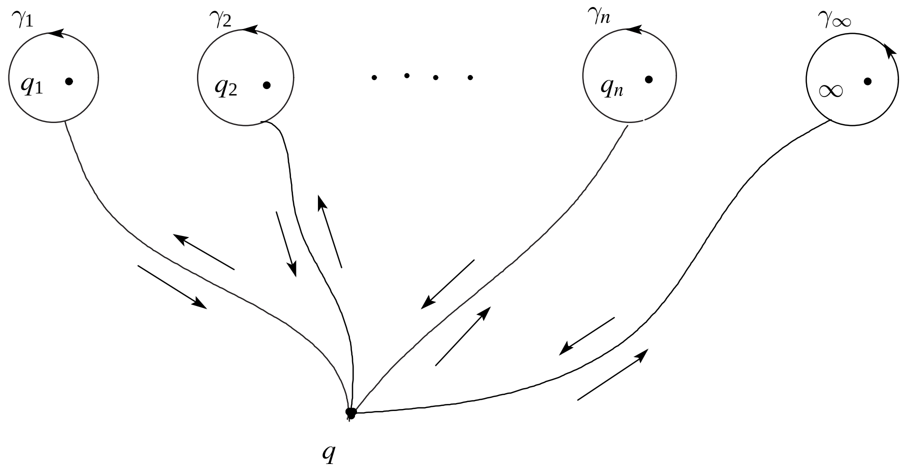

Fig. 3.

by

$$\Phi(\gamma_j) = A_j \quad (j = 1, \ldots, n), \qquad \Phi(\gamma_\infty) = A_\infty.$$

Then the branched covering

$$\hat{f} \colon \hat{X} \longrightarrow \mathbb{P}^1$$

corresponding to Φ (see Theorem 1) is an extension of f.

Note that if $A_\infty = 1$, then $\hat{f}$ does not branch at the point ∞.

Let

$$f \colon X \longrightarrow \Delta(0, b)$$

be a branched covering of the disc $\Delta(0, b)$ of degree 3. Let $\gamma_j (j = 1, \ldots, n)$ be the lassos as in Fig. 2. Put $A_j = \Phi_f(\gamma_j)$ $(j = 1, \ldots, n)$. Suppose that every A_j is a transposition in the 3rd symmetric group S_3. As above, we extend the covering to that of $\mathbb{P}^1$ which is denoted by the same notation f for simplicity. Let γ_∞ be the lasso around the point ∞ and put

$$A_\infty = (A_n \cdots A_1)^{-1} = \Phi_f(\gamma_\infty)$$

as above. There are three cases:

CASE 1. $A_\infty = 1$. In this case, the extended covering does not branch at ∞.

CASE 2. A_∞ is a transposition. In this case, the point ∞ is a branch point, that is, there is a point over ∞ with the ramification index 2. Since we may change the monodromy with an equivalent representation, we may assume that $A_\infty = (1\ 2)$.

CASE 3. A_∞ is a cyclic permutation. In this case, the point ∞ is a branch point. We may assume that $A_\infty = (1\ 3\ 2)$.

Under these assumptions, we have the following theorem:

Theorem 3. *Under the above assumptions, the covering f is topologically equivalent to one of the following canonical forms: Arranging $A_1, A_2, \dots, A_n$ in this order:*

CASE 1: $(1\ 2), (1\ 2), (2\ 3), (2\ 3), \underbrace{(2\ 3), (2\ 3), \dots, (2\ 3)}_{2g}$

CASE 2: $(1\ 2), (2\ 3), (2\ 3), \underbrace{(2\ 3), (2\ 3), \dots, (2\ 3)}_{2g}$

CASE 3: $(1\ 2), (2\ 3), \underbrace{(2\ 3), (2\ 3), \dots, (2\ 3)}_{2g}$

where g is the genus of the Riemann surface X.

Theorem 3 can be proved along a similar line to that of Birman-Wajnryb [3] or Bauer-Catanese [1], so we omit it.

4. Isotropy subgroups of the braid groups

Let

$$\Phi\colon \langle \gamma_1, \dots, \gamma_n \rangle \longrightarrow S_d$$

be a representation of the free group $\langle \gamma_1, \dots, \gamma_n \rangle$ of n generators into the d-th symmetric group S_d whose image $\operatorname{Im}\Phi$ is transitive.

By the discussion in Section 2, it is important to consider the braid $\sigma \in B_n$ such that $\Phi \circ \sigma = \Phi$, where the equality is not as representation classes but is just as representations. (The action of the braid σ on the free group $\langle \gamma_1, \dots, \gamma_n \rangle$ is defined in Section 3.) Put

$$\mathrm{I}(\Phi) = \{\sigma \in B_n \mid \Phi \circ \sigma = \Phi\},$$

the isotropy subgroup of B_n for Φ.

Since the number of representations Φ is finite (in fact is less than $(d!)^n$), $\mathrm{I}(\Phi)$ is a subgroup of B_n of finite index.

Note that the following equality holds:

$$\mathrm{I}(\Phi \circ \tau) = \tau^{-1}\mathrm{I}(\Phi)\tau.$$

Put

$$\Phi(\gamma_j) = A_j \quad (j = 1, 2, \dots, n).$$

Now, let Φ be the representation of one of the canonical forms as in Theorem 3.

The following theorem is due to Birman-Wajnryb [3].

Theorem 4 (Birman-Wajnryb [3]). *For* Cases 2 *and* 3 (*i.e*, $A_1 = (1\ 2)$, $A_2 = \cdots = A_n = (2\ 3)$), $\mathrm{I}(\Phi)$ *is generated by the following elements*:

$$\sigma_1^3, \sigma_2, \dots, \sigma_{n-1},$$
$$\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-1}\sigma_3^{-1}\sigma_4\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3^2\sigma_2\sigma_1\ (n \geq 5).$$

The following theorem for Case 1 (i.e, $A_1 = A_2 = (1\ 2)$, $A_3 = \cdots = A_n = (2\ 3)$, where n is even) is our main result in this paper.

Theorem 5. *For* Case 1, $\mathrm{I}(\Phi)$ *is generated by the following elements*:

$$\sigma_1, \sigma_2^3, \sigma_3, \dots, \sigma_{n-1}, \sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^2\sigma_2,$$
$$\sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-2}\sigma_3^{-1}\sigma_2^{-2}\sigma_3^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_3\sigma_2^2\sigma_3\sigma_4^2\sigma_3\sigma_2\ (n \geq 6).$$

REMARK 1. For Case 1, the generators of the isotropy subgroup $\mathrm{I}(\Phi)$ of $B_n(S^2)$ are described in Birman-Wajnryb ([3]) but not of B_n.

5. Preliminary for proof of Theorem 5

In this section we recall some notations and results in the papers of Birman-Wajnryb [3] and [4].

Let $\Delta \subset \mathbb{C}$ be a disc, X a Riemann surface with boundary and $f\colon X \longrightarrow \Delta$ a branched covering of degree 3. We assume that f is simple i.e., the inverse image of every point in Δ contains at least two distinct points.

Let $B = \{q_1, \dots, q_n\}$ be the set of the branch points of f and q a fixed base point on the boundary $\partial\Delta$. Let

$$\Phi\colon \pi_1(\Delta - \{q_1, \dots, q_n\}, q) \longrightarrow S_3$$

be the monodromy homomorphism of f. The *total monodromy* is by definition the monodromy of the loop $\partial\Delta$ in the clockwise direction.

Let us recall that B_n can be identified with the group of isotopy classes of the homeomorphisms of Δ that leave B invariant and $\partial\Delta$ pointwise fixed. For an element x in B_n, we denote by $\overline{x}$ the inverse of x in B_n. We say that $h \in B_n$ is *liftable* if it has a representative that can be lifted to a fiber-preserving homeomorphism of X which fixes every point of the fiber $f^{-1}(q)$. Note that $h \in B_n$ is liftable if and only if $h \in \mathrm{I}(\Phi)$.

By a *curve* in Δ we mean a simple path in Δ such that (i) the initial point is q, (ii) the terminal point is some branch point q_j, (iii) it does not pass through the other branch points than q_j and (iv) it does not pass through the boundary points of Δ. By

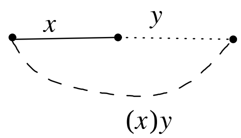

Fig. 4.

the *monodromy of a curve* α we mean $\Phi(\gamma)$ ($\in S_3$), where γ is a simple closed path in Δ which bounds a region U such that (i) U contains α and (ii) the closure $\overline{U} = U \cup \gamma$ of U does not contain the other branch points than the terminal point of α.

By a *Hurwitz system* we mean an ordered set of curves $\alpha_1, \dots, \alpha_n$ which meet only at q in the clockwise order. The *monodromy sequence of a Hurwitz system* $\alpha_1, \dots, \alpha_n$ is by definition the sequence of the monodromy of the curves $\alpha_j (1 \leq j \leq n)$. The *total monodromy* of a Hurwitz system is by definition the product of the monodromy sequence of a Hurwitz system.

The following lemma is fundamental (cf. Birman-Wajnryb [3] p. 27):

Lemma 1. *A homeomorphism $h \in B_n$ is liftable if and only if it preserves the monodromy sequence of some Hurwitz system.*

By an *interval* in Δ we mean a simple path such that (i) it connects two branch points and (ii) it meets neither other branch points nor boundary points. Let x be an interval. Let U be small neighborhood of x which is homeomorphic to a disc. By a *rotation around* x we mean a homeomorphism h of Δ or the element of B_n corresponding to h such that (i) h is equal to the identity mapping outside U, (ii) h rotates U by 180 degrees counterclockwise (up to isotopy), (iii) h maps x onto itself and (iv) h reverses the ends of x. Rotations around isotopic intervals represent the same element of B_n. Hence we do not distinguish between isotopic intervals. Thus the action of an element of B_n on an interval can be defined. We denote by $(x)y$ the image of an interval x under a rotation around an interval y. (see Fig. 4.)

The following two lemmas can be deduced from Lemma 1 immediatly (cf. Birman-Wajnryb [3] p. 28).

Lemma 2. *Let x be an interval and α a curve. Assume that α meets x only at its end point. Then x is liftable if and only if $\Phi(\alpha) = \Phi((\alpha)x)$.*

Lemma 3. *Let x and y be intervals which meet only at one common end point. Assume that x and y are not liftable. Then $z = (x)y$ is liftable $\Longleftrightarrow$ $z_1 = (x)y^2$ is not liftable $\Longleftrightarrow$ $z_2 = (x)\overline{y}$ is not liftable.*

We say that a sequence of intervals $x_1, \dots, x_k$ makes up a *chain* if (i) the consecutive intervals have the common end points and no other intersection points and (ii)

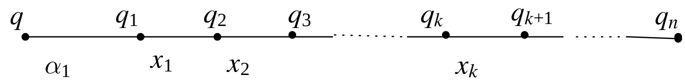

Fig. 5.

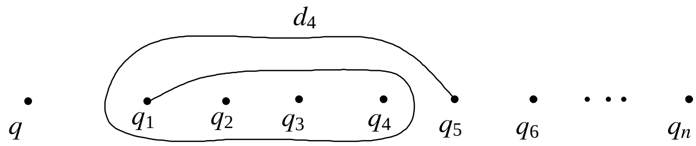

Fig. 6.

other pairs of intervals have no intersection points. A chain is said to be *maximal* if it contains all the branch points. For a maximal chain of intervals, B_n is generated by rotations around its elements ([2]). For a Hurwitz system $\alpha_1, \dots, \alpha_n$, there corresponds a maximal chain of intervals $x_1, \dots, x_{n-1}$ such that x_i is homotopic to $\alpha_i \cup \alpha_{i+1}$. Note that α_{i+1} is isotopic to $(\alpha_i)x_i$ in this case. Conversely, for a maximal chain of intervals $x_1, \dots, x_{n-1}$ and a curve α_1 which meets the chain only at the initial point of x_1, there corresponds a Hurwitz system $\alpha_1, \dots, \alpha_n$ such that $\alpha_{i+1} = (\alpha_i)x_i$ for $i = 1, 2, \dots, n-1$. A chain of intervals $x_1, \dots, x_k$ is said to be *regular* if x_1 is not liftable and $x_j (j = 2, \dots, k)$ are liftable.

A curve α in Δ is said to be *separating* if every interval in the complement of α is liftable. A curve α in Δ is said to be *regular* if the complement of α contains a maximal regular chain of intervals.

Let q be a fixed point on $\partial\Delta$. Let

$$\hat{f} \colon \hat{X} \longrightarrow \mathbb{P}^1$$

be an extension of f. If $\hat{f}$ branches at the point ∞ then by Theorem 3 (Cases 2 and 3) we can choose a Hurwitz system of curves $\alpha_1, \dots, \alpha_n$ with the monodromy sequence (1 2), (2 3), $\dots$, (2 3). Let q_i be the end point of α_i, and $x_1, \dots, x_{n-1}$ a maximal chain of intervals corresponding to the Hurwitz system. We note that $x_1, \dots, x_{n-1}$ is regular. By replacing f by its suitable topological equivalent branched covering, we may assume that Δ is the unit disc in $\mathbb{C}$, $q = -1$, and the paths α_1, $x_1, \dots, x_{n-1}$ lie on the real axis from left to right. (see Fig. 5.) Note that B_n is generated by the rotations around $x_1, \dots, x_{n-1}$. Let us denote by L_n the subgroup of the liftable elements of B_n. Let d_4 be the interval $(x_4)x_3x_2x_1^2x_2x_3^2x_2x_1$ in Fig. 6.

REMARK 2. The rotation x_i corresponds to the braid σ_i^{-1}.

In the notations above Theorem 4 is expressed as follows:

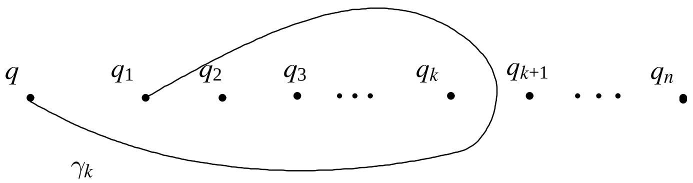

Fig. 7.

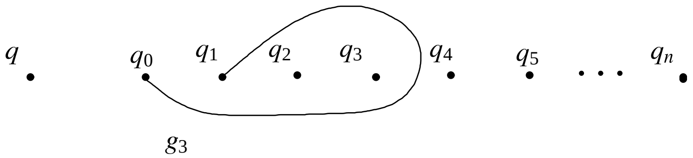

Fig. 8.

Theorem 4 (restated). *The group L_n is generated by the rotations*

$$x_1^3, x_2, \ldots, x_{n-1} \quad \text{and} \quad d_4 \ (n \geq 5).$$

Let G be a subgroup of B_n. Intervals (or curves) x, y are said to be *G-equivalent* if there exists $g \in G$ such that $(x)g = y$. If x and y are intervals, then the rotation $(x)g = y$ implies that the rotation y is equal to $\overline{g}xg$. For a curve or an interval x, we denote by x' the path symmetric to x with respect to the real axis. For $k = 2, \ldots, n$ let γ_k be the curve $(\alpha_1)x_1 \cdots x_{k-2}x_{k-1}^2 x_{k-2} \cdots x_1$ represented in Fig. 7.

Let $\alpha_{j+1} = (\alpha_j)x_j (j = 1, \ldots, n-1)$.

Proposition 1 (Birman-Wajnryb). *Every curve in Δ is L_n-equivalent to some of the curve α_i, γ_i, α_i' or γ_i'.*

6. Proof of Theorem 5

In this section we treat the case where $\hat{f}$ does not branch at the point ∞, i.e. Case 1.

Let $\tilde{B} := \{q_0, q_1, \ldots, q_n\}$ be the set of branch points lying on the real axis, in this order. In Case 1, the number of branch points is even. Hence we may assume that n is odd. By Theorem 3 we can find a Hurwitz system of curves $\alpha_0, \alpha_1, \ldots, \alpha_n$ with the monodromy sequence (1 2), (1 2), (2 3), $\ldots$, (2 3). Let $x_0, x_1, \ldots, x_{n-1}$ be a maximal chain of intervals corresponding to the Hurwitz system. The group $\tilde{B}_n \simeq B_{n+1}$ of the isotopy classes of homeomorphisms of Δ that leave $\tilde{B}$ invariant and $\partial\Delta$ pointwise fixed, is generated by the rotations $x_0, x_1, \ldots, x_{n-1}$. We denote by $\tilde{L}_n$ the subgroup of the liftable elements of $\tilde{B}_n$. Let g_3 be the interval $(x_0)x_1x_2^2x_1$ in Fig. 8.

REMARK 3. The rotation x_i corresponds to the braid σ_{i+1}^{-1}.

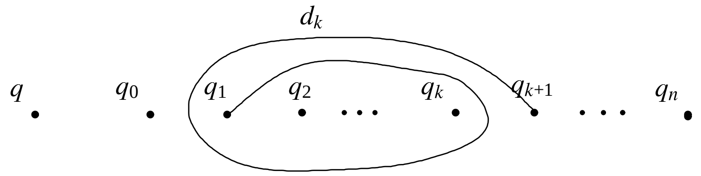

Fig. 9.

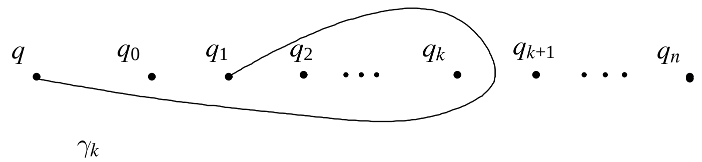

Fig. 10.

Let d_k be the interval $(x_k)x_{k-1}\cdots x_2x_1^2x_2\cdots x_{k-2}x_{k-1}^2x_{k-2}\cdots x_1$ represented in Fig. 9:

Theorem 5 can be expressed as follows:

Theorem 5 (restated). *The group $\tilde{L}_n$ is generated by the rotations*

$$x_0, x_1^3, x_2, \ldots, x_{n-1}, g_3 \quad \textit{and} \quad d_4 \ (n \geq 5).$$

Let us denote by $\tilde{N}$ the group generated by the rotations

$$x_0, x_1^3, x_2, \ldots, x_{n-1}, g_3 \text{ and } d_4.$$

Let γ_k be the curve $(\alpha_1)x_1\cdots x_{k-2}x_{k-1}^2x_{k-2}\cdots x_1$ represented in Fig. 10.

A curve is said to be *admissible* if it is $\tilde{N}$-equivalent to some of the curves α_i, γ_i, α_i' or γ_i'. An interval x is said to be *admissible* if either (i) $x \in \tilde{N}$ or (ii) $x \notin \tilde{L}_n$ but $x^3 \in \tilde{N}$. Note that if x is an admissible interval or an admissible curve and if y is $\tilde{N}$-equivalent to x, then y is admissible.

Theorem 5 is clearly a consequence of the following:

Proposition 2. *$\tilde{N} = \tilde{L}_n$. Moreover every curve in Δ is admissible.*

We prove Proposition 2 in a similar way to Birman-Wajnryb [3].

By Theorem 4 we get

Lemma 4. *If h is liftable and $(\alpha_0)h = \alpha_0$, then $h \in \tilde{N}$.*

REMARK 4. If k is even, then d_k and d_k' are liftable. Therefore d_k, $d_k' \in \tilde{N}$ by Lemma 4.

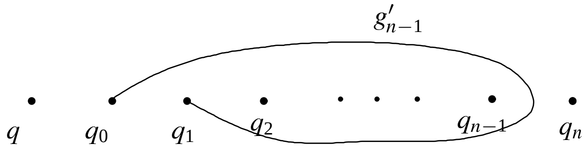

Fig. 11.

Lemma 5. *If h is liftable and $(\alpha_n)h = \alpha_n$, then $h \in \tilde{N}$.*

Proof. Since h is liftable, h preserves the monodromy sequence of some Hurwitz system. Now, we consider a Hurwitz system

$$(\alpha_0)g'_{n-1}, (\alpha_1)g'_{n-1}, \dots, (\alpha_{n-1})g'_{n-1}.$$

The monodromy sequence of this system is (2 3), (1 3), …, (1 3), where g'_{n-1} means the interval $(x_0)\overline{x_1}\cdots\overline{x_{n-3}}\,\overline{x_{n-2}}^2\,\overline{x_{n-3}}\cdots\overline{x_1}$. (see Fig. 11.)

Let y_i be an interval which is homotopic to the union $(\alpha_{i-1})g'_{n-1} \cup (\alpha_i)g'_{n-1}$. By Theorem 4, h belongs to the group N_1 which is generated by the rotations

$$y_1^3, y_2, \dots, y_{n-1}, (y_4)y_3y_2y_1^2y_2y_3^2y_2y_1.$$

Note that $h \in N_1$. We prove $N_1 \subset \tilde{N}$.

We can check that

$$(y_1)\overline{d'_{n-1}}\,\overline{x_0}\,x_{n-1}\cdots x_2 = x_1.$$

Hence y_1 is $\tilde{N}$-equivalent to x_1; moreover y_1^3 is $\tilde{N}$-equivalent to x_1^3. It follows $y_1^3 \in \tilde{N}$.

We can check also that

$$(y_2)\overline{x_2}\cdots\overline{x_{n-3}}\,d_{n-3}x_{n-3}x_{n-2}x_{n-4}x_{n-3}\cdots x_2x_3\overline{x_0} = g_3,$$
$$y_k = x_{k-1} \quad \text{for } k \neq 1,\ 2$$

and

$$\{(y_4)y_3y_2y_1^2y_2y_3^2y_2y_1\}\overline{x_4}\cdots\overline{x_{n-3}}\,\overline{x_{n-2}}^2\,\overline{x_{n-3}}\cdots\overline{x_4}\,\overline{x_3}\,\overline{x_4}\cdots\overline{x_{n-2}}\,\overline{x_2}\,\overline{x_3}\cdots\overline{x_{n-3}}\,d'_{n-3}$$
$$\overline{x_2}\cdots\overline{x_{n-5}}\,\overline{x_{n-4}}^2\,\overline{x_{n-5}}\cdots\overline{x_2}\,x_{n-2}\,\overline{s'_{n-1}}^3\,h_1^2\,\overline{g'_3}\,\overline{x_3}\,\overline{x_2}\,\overline{x_4}\,\overline{x_3}\cdots\overline{x_{n-4}}\,\overline{x_{n-5}} = d_{n-5},$$

where s'_{n-1} denotes the interval $(x_0)\overline{x_1}\cdots\overline{x_{n-2}}$ (see Fig. 12) and h_1 is a Dehn twist around the loop u_1, which points q_0, q_{n-1} and q_n are outside u_1 and the union $x_1 \cup \cdots \cup x_{n-3}$ is inside u_1. Then s'_{n-1} is $\tilde{N}$-equivalent to x_1. Hence $s'^3_{n-1} \in \tilde{N}$, and h_1^2 is liftable (this can be checked by using Lemma 1). Hence $h_1^2 \in \tilde{N}$ by Lemma 4.

Since all generators of N_1 belong to $\tilde{N}$, it follows that $N_1 \subset \tilde{N}$. $\square$

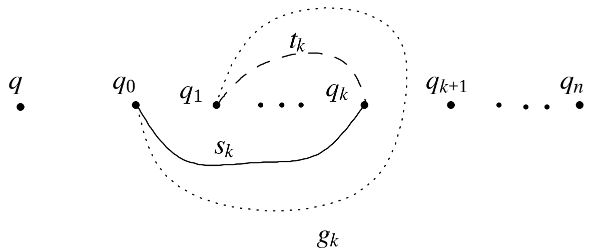

Fig. 12.

Lemma 6. *An interval x is admissible if it does not meet some α_i.*

Proof. If x does not meet α_0 then it is admissible by Lemma 4. If x does not meet α_n, then it is admissible by Lemma 5. The curve α_1 is $\tilde{N}$-equivalent to α_0 and for $i \neq 0, 1$ the curve α_i is $\tilde{N}$-equivalent to α_n which proves the lemma. □

Lemma 7. *Let $s_k = (x_0)x_1 \cdots x_{k-1}$, $t_k = (x_{k-1})x_{k-2} \cdots x_1$ and $g_k = (s_k)t_k$ for $k = 2, \ldots, n$. (see* Fig. 12.) *Then g_k is liftable for k odd, g_k is not liftable for k even, and g_k and g_k' are admissible for each k.*

Proof. It is easy to see that s_k and t_k are $\tilde{L}_n$-equivalent to x_1. Hence they are not liftable. $g_2 = (x_0)x_1^2$ is not liftable. $g_{k-1} = (s_k)\overline{t_k}$ for all k. Hence, by Lemma 3, g_k is liftable for k odd and is not liftable for k even. If $k < n$, then g_k is admissible by Lemma 5.

For g_n, we can check that

$$(\alpha_0)g_n\,\overline{g_{n-2}}\,\overline{x_{n-2}}\,\overline{x_{n-1}}\,\overline{x_{n-3}}\,\overline{x_{n-2}}\,g_{n-4}\,\overline{g_{n-2}} = \alpha_0.$$

Hence, by Lemma 4, the product on the left side belongs to $\tilde{N}$. Since all factors different from g_n belong to $\tilde{N}$, g_n belongs to $\tilde{N}$.

Finally we have $g_k' = (g_k)x_0$. Hence g_k' is admissible for all k. □

Let w be a Dehn twist around the boundary $\partial\Delta$. Then $w = (x_0x_1 \cdots x_{n-1})^{n+1}$ and w is a generator of the center of $\tilde{B}_n$.

REMARK 5. Note that $w \in \tilde{L}_n$ and $(\alpha_0)w\overline{x}_0\overline{g}_n = \alpha_0$. Hence, by Lemma 4, $w \in \tilde{N}$.

For $j = 2, \ldots, n$, we denote by $\tilde{\alpha}_j$ the curve $(\alpha_0)\overline{x}_0x_1 \cdots x_{j-1}$ and by δ_j the curve $(\alpha_0)\overline{x}_0x_1 \cdots x_{j-2}x_{j-1}^2x_{j-2} \cdots x_1$.

Lemma 8. *An admissible curve β is $\tilde{N}$-equivalent to some of the curves α_0, α_n or $\tilde{\alpha}_2'$.*

Proof. We show that every curve in α_i, α'_i, γ_i, γ'_i, $(i = 0, \dots, n)$ is $\tilde{N}$-equivalent to some of the curves α_0, α_n or $\tilde{\alpha}'_2$. The curves α_1 and α'_1 are $\tilde{N}$-equivalent to α_0. For k odd, we have g_k and $g'_k \in \tilde{N}$. Since $\gamma_k = (\alpha_0)g_k$ and $\gamma'_k = (\alpha_0)\overline{g'_k}$, γ_k and γ'_k are $\tilde{N}$-equivalent to α_0. For $i = 2, \dots, n-1$, the curve α_i is $\tilde{N}$-equivalent to α_n and the curve α'_i is $\tilde{N}$-equivalent to α'_n. Note that $\alpha_n = (\alpha'_n)w$. Hence, α'_i is $\tilde{N}$-equivalent to α_n.

For k even, we have $d_k \in \tilde{N}$ and $(\gamma_k)\overline{d}_k x_k \cdots x_2 = \tilde{\alpha}'_2$. Hence γ_k is $\tilde{N}$-equivalent to $\tilde{\alpha}'_2$.

Since $(\gamma'_k)w$ is $\tilde{N}$-equivalent to γ_{n-k+1}, γ'_k is also $\tilde{N}$-equivalent to $\tilde{\alpha}'_2$. □

Lemma 9. *If a curve β meets some α_i only at q, then β is admissible.*

Proof. If β meets α_1 only at q, then β is $\tilde{N}$-equivalent to a curve which meets α_0 only at q. Hence, we may assume that β meets α_0 only at q. By Proposition 1, β is admissible or $\tilde{N}$-equivalent to some of the curves $\tilde{\alpha}_j$, $\tilde{\alpha}'_j$, δ_j or δ'_j, $j = 2, \dots, n$.

$\tilde{\alpha}_j$, $j = 3, \dots, n-1$, is $\tilde{N}$-equivalent to $\tilde{\alpha}_2$. We have $(\tilde{\alpha}_2)\overline{x_1}^3 = \gamma'_2$. Therefore $\tilde{\alpha}_j$ is admissible for any j. Similarly we can show that $\tilde{\alpha}'_j$ is admissible for any j.

If k is odd, then g_k and g'_k belong to $\tilde{N}$. Since $(\delta_k)g_k = \alpha_0$ and $(\delta'_k)\overline{g'_k} = \alpha_0$, δ_k and δ'_k are admissible. If k is even, then d_k and d'_k belong to $\tilde{N}$. Since $(\delta_k)\overline{d_k} = \alpha'_{k+1}$ and $(\delta'_k)d'_k = \alpha_{k+1}$, δ_k and δ'_k are admissible.

If β meets α_i, $i \neq 0, 1$, only at q, then it is $\tilde{N}$-equivalent to a curve which meets α_n only at q. If β starts on the right side of α_n, then $(\beta)\overline{w}$ starts on the left side of α_n. So we may assume that β starts on the left side of α_n.

We consider the restriction of f to $\Delta - \alpha_n$. The total monodromy of the complement of α_n is (2 3). If we take the Hurwitz system

$$(\alpha_0)g'_{n-1}, (\alpha_1)g'_{n-1}, \dots, (\alpha_{n-1})g'_{n-1}$$

as in the proof of Lemma 5, then by Proposition 1 β is N_1-equivalent to some of the curves $(\alpha_i)g'_{n-1}$, $(\alpha'_i)g'_{n-1}$, $(\gamma_i)x_0g'_{n-1}$ or $(\gamma'_i)\overline{x_0}g'_{n-1}$. Since $N_1 \subset \tilde{N}$ (see the proof of Lemma 5), β is $\tilde{N}$-equivalent to some of the curves $(\alpha_i)g'_{n-1}$, $(\alpha'_i)g'_{n-1}$, $(\gamma_i)x_0g'_{n-1}$ or $(\gamma'_i)\overline{x_0}g'_{n-1}$.

We can check that

$$\{(\alpha_0)g'_{n-1}\}d'_{n-1} = \alpha_n,$$
$$\{(\alpha_1)g'_{n-1}\}\overline{x_0} = \gamma_{n-1}$$

and

$$\{(\alpha_k)g'_{n-1}\}\,\overline{x_{k-1}}\cdots\overline{x_2}\,x_1^3x_2\cdots x_{n-2}\overline{g'_{n-2}}\,\overline{x_0} = \gamma_{n-1} \quad \text{for } k \neq 0,\ 1.$$

Since the interval $(g'_{n-1})x_1$ is not liftable, $(g'_{n-1})\overline{x_1}$ is liftable by Lemma 3, and $(g'_{n-1})\overline{x_1}$ belongs to $\tilde{N}$ by Lemma 5.

We can also check that

$$
\begin{aligned}
&\{(\alpha_1')g_{n-1}'\}\overline{((g_{n-1}')\overline{x_1})}\,\overline{x_2}\cdots\overline{x_{n-3}}\,\overline{x_{n-2}}^2\,\overline{x_{n-3}}\cdots\overline{x_2}\,x_1^3x_2\cdots x_{n-2}\,\overline{g_{n-2}'}=\alpha_0,\\
&\{(\alpha_k')g_{n-1}'\}\overline{x_k}\cdots\overline{x_{n-3}}\,\overline{x_{n-2}}^2\,\overline{x_{n-3}}\cdots\overline{x_2}\,x_1^3x_2\cdots x_{n-2}\,\overline{g_{n-2}'}=\alpha_0\quad\text{for } k\neq 0,\ 1,\\
&\{(\gamma_k)x_0g_{n-1}'\}(\overline{x_k}\cdots\overline{x_{n-2}})(\overline{x_{k-1}}\cdots\overline{x_{n-3}})\cdots(\overline{x_2}\cdots\overline{x_{n-k}})d_{n-k}'x_{n-k+1}\cdots x_{n-3}\\
&x_{n-2}^2x_{n-3}\cdots x_2\overline{x_0}\,\overline{x_1}^3\,\overline{x_2}\cdots\overline{x_{n-2}}=\gamma_{n-2}\quad\text{for } k\text{: odd},\\
&\{(\gamma_k)x_0g_{n-1}'\}(\overline{x_k}\cdots\overline{x_{n-2}})(\overline{x_{k-1}}\cdots\overline{x_{n-3}})\cdots(\overline{x_2}\cdots\overline{x_{n-k}})h_2^2(x_{n-k}\cdots x_{n-2})\\
&(x_{n-k-1}\cdots x_{n-3})\cdots(x_2\cdots x_k)d_k'x_{k+1}\cdots x_{n-2}\overline{h_3}^2=\alpha_{n-1}'\quad\text{for } k\text{: even}, k\neq n-1
\end{aligned}
$$

and

$$\{(\gamma_{n-1})x_0g_{n-1}'\}\overline{h_3}^2\overline{d_{n-1}'}=\alpha_n'.$$

Here h_2 (resp. h_3) is a Dehn twist around the loop u_2 (resp. u_3), which points q_0, $q_{n-k+1},\dots,q_{n-1}$ and q_n (resp. q_n) are outside u_2 (resp. u_3) and the union $x_1\cup\cdots\cup x_{n-k-1}$ (resp. $x_0\cup x_1\cup\cdots\cup x_{n-2}$) is inside u_2 (resp. u_3). Finally $(\gamma_k')\overline{x_0}g_{n-1}'w$ is $\tilde{N}$-eqivalent to $(\gamma_{n-k+1})x_0g_{n-1}'$.

Hence β is admissible. □

Lemma 10. *Let x be an interval which meets α_0 only at q_0. Suppose that every interval in the complement of $x\cup\alpha_0$ is liftable. Then x is admissible.*

Proof. We may slide the end q_0 of x along α_0 on the right side of α_0. We then get a curve β such that (i) β meets α_0 only at q and (ii) β is separating in the complement of α_0. Hence, by Proposition 1, there exists $h\in\tilde{N}$ which leaves α_0 fixed and takes β onto a curve $\tilde{\beta}$, isotopic to one of the curves α_1, γ_n or δ_n'. If we slide back the initial point of $\tilde{\beta}$ along α_0, then we get one of the intervals x_0, g_n or g_n'. These intervals are admissible by Lemma 7. Hence x is admissible. □

Lemma 11. *Let x be an interval which meets α_n only at q_n. Suppose that every interval in the complement of $x\cup\alpha_n$ is liftable. Then x is admissible.*

Proof. We may slide the end q_n of x along α_n on the left side of α_n. We then have a curve β such that (i) β meets α_n only at q and (ii) β is separating in the complement of α_n. Hence, by Lemma 9, β is N_1-equivalent to a curve $\tilde{\beta}$, isotopic to one of the curves $(\alpha_0)g_{n-1}'$, $(\gamma_{n-1})x_0g_{n-1}'$ or $(\gamma_{n-1}')\overline{x_0}g_{n-1}'$. If we slide back to the initial point of $\tilde{\beta}$ along α_n, then we get one of the intervals d_{n-1}', v_1 or v_2 in Fig. 13.

We can check that

$$
\begin{aligned}
&(v_1)\overline{x_0}x_{n-1}\cdots x_2x_1^3=g_n,\\
&(v_2)h_3^2=v_1.
\end{aligned}
$$

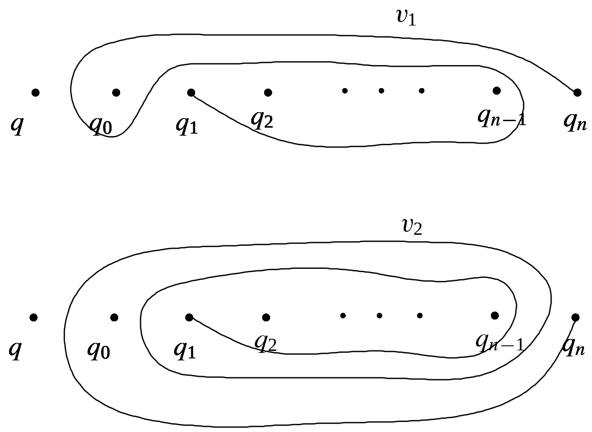

Fig. 13.

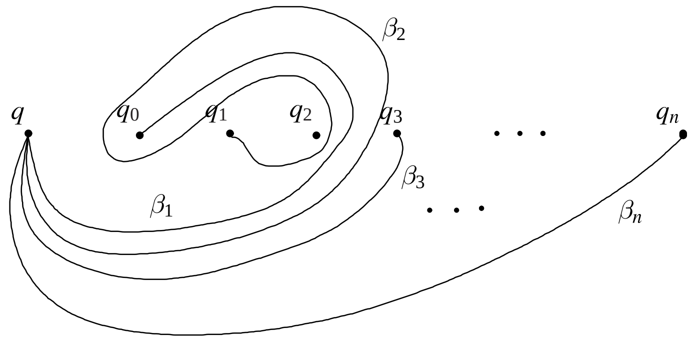

Fig. 14.

Hence, these intervals are admissible. Hence x is admissible. □

Lemma 12. *If h is liftable and $(\tilde{\alpha}_2')h = \tilde{\alpha}_2'$ then $h \in \tilde{N}$.*

Proof. Since h is liftable, h preserves the monodromy sequence of some Hurwitz system. Now, if we consider a Hurwitz system of curves $\beta_1, \dots, \beta_n$ in the complement of $\tilde{\alpha}_2'$, as in Fig. 14, then the monodromy sequence of this system is (1 3), (2 3), (2 3), …, (2 3).

Let z_i be an interval which is homotopic to the union $\beta_i \cup \beta_{i+1} (i = 1, \dots, n-1)$. By Theorem 4, h belongs to a group N_2 which is generated by the rotations

$$z_1^3, z_2, \dots, z_{n-1}, (z_4)z_3 z_2 z_1^2 z_2 z_3^2 z_2 z_1.$$

Note that $h \in N_2$. We prove that $N_2 \subset \tilde{N}$.

We can check that

$$
\begin{aligned}
&z_1 = g_2', \\
&(z_2)\overline{x_0}x_2x_1^3 = g_3, \\
&z_i = x_i \quad \text{ for } i \neq 1,\ 2
\end{aligned}
$$

and

$$\{(z_4)z_3z_2z_1^2z_2z_3^2z_2z_1\}\overline{g_2'}^3 x_4x_3x_2{s_2'}^3 g_3' = g_5'.$$

Since all generators of N_2 belong to $\tilde{N}$, it follows that $N_2 \subset \tilde{N}$. □

Lemma 13. *Let x be an interval which meets $\tilde{\alpha}_2'$ only at q_2. Suppose that every interval in the complement of $x \cup \tilde{\alpha}_2'$ is liftable. Then x is admissible.*

Proof. We can slide the end q_2 of x along $\tilde{\alpha}_2'$ on the right side of $\tilde{\alpha}_2'$. We then have a curve β such that (i) β meets $\tilde{\alpha}_2'$ only at q and (ii) β is separating in the complement of $\tilde{\alpha}_2'$.

By Proposition 1, β is N_2-equivalent to a curve $\tilde{\beta}$, isotopic to one of the curves β_1, $(\beta_1)z_1\cdots z_{n-2}z_{n-1}^2z_{n-2}\cdots z_1$ or $(\beta_1)\overline{z_1}\cdots\overline{z_{n-2}}\,\overline{z_{n-1}}^2\,\overline{z_{n-2}}\cdots\overline{z_1}$. (see the proof of Lemma 12.) If we slide back the initital point of $\tilde{\beta}$ along $\tilde{\alpha}_2'$, then we get one of the intervals v_3, v_4 or v_5 in Fig. 15.

Note that $N_2 \subset \tilde{N}$. (see the proof of Lemma 12). We prove that these intervals are admissible. We can check that

$$
\begin{aligned}
&(v_3)x_1^3 = x_0, \\
&(v_4)\overline{g_2'}^3\,\overline{x_2}\cdots\overline{x_{n-1}} = d_{n-1}
\end{aligned}
$$

and

$$(v_5)\overline{x_0}x_2\cdots x_{n-1} = d_{n-1}'.$$

These intervals are admissible. Hence x is admissible. □

By the *index* of an interval or a curve x we mean the number (minimal in the isotopy class of x) of the intersection points of x with the union $\alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_n$.

Lemma 14. *Let x be a curve or an interval such that* (i) *x has the minimal index in its $\tilde{N}$-equivalence class,* (ii) *x is not admissible and* (iii) *every interval with index smaller than the index of x is admissible. Then* (a) *every interval in the complement of x is liftable and* (b) *every interval which meets x at its end points is not liftable.*

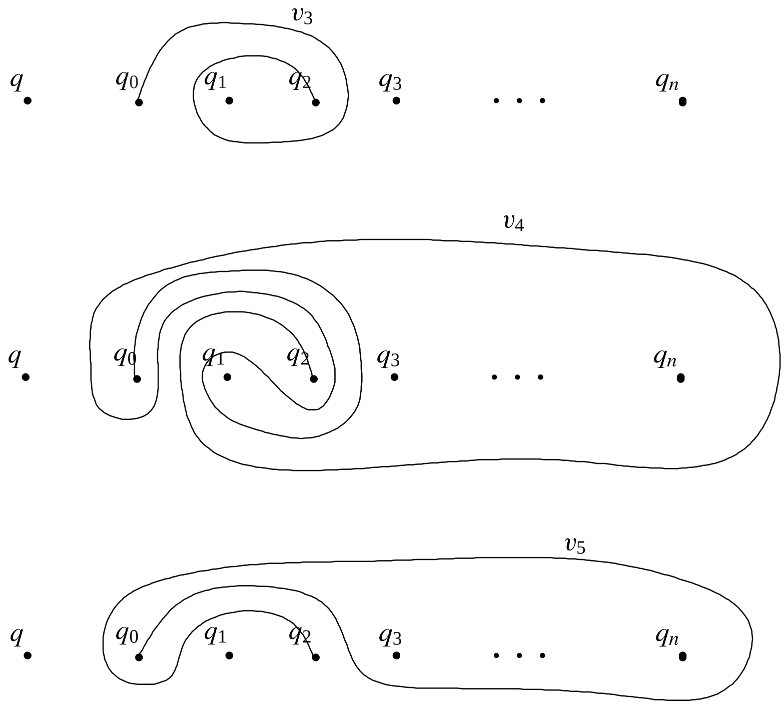

Fig. 15.

Proof. This follows from Lemma 6, Lemma 9 and Lemma 3.10 of Birman-Wajnryb [3]. □

Lemma 15. *Assume that every interval and every curve with index smaller than k is admissible. Then every curve and every interval with index k is admissible.*

Proof. Since the total monodromy is trivial, every curve is not separating. By Lemma 14, every curve of index k is either admissible or $\tilde{N}$-equivalent to a curve with smaller index. Hence every curve is admissible.

Let x be an interval with index k. By Lemma 14, we can assume that every interval in the complement of x is liftable. Note that x intersects every curve α_i. Let p be the first point of α_0 which belongs to x. Let β be a curve isotopic to the union of the piece of α_0 from q to p and the piece of x from p to an end point of x. Then β has index smaller than k. Hence β is an admissible curve. Hence β meets x only at its end point. By Lemma 8, β is $\tilde{N}$-equivalent to one of the curves α_0, α_n or $\tilde{\alpha}'_2$. Hence x is $\tilde{N}$-equivalent to an interval which meets one of the curves α_0, α_n or $\tilde{\alpha}'_2$ only at its end point. Hence, by Lemma 10, Lemma 11 and Lemma 13, x is admissible. □

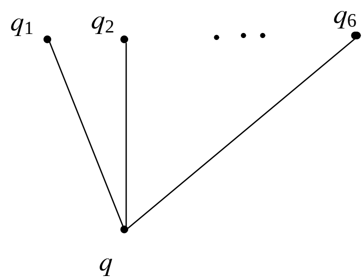

Fig. 16.

Proof of Proposition 2. By Lemma 15, every curve and every interval is admissible. Let h be an arbitrary liftable homeomorphism in $\tilde{B}_n$. Then $(\alpha_0)h$ is an admissible curve with monodromy (1 2). By Lemma 8 it is $\tilde{N}$-equivalent to one of the curves α_0, α_n or $\tilde{\alpha}'_2$. But only α_0 has the monodrmomy (1 2) among these. Hence there exists g in $\tilde{N}$ such that $(\alpha_0)hg = \alpha_0$. By Lemma 4, h belongs to $\tilde{N}$.

This completes the proof of Proposition 2 and Theorem 5. □

7. Riemann pictures and symplectic basis for canonical forms

In this section, we introduce a picture, (we call it a Riemann picture), which represents a finite branched covering of a disc topologically (see Namba-Takai [13]). We explain it by an example:

Let us consider Case 1 of genus 1.

Let X be a Riemann surface of genus 1. Let $f: X \longrightarrow \Delta$ be a branched covering of degree 3 with the monodromy Φ of canonical form of Case 1. Put $B_f = \{q_1, q_2, \ldots, q_6\}$. Let q be a reference point. We take the lassos γ_j around q_j as in Fig. 2. We extend the covering to the branched covering of $\mathbb{P}^1$ in a canonical way as in Section 3. In this case, we have

$$\pi_1(\mathbb{P}^1 - B_f, q) = \langle \gamma_1, \gamma_2, \ldots, \gamma_6, \gamma_\infty \mid \gamma_\infty\gamma_6 \cdots \gamma_2\gamma_1 = 1 \rangle\,,$$
$$A_1 = A_2 = (1\ 2), \quad A_2 = \cdots A_6 = (2\ 3), \quad A_\infty = id \quad (A_j = \Phi(\gamma_j)).$$

Consider the picture (Fig. 16) in which the circle part of every lasso γ_j in Fig. 2 is degenerated to the point q_j:

We then pull the picture in Fig. 16 back over the covering f and get the following picture in Fig. 17 which we call the Riemann picture of f:

In Fig. 17, the points ①, ②, ③ are the inverse images of the reference point q while the points $1, \ldots, 6$ and ∞ are the inverse images of $q_1, \ldots, q_6$ and ∞ respectively. Note that around every point ①, ②, ③, the paths connecting to the points $1, \ldots, 6$ and ∞ in this order are arranged clockwise. On the other hand, around every point $1, \ldots, 6$ and ∞, the paths connecting to the points ①, ②, ③ are arranged

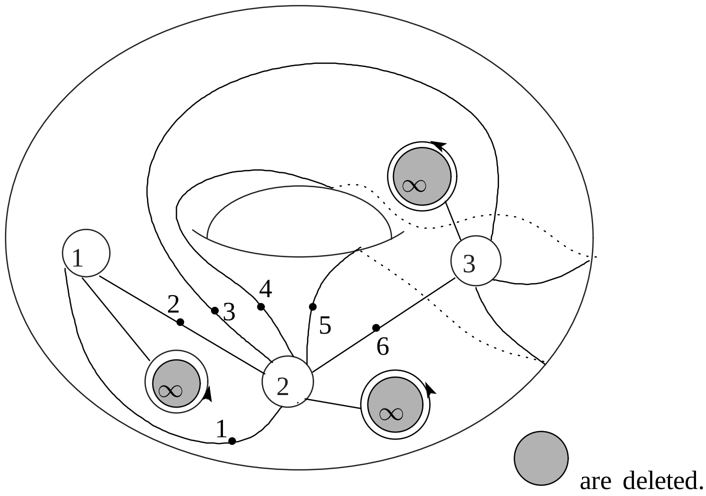

Fig. 17.

counterclockwise in order to be compatible with the monodromy. (We omit unramified points and paths connecting to them in the picture.)

The covering (X, f) can be topologically expressed by this picture.

Put

$$\begin{aligned} \xi_3 &= [1, 21][\infty, 11][1, 12], \\ \xi_2 &= [\infty, 22], \\ \xi_1 &= [6, 23][\infty, 33][6, 32], \\ \alpha &= [3, 23][4, 32], \\ \beta &= [5, 23][4, 32]. \end{aligned}$$

Here the notation $[6, 23]$ for example means the path in Fig. 17 whose initial point is ② and the terminal point is ③ passing through the branch point 6. Then these are loops with the initial point ②. We can observe the following relations:

$$\begin{aligned} \beta\alpha\beta^{-1}\alpha^{-1}\xi_3\xi_2\xi_1 &= 1, \\ \langle\alpha, \beta\rangle &= 1, \end{aligned}$$

where the notation $\langle , \rangle$ means the intersection number. We pull back the relation

$$\gamma_\infty\gamma_6\cdots\gamma_1 = 1$$

over f and get the following three relations:

$$[\infty, 11][2, 12][1, 21] = 1,$$

$$[\infty, 22][6, 23][5, 32][4, 23][3, 32][2, 21][1, 12] = 1,$$
$$[\infty, 33][6, 32][5, 23][4, 32][3, 23] = 1.$$

The above relation

$$\beta\alpha\beta^{-1}\alpha^{-1}\xi_3\xi_2\xi_1 = 1$$

can be induced from these three relations.

The Riemann picture of a general (X, f) is defined as in the above example, that is, a pull-back over f of the graph on $\mathbb{P}^1$ of Fig. 2 degenerated the circle part of every lasso to the branch point.

REMARK 6. 1. The Riemann picture is determined by (X, f) up to orientation preserving homeomorphisms of X.
2. As noted above, we can draw the Riemann picture of (X, f) even when only the monodromy $\Phi = \Phi_f$ is given and (X, f) is not explicitly given.
3. In Namba-Takai [13], we have introduced another picture in order to express (X, f) topologically, which we called a Klein picture. Klein pictures and Riemann pictures are dual in a sense. Klein pictures are useful to observe the degeneration of branched coverings, while Riemann picutres are useful to compute fundamental groups as will be seen in Section 8.

We draw the Riemann pictures of the canonical forms in Theorem 3 (see Figs. 18, 19 and 20 for $g = 3$), from which we easily find canonical generators $\{\alpha_i, \beta_i, \xi_i\}$ of the fundamental group of X such that
CASE 1: $\beta_g\alpha_g\beta_g^{-1}\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\alpha_{g-1}^{-1}\cdots\beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1}\xi_3\xi_2\xi_1 = 1$,
CASE 2: $\beta_g\alpha_g\beta_g^{-1}\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\alpha_{g-1}^{-1}\cdots\beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1}\xi_2\xi_1 = 1$,
CASE 3: $\beta_g\alpha_g\beta_g^{-1}\alpha_g^{-1}\beta_{g-1}\alpha_{g-1}\beta_{g-1}^{-1}\alpha_{g-1}^{-1}\cdots\beta_1\alpha_1\beta_1^{-1}\alpha_1^{-1}\xi = 1$.
Here $\{\alpha_i, \beta_i (i = 1, \ldots, g)\}$ is a symplectic basis in homology level of the extension $\hat{X}$ of X:

$$\langle\alpha_i, \beta_j\rangle = \delta_{ij}, \quad \langle\alpha_i, \alpha_j\rangle = 0, \quad \langle\beta_i, \beta_j\rangle = 0$$

$(i,\ j = 1, \ldots, g)$, where $\langle, \rangle$ means the intersection number.

In fact we may take them as follows:
CASE 1:

$$\begin{aligned} \xi_3 &= [1, 21][\infty, 11][1, 12] \\ \xi_2 &= [\infty, 22] \\ \xi_1 &= [2g + 4, 23][\infty, 33][2g + 4, 32] \\ \alpha_1 &= [3, 23][4, 32] \\ \beta_1 &= [5, 23][4, 32] \end{aligned}$$

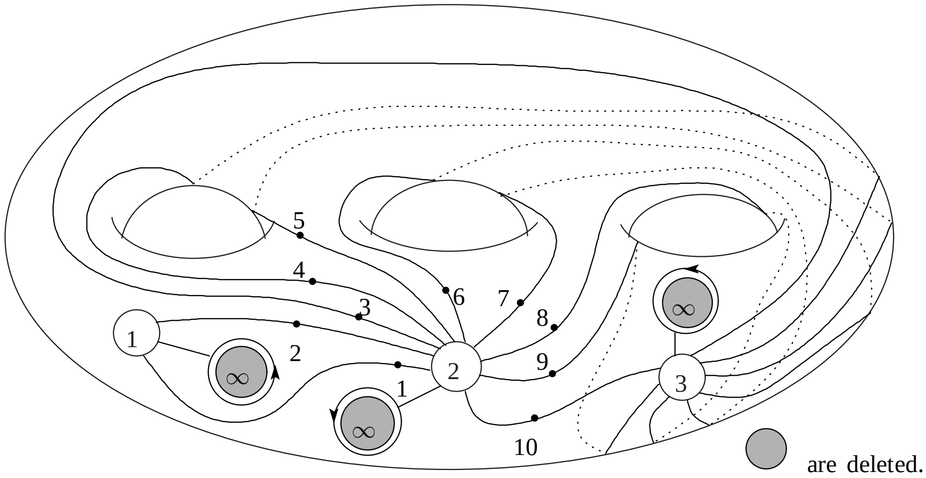

Fig. 18. Case 1

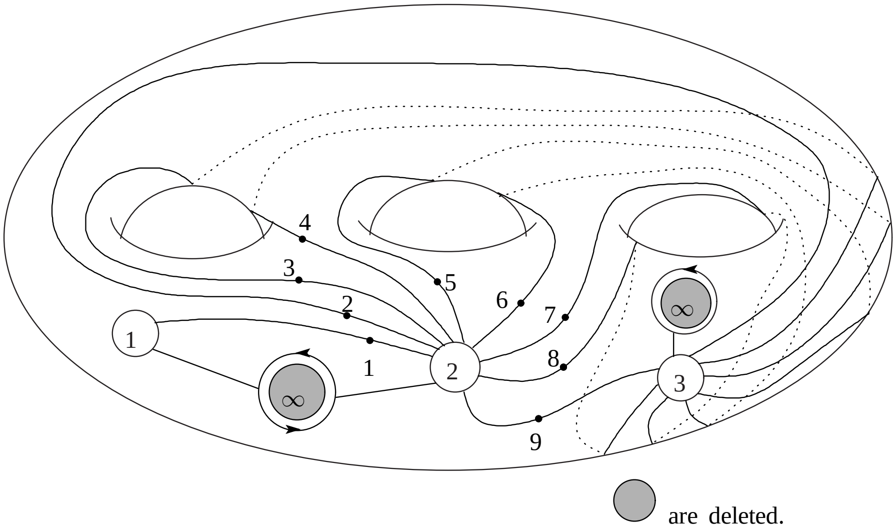

Fig. 19. Case 2

$$\cdots\cdots$$

$$\alpha_j = [2j+1, 23][2j, 32]\cdots[3, 23][2j+2, 32]$$

$$\beta_j = [2j+3, 23][2j+2, 32]$$

$$\cdots\cdots$$

$$\alpha_g = [2g+1, 23][2g, 32]\cdots[3, 23][2g+2, 32]$$

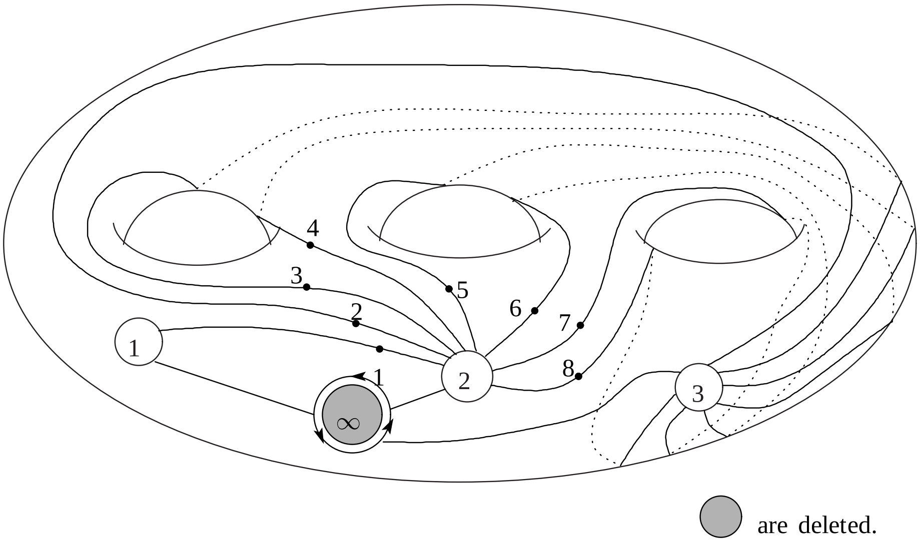

Fig. 20. Case 3

$$\beta_g = [2g+3, 23][2g+2, 32].$$

CASE 2:

$$\xi_2 = [\infty, 21][\infty, 12]$$
$$\xi_1 = [2g+3, 23][\infty, 33][2g+3, 32]$$
$$\alpha_1 = [2, 23][3, 32]$$
$$\beta_1 = [4, 23][3, 32]$$
$$\cdots\cdots$$
$$\alpha_j = [2j, 23][2j-1, 32]\cdots[2, 23][2j+1, 32]$$
$$\beta_j = [2j+2, 23][2j+1, 32]$$
$$\cdots\cdots$$
$$\alpha_g = [2g, 23][2g-1, 32]\cdots[2, 23][2g+1, 32]$$
$$\beta_g = [2g+2, 23][2g+1, 32].$$

CASE 3:

$$\xi = [\infty, 21][\infty, 13][\infty, 32]$$
$$\alpha_1 = [2, 23][3, 32]$$
$$\beta_1 = [4, 23][3, 32]$$
$$\cdots\cdots$$

$$\begin{aligned}
\alpha_j &= [2j, 23][2j-1, 32]\cdots[2, 23][2j+1, 32] \\
\beta_j &= [2j+2, 23][2j+1, 32] \\
&\quad \cdots\cdots \\
\alpha_g &= [2g, 23][2g-1, 32]\cdots[2, 23][2g+1, 32] \\
\beta_g &= [2g+2, 23][2g+1, 32].
\end{aligned}$$

8. Calculations of fundamental groups

In this section, we compute fundamental groups of some 3-dimensional compact oriented manifolds using the local version of the theorem of Zariski-van Kampen (see Dimca [5], Matsuno [10]) and the method of Reidemeister-Schreier (see Rolfsen [14]). One can compute the fundamental group rigorously if one uses the Riemann picture. We explain this using a concrete example:

EXAMPLE 1. Let us consider Case 1 of genus 1 for simplicity. If we take the braid σ as

$$\sigma = \sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_5\sigma_4\sigma_3\sigma_2^3$$

(σ induces a knot), then we have the equality

$$\Phi \circ \sigma = \Phi$$

where Φ is the monodromy of the canonical form. Hence we may construct a topological degenerating family

$$f\colon X \longrightarrow \Delta(0, a) \times \Delta(0, b)$$

of branched coverings of discs constructed from the pair (Φ, σ) (see Section 2). Let B_f be the branch locus of f. Let $\gamma_j (j = 1, \ldots, 6)$ be the lassos as in Fig. 2. The local version of the theorem of Zariski-van Kampen asserts that the fundamental group of $\Delta(0, a) \times \Delta(0, b) - B_f$ is generated by $\gamma_j (j = 1, \ldots, 6)$ whose generating relations are $\sigma(\gamma_j) = \gamma_j (j = 1, \ldots, 6)$. That is to say

$$\begin{aligned}
&\pi_1(\Delta(0, a) \times \Delta(0, b) - B_f, q) = \langle \gamma_1, \ldots, \gamma_6 \mid \sigma(\gamma_j) = \gamma_j (j = 1, \ldots, 6)\rangle \\
&= \langle \gamma_1, \ldots, \gamma_6 \mid (\sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_5\sigma_4\sigma_3\sigma_2^3)\gamma_j = \gamma_j (j = 1, \ldots, 6)\rangle \\
&= \langle \gamma_1, \ldots, \gamma_6 \mid \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_1^{-1} = 1, \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_5^{-1}\gamma_6^{-1}\gamma_5 \\
&\gamma_1^{-1}\gamma_5^{-1}\gamma_6\gamma_5\gamma_1\gamma_5^{-1}\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_2^{-1} = 1, \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1} \\
&\gamma_5^{-1}\gamma_6^{-1}\gamma_5\gamma_1\gamma_5^{-1}\gamma_6\gamma_5\gamma_4\gamma_3\gamma_2\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_3^{-1} = 1, \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4^{-1}\gamma_3\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1} \\
&\gamma_4^{-1}\gamma_1\gamma_4^{-1} = 1, \gamma_1^{-1}\gamma_4\gamma_3\gamma_2\gamma_3^{-1}\gamma_4\gamma_3\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_1\gamma_5^{-1} = 1, \gamma_5\gamma_6^{-1} = 1\rangle.
\end{aligned}$$

Now, for fixed $t \neq 0$, the restriction of f is

$$f_t \colon X_t \longrightarrow t \times \Delta(0, b).$$

This is a covering of degree 3 and the genus of X_t is 1. We extend the covering to the branched covering of $\mathbb{P}^1$ in the caconincal way as in Section 3 which is denoted by the same notation for simplicity.

Now the method of Reidemeister-Schreier says that the fundamental group $\pi_1(X - \{x\}, p)$, $(x = f^{-1}((0,0)))$ is generated by these loops ξ_1, ξ_2, ξ_3, α and β (see Section 7) and their generating relations are pull-back over f_t of these of the fundamental group $\pi_1(\Delta(0, a) \times \Delta(0, b) - \boldsymbol{B}_f, q)$, expressed by the generators ξ_1, ξ_2, ξ_3, α and β. We can carry this out observing the Riemann picture in Fig. 17.

For example, we consider pull-back over f_t of the relation $\gamma_5\gamma_6^{-1} = 1$. A loop $[5, 23][6, 32]$ in Riemann picture is pull-back over f_t of the path $\gamma_5\gamma_6^{-1}$ in $\Delta(0, b) - \boldsymbol{B}_{f_t}$ and expressed $\alpha^{-1}\xi_3\xi_2$ by the generators. Then we get a relation of the fundamental group: $\alpha^{-1}\xi_3\xi_2 = 1$.

The result is as follows:

$$\pi_1(X - \{x\}, p) = \langle \alpha, \beta, \xi_1, \xi_2, \xi_3 \mid \xi_3^{-1}\alpha = 1, \xi_3^{-1}\alpha\xi_3\alpha^{-1}\xi_3\xi_2\beta = 1, \xi_3^{-1}\alpha^2 = 1,$$
$$\xi_3^{-1}\alpha\beta^{-1} = 1, \alpha^{-1}\xi_3\xi_2 = 1, \beta\alpha\beta^{-1}\alpha^{-1}\xi_3\xi_2\xi_1 = 1\rangle = \{1\}.$$

Therefore

$$\pi_1(Y, p) \simeq \pi_1(X - \{x\}, p) = \{1\},$$

where Y is the 3-dimensional compact oriented manifold on which X is a cone (see Section 2).

EXAMPLE 2. We consider Case 1 of genus 1 again. If we take the braid σ as

$$\sigma = \sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-2}\sigma_3^{-1}\sigma_2^{-2}\sigma_3^{-1}\sigma_4^{-1}\sigma_5\sigma_4\sigma_3\sigma_2^2\sigma_3\sigma_4^2\sigma_3\sigma_2\sigma_1\sigma_3\sigma_4\sigma_5$$

(σ induces a knot), then we have the equality

$$\Phi \circ \sigma = \Phi.$$

Hence we can calculate the fundamental group of the 3-dimensional compact oriented manifold Y constructed from the pair (Φ, σ) as Example 1. The result is as follows:

$$\pi_1(Y, p) = \left\langle \alpha \mid \alpha^3 = 1 \right\rangle.$$

EXAMPLE 3. Let us consider Case 1 of genus 2. If we take the braid σ as

$$\sigma = \sigma_2^{-1}\sigma_3^{-2}\sigma_2^{-1}\sigma_1\sigma_2\sigma_3^2\sigma_2\sigma_7\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2^3$$

(σ induces a knot), then we have the equality

$$\Phi \circ \sigma = \Phi.$$

Hence we can calculate the fundamental group of the 3-dimensional compact oriented manifold Y constructed from the pair (Φ, σ). The result is as follows:

$$\pi_1(Y, p) = \{1\}.$$

References

[1] I. Bauer and F. Catanese: Generic lemniscates of algebraic functions, Math. Ann. **307** (1997), 417–444.

[2] J.S. Birman: *Braids, Links, and Mapping Class Group*, Ann. Math. Studies, **82**, Princeton, (1974).

[3] J.S. Birman and B. Wajnryb: 3-Fold branched coverings and the mapping class group of a surface, Lect. Notes in Math. **1167**, Springer, (1985), 24–46.

[4] J.S. Birman and B. Wajnryb: Errata: Presentations of the mapping class group, Israel J. Math. **88** (1994), 425–427.

[5] A. Dimca: Singurarities and Topology of Hypersurfaces, Springer-Verlag, 1992.

[6] O. Forster: Riemannsche Flächen, Springer-Verlag, 1977.

[7] R.H. Fox: Covering spaces with singularities, Lefschetz Symposium, Princeton Univ. Press, (1957), 243–262.

[8] H. Grauert and R. Remmert: Komplexe Räume, Math. Ann. **136** (1958), 245–318.

[9] H.M. Hilden: *Every closed orientable* 3*-manifold is a* 3*-fold branched covering space of* S^3, Bull. Amer. Math. Soc. **80** (1974), 1243–1244.

[10] T. Matsuno: *On a theorem of Zariski-Van Kampen type and its applications*, Osaka J. Math. **32** (1995), 645–658.

[11] J.M. Montesinos: *A representation of closed, orientable* 3*-manifolds as* 3*-fold branched coverings of* S^3, Bull. Amer. Math. Soc. **80**, (1974), 845–846.

[12] M. Namba: *Degenerationg families of meromorphic functions*, Proc. Internat. Conf. "Geometry and Analysis in Several Complex Variables", Kyoto Univ. (1997), RIMS Kokyuroku, **1058** (1998), 77–94.

[13] M. Namba and M. Takai: *Degenerating families of branched coverings*, Osaka J. Math. **40** (2003), 139–170.

[14] D. Rolfsen: Knots and Links, Publish or Perisch Inc., 1976.

Department of Mathematics
Osaka University
Toyonaka City, 560-0043, Japan
e-mail: takai@gaia.math.wani.osaka-u.ac.jp