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TRANSFORMS OF CURRENTS BY MODIFICATIONS AND 1-CONVEX MANIFOLDS

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1. Introduction

Let X' and X be complex manifolds (not compact, a priori), and $X' \overset{\alpha}{\to} X$ a proper modification with center Z and exceptional divisor E, whose irreducible components are $\{E_k\}$. Let Y be an analytic subset of X without irreducible components in Z: then its strict (proper) transform Y' is a well-defined analytic subset of X'. In particular, when D is a complex hypersurface of X, we can define the strict transform D' and also the total transform

$$\alpha^* D = D' + \sum_k n_k E_k, \quad n_k \geq 0. \tag{1.1}$$

In the first part of this paper we shall extend these notions to the case of currents on X, and ask for the existence and uniqueness of strict and total transforms.

We can look for a strict transform T' of a current T on X (of every bidegree) when T is of order zero and $\chi_Z T = 0$ (see Definition 3.1); moreover, if a strict transform exists, it is unique (see Proposition 3.2).

On the other hand, to define the total transform $\alpha^* T$ of a current T on X (Definition 3.3), T must be "closed" in some sense: in fact, the idea is that if φ is a smooth form on X, cohomologous to T, then $\alpha^* T$ should be cohomologous to $\alpha^* \varphi$. The classical case is that of d-closed currents, while the most general context seems to be that of $\partial\overline{\partial}$-closed currents (i.e. pluriharmonic currents); moreover, we would like to generalize (1.1) as:

$$\alpha^* T = T' + L \tag{1.2}$$

where L is a current supported on E. As for existence results, since we have to estimate locally the mass of $T_\alpha := (\alpha|_{X'-E})_*^{-1}(T|_{X-Z})$, we shall assume $T \geq 0$ (in the sense of Lelong).

But notice that defining a "good" total transform, besides bidegree $(1,1)$, seems hopeless: for instance, if Y is a line through the origin in $\mathbf{C}^3$ and $X' \overset{\alpha}{\to} X := \mathbf{C}^3$ is the blow-up with center in the origin, what could be the "true" meaning of $\alpha^* Y$?

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Thus, in general, we need to take into account the bidegree of the current: we shall give most results when the bidegree is $(1, 1)$, only a few results when the bidimension is $(1, 1)$ (that is, the bidegree is $(n-1, n-1)$, with $n := \dim_{\mathbb{C}} X$); nothing is known in the general case, besides the uniqueness of the strict transform.

Let us now explain our results when the bidegree of T is $(1, 1)$, i.e. T is a $(1, 1)$-current. First of all, as regards *uniqueness*.

Proposition 1.1 (see Theorem 3.9). *Let T be a pluriharmonic* $(1, 1)$*-current of order zero on X. Then, if the strict transform T' and the total transform α^*T exist, they are unique and* (1.2) *holds.*

This result is not obvious, since L, the part of the total transform which is supported on E, is not, in general, of the form $\sum_k f_k[E_k]$, nor a current on E (see Examples 3.10 and 3.15).

As for the *existence*, the "classical" case is not difficult: if T is a closed, positive $(1, 1)$-current on X, then T' and α^*T exist, they are closed and positive, and moreover

$$\alpha^*T = T' + \sum_k c_k[E_k],$$

where every c_k is a non-negative constant. In general, we have:

Theorem 1.2 (see Theorem 3.11, but also Theorem 3 in [2]). *Let T be a positive pluriharmonic* $(1, 1)$*-current on X. Then the strict transform T' and the total transform α^*T exist and are positive; moreover, $\alpha^*T = T' + \sum_k f_k[E_k]$, where every f_k is a non-negative weakly plurisubharmonic function on E_k.*

Hence, while α^*T is pluriharmonic, T' turns out to be only plurisuperharmonic, i.e. $i\partial\overline{\partial}T' \leq 0$ (see Example 3.12).

As for the existence of the strict transform, we get:

Proposition 1.3 (see Proposition 3.13 but also Corollary 3.6 in [3]). *Let T be a positive plurisubharmonic* $(1, 1)$*-current on X (i.e. $i\partial\overline{\partial}T \geq 0$). Then the strict transform T' exists.*

For *currents of bidimension* $(1, 1)$, the analogue of complex curves, we have the following result (notice that a compactness hypothesis cannot be avoided, see Remark 4.2):

Theorem 1.4 (see Theorem 4.1 and Proposition 4.5). *Let T be a positive plurisubharmonic current on X of bidimension* $(1, 1)$ *and such that $\chi_Z T = 0$. If E is compact and has a Kähler neighborhood in X', or if T has compact support and* *there exists a Kähler current on a neighborhood of* $\alpha^{-1}(\operatorname{Supp} T)$*, then the strict transform* T' *exists (and is unique).*

From this theorem we get the following:

Theorem 1.5 (see Theorem 4.7). *Let* X *be a complex manifold which is an open subset of a manifold in the class* $\mathcal{C}$ *of Fujiki and let* S *be a compact analytic subset of* X*. If* T *is a positive pluriharmonic current on* X*, of bidimension* $(1, 1)$ *and supported on* S*, then there exist suitable currents* R *and* P *on* X*, supported on* S*, such that* R *is closed and of bidimension* $(1, 1)$ *and* $T = R + \overline{\partial} P + \partial \overline{P}$*.*

Roughly speaking, the meaning of the Theorem is the following: if S is smooth, the hypothesis concerning the class $\mathcal{C}$ of Fujiki implies that the De Rham cohomology of S coincides with the Aeppli cohomology of S ($i\partial\overline{\partial}$-closed forms modulo $(\partial + \overline{\partial})$-exact forms); the Theorem asserts that a similar statement also holds in the singular case (this result is needed in the proof of Theorem 5.4).

The second part of the paper concerns 1-*convex manifolds*.

A complex analytic space X is 1-convex when it is a proper modification of a Stein space Y in a finite number of points. In the present paper we consider only the case of a complex *manifold* X, hence Y has only a finite number of (isolated) singularities; we shall always indicate with S the exceptional set of the modification, which is also the maximal compact analytic subset of X.

An old question is to establish when a 1-convex space is *embeddable*, that is when there is an embedding of X in $\mathbf{C}^p \times \mathbf{CP}_m$ for suitable p and m.

It is well-known that 1-convex surfaces are embeddable (see [7]). More recentely it has been shown that 1-convex manifolds X whose exceptional set S is 1-dimensional are certainly embeddable when $\dim X > 3$, while if $\dim X = 3$ there could be some exceptional cases which are listed in [11]. More precisely, if X is not embeddable, then S contains an irreducible component which is a rational curve of type $(-1, -1)$, $(0, -2)$ or $(1, -3)$; as a matter of fact, examples are known only for the first two cases (see [30], [11] and [9]).

Another problem is the tie between the Kähler property and the embeddability: every embeddable 1-convex manifold is Kähler, but the converse is unknown. A partial result (see [5]) says that, when S is a curve, a possible counterexample should satisfy the condition that $H_2(X, \mathbf{Z})$ is not finitely generated. When $\dim S > 1$, very few is known.

Both the known examples of non-embeddable 1-convex manifolds have been built starting from a Stein space Y which is the affine part of a projective hypersurface. This kind of construction has been recently generalized by Vâjâitu, who proved the following:

Proposition 1.6 ([27]). *Let* $N \subset \mathbf{CP}_m$ *be a hypersurface with isolated singularities,* $M \xrightarrow{f} N$ *be a resolution of singularities, and* $H \subset \mathbf{CP}_m$ *be a hyperplane which avoids the singular locus of* N *and such that* $\Sigma := H \cap N$ *is smooth. Set* $X := M - f^{-1}(\Sigma)$*. Then, for* $m \geq 4$*, the following statements are equivalent*: (i) X *is Kähler.* (ii) X *is embeddable.* (iii) M *is projective.*

The main goal of this paper is to generalize the above result as follows:

Theorem 1.7 (Theorem 5.4). *Let* N *be a projective variety of dimension at least three and with isolated singularities. Let* $M \xrightarrow{f} N$ *be a resolution of singularities, and* Σ *a hypersurface of* N *which avoids the singular locus of* N *and such that* $N - \Sigma$ *is Stein. Let* $X := M - f^{-1}(\Sigma)$*, which is a* 1*-convex manifold. Then, if the map*:

$$H_2(X, \mathbf{R}) \xrightarrow{i^*} H_2(M, \mathbf{R}) \tag{1.3}$$

is injective, the following properties are equivalent:
(i) X *is Kähler.*
(ii) X *is embeddable.*
(iii) M *is projective.*

In general, we don't know when the hypothesis (1.3) is really necessary (if $\dim S = 1$, see Remark 5.6); but when Σ is smooth, we can replace it with some other hypotheses, which are stronger but easier to check, precisely with one of the following:
(i) $H_1(\Sigma, \mathbf{R}) = 0$;
(ii) $\dim S < \dim X - 1$ and $H^1(M, \mathbf{R}) = 0$ (or $H^1(X, \mathbf{R}) = 0$);
(iii) Σ is a complete intersection in some $\mathbf{CP}_q$;
(iv) Σ is embeddable in some $\mathbf{CP}_m$, with $m \leq 2 \dim X - 3$;
(v) N is a complete intersection in some $\mathbf{CP}_q$;

In particular, from (i) it follows that this is a true generalization of Vâjâitu's result.

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2. Preliminaries

We cannot report here all the preliminaries concerning the theory of currents that are needed in what follows: so we shall only recall some results about **C**-flat currents, currents supported on analytic subsets and Aeppli cohomology.

Let X be an n-dimensional complex manifold; $\mathcal{E}^{p,q}(X)$ and $\mathcal{D}^{p,q}(X)$ are respectively the space of (p, q)-forms on X and its subspace of compactly supported ones. The space of currents on X with *bidimension* (p, q) is denoted by $\mathcal{D}'_{p,q}(X)$ and is the dual space of $\mathcal{D}^{p,q}(X)$ with respect to its natural topology. Since a current $T \in$

$\mathcal{D}'_{p,q}(X)$ is locally given by a $(n-p, n-q)$-form with distribution coefficients, we shall say that T has *bidegree* $(n-p, n-q)$ or that it is an $(n-p, n-q)$-current. A subscript $\mathbf{R}$, like for instance $\mathcal{E}^{p,p}_{\mathbf{R}}(X)$, denotes the spaces of *real* forms or currents.

The space of currents of bidimension (p, q) and of order zero, that is, such that all coefficients are complex measures, is denoted by $\mathcal{M}_{p,q}(X)$.

If Y is an analytic subset of X, and $T \in \mathcal{M}_{p,q}(X - Y)$, then T can be extended to a current $S \in \mathcal{M}_{p,q}(X)$ if and only if T has locally finite mass across Y; among all these extensions, the trivial extension T^0 is characterized by $\chi_Y T^0 = 0$.

When a real (k, k)-current is positive in the sense of Lelong, we shall write $T \geq 0$. Every positive current is real and of order zero.

DEFINITION 2.1. Let T be a real (k, k)-current on X. T is said pluriharmonic if $\partial\overline{\partial}T = 0$, plurisubharmonic if $i\partial\overline{\partial}T \geq 0$ and plurisuperharmonic if $i\partial\overline{\partial}T \leq 0$.

DEFINITION 2.2. A current T on X is $\mathbf{C}$-flat if locally $T = F + \overline{\partial}G + \partial H$ for some currents F, G, and H with coefficients in L^1_{loc} (see [8, Definition 1.1]).

For $\mathbf{C}$-flat currents, we shall refer to [8]; in particular, we shall often use the following result, which is not explicitly proved there:

Proposition 2.3. *Let T be a real plurisubharmonic current in $\mathcal{M}_{p,p}(X)$. If Y is an analytic subset of X, with* $\dim Y < p$, *then* $\chi_Y T = 0$ (*as usual, χ_Y is the characteristic function of the set Y*).

Proof. Also $i\partial\overline{\partial}T$ is of order zero, for it is positive. By Corollary 1.16 in [8], T is $\mathbf{C}$-flat, and by the Cut-Off Lemma 1.11 in the same paper, $\chi_Y T$ is also $\mathbf{C}$-flat. Therefore, since the $2p$-dimensional Hausdorff measure of Y vanishes, we get $\chi_Y T = 0$ by the Federer-type Support Theorem 1.13 in [8]. □

In the present paper, we can avoid to use the full notation of forms and currents on an analytic subset (nevertheless, see [8, pp. 576–577]), since we shall be always in the following particular case:

Y is an analytic subset of X of pure dimension p, and T is a real $\mathbf{C}$-flat current in $\mathcal{M}_{p,p}(X)$ such that $\mathrm{Supp}(T) \subseteq Y$.

In this situation we say that T *is a current on* Y if there is $f \in L^1_{\mathrm{loc}}(Y)$ such that $T = f[Y]$. As a matter of fact, if we agree that this definition is correct when Y is smooth, we can argue as follows: by the previous Proposition, $\chi_{\mathrm{Sing}(Y)}T = 0$, hence T is the trivial extension of the current $R := T|_{X-\mathrm{Sing}(Y)}$ across $\mathrm{Sing}(Y)$. Then R, being a current on $\mathrm{Reg}(Y)$, is of the form $R(\varphi) = \int_{\mathrm{Reg}(Y)} f\varphi$ for every $\varphi \in \mathcal{D}^{p,p}(X - \mathrm{Sing}(Y))$, where $f \in L^1_{\mathrm{loc}}(\mathrm{Reg}(Y))$. But, since R has locally finite mass across $\mathrm{Sing}(Y)$, f is integrable not only on compact sets in $\mathrm{Reg}(Y)$, but also on $\mathrm{Reg}(Y) \cap K$, for every compact K in X, so that:

$$T(\varphi) = \int_{\mathrm{Reg}(Y)} f\varphi \quad \forall \varphi \in \mathcal{D}^{p,p}(X).$$

This means that $T = f[Y]$.

Let us recall that the Aeppli groups are defined by:

$$V_{\mathbf{R}}^{p,p}(X) := \frac{\{\varphi \in \mathcal{E}_{\mathbf{R}}^{p,p}(X) : i\partial\overline{\partial}\varphi = 0\}}{\{\partial\overline{\psi} + \overline{\partial}\psi : \psi \in \mathcal{E}^{p,p-1}(X)\}}$$

$$\Lambda_{\mathbf{R}}^{p,p}(X) := \frac{\{\varphi \in \mathcal{E}_{\mathbf{R}}^{p,p}(X) : d\varphi = 0\}}{\{i\partial\overline{\partial}\psi : \psi \in \mathcal{E}_{\mathbf{R}}^{p-1,p-1}(X)\}}$$

The inclusion $\mathcal{E}_{\mathbf{R}}^{p,p}(X) \to \mathcal{D}'_{n-p,n-p}(X)_{\mathbf{R}}$ induces the following isomorphisms:

$$V_{\mathbf{R}}^{p,p}(X) \simeq \frac{\{T \in \mathcal{D}'_{n-p,n-p}(X)_{\mathbf{R}} : i\partial\overline{\partial}T = 0\}}{\{\partial\overline{P} + \overline{\partial}P : P \in \mathcal{D}'_{n-p,n-p+1}(X)\}}$$

$$\Lambda_{\mathbf{R}}^{p,p}(X) \simeq \frac{\{T \in \mathcal{D}'_{n-p,n-p}(X)_{\mathbf{R}} : dT = 0\}}{\{i\partial\overline{\partial}P : P \in \mathcal{D}'_{n-p+1,n-p+1}(X)_{\mathbf{R}}\}}.$$

REMARK 2.4. If φ is a real $\partial\overline{\partial}$-closed (p, p)-form on X and T is a real $\partial\overline{\partial}$-closed (p, p)-current on X, we shall denote by $\langle\varphi\rangle$ and $\langle T\rangle$ their classes in $V_{\mathbf{R}}^{p,p}(X)$. In particular, when $\langle T\rangle = 0$, we shall say that T is a *component of a boundary* (for there is a current P such that $T = \partial\overline{P} + \overline{\partial}P$, thus T is the component of bidegree (p, p) of $d(P + \overline{P})$).

Finally, if $X \overset{\Phi}{\to} Y$ is a map between complex manifolds, the map $\mathcal{E}_{\mathbf{R}}^{p,p}(Y) \overset{\Phi^*}{\to} \mathcal{E}_{\mathbf{R}}^{p,p}(X)$ induces a map $V_{\mathbf{R}}^{p,p}(Y) \overset{\Phi^*}{\to} V_{\mathbf{R}}^{p,p}(X)$. It follows that if T is a $\partial\overline{\partial}$-closed (p, p)-current on Y, then the classes $\langle T\rangle \in V_{\mathbf{R}}^{p,p}(X)$ and $\Phi^*\langle T\rangle \in V_{\mathbf{R}}^{p,p}(Y)$ are well-defined.

3. Transforms of currents of degree $(1, 1)$

In the present chapter, X and X' always denote complex manifolds, $X' \overset{\alpha}{\to} X$ is a proper modification with exceptional divisor E whose irreducible components (necessarily of codimension 1) are denoted by $\{E_k\}$; $Z := \alpha(E)$ is the center of the modification, so that $\alpha|_{X'-E} : X' - E \to X - Z$ is a biholomorphic map. We are interested in the study of the strict transform and of the total transform of a current T on X. As we shall see, the bidegree of the current is important; moreover, the case when T is pluriharmonic (which is needed in Theorem 4.7, and to study 1-convex manifolds) will be a little more difficult than the classical case (when T is closed).

Let us start with an easy consideration. If Y is an analytic subset of X, with no irreducible component contained in the center Z , then the strict transform of Y is nothing but the topological closure $Y' := \overline{\alpha^{-1}(Y - Z)}$ in X'. In particular, in the case of a (irreducible, for simplicity) hypersurface D of X , besides the strict transform D', we can also define the total transform $\alpha^* D$; if D is locally defined by a holomorphic function f, then $\alpha^* D$ is the divisor defined by $f \circ \alpha$, and it holds:

$$\alpha^* D = D' + \sum_k n_k E_k \tag{3.1}$$

where every n_k is a non negative integer.

Let us extend the notion of strict transform to currents (of order zero, because we need characteristic functions).

DEFINITION 3.1. Let T be a current of order zero on X. We say that a current T' of order zero on X' is the strict transform of T by α if $\chi_E T' = 0$ and $\alpha_* T' = T$.

Since $\alpha|_{X'-E} \colon X' - E \to X - Z$ is a biholomorphic map, the current $T_\alpha := (\alpha|_{X'-E})_*^{-1}(T|_{X-Z})$ is well-defined on $X' - E$.

Proposition 3.2. *Let T be a current of order zero on X. There exists a strict transform of T if and only if $\chi_Z T = 0$ and $T_\alpha := (\alpha|_{X'-E})_*^{-1}(T|_{X-Z})$ has locally finite mass across E. If a strict transform exists, then it coincides with the trivial extension of T_α across E, thus it is unique.*

Proof. If T' is a strict transform of T, from $\alpha_* T' = T$ it follows that $T_\alpha = T'|_{X'-E}$; since, by hypothesis, $\chi_E T' = 0$, T' turns out to be the trivial extension $(T_\alpha)^0$ of T_α across E. Moreover, $\chi_Z T = \chi_Z \alpha_* T' = \alpha_*(\chi_E T') = 0$. On the contrary, let us suppose that T_α has locally finite mass across E (hence there exists $(T_\alpha)^0$) and that $\chi_Z T = 0$. To show that $(T_\alpha)^0$ is the strict transform of T, we have only to check that $\alpha_*(T_\alpha)^0 = T$ (notice that the fact that $\chi_E(T_\alpha)^0 = 0$ follows from the definition of trivial extension). The currents $\alpha_*(T_\alpha)^0$ and T coincide on $X - Z$ (by definition of T_α); moreover, $\chi_Z \alpha_*(T_\alpha)^0 = \alpha_*(\chi_E(T_\alpha)^0) = 0$ and $\chi_Z T = 0$; hence $\alpha_*(T_\alpha)^0 = T$. □

The above Proposition emphasizes the analogy between currents and analytic subsets: the absence of irreducible components of Y contained in Z corresponds to the condition $\chi_Z T = 0$, and moreover, as $Y' = \overline{\alpha^{-1}(Y - Z)}$, T' is a sort of "closure" of $T_\alpha := (\alpha|_{X'-E})_*^{-1}(T|_{X-Z})$.

Let us now extend the definition of total transform.

DEFINITION 3.3. Let T be a pluriharmonic $(1,1)$-current of order zero on X. We shall say that a pluriharmonic $(1,1)$-current R of order zero on X' is a total transform of T if $\alpha_* R = T$ and $R \in \alpha^* \langle T \rangle$ (see 2.4).

Le us notice that the above definition generalizes the case of a divisor:

REMARK 3.4. Let D be an effective divisor of X; then $[\alpha^* D]$ is a total transform of $[D]$ (recall that $[D]$ is the current associated to the divisor D).

Proof. Locally, in an open set U of X, D is defined by the holomorphic function f and $\alpha^* D$ is defined by $f \circ \alpha$ in $\alpha^{-1}(U)$. By Lelong's formula, $[D] = (i/\pi)\partial\overline{\partial}\log|f|$ in U and $[\alpha^* D] = (i/\pi)\partial\overline{\partial}\log|f\circ\alpha|$ in $\alpha^{-1}(U)$. Let us check that $\alpha_*[\alpha^* D] = [D]$.

Call $A := \mathrm{Supp}(D) \cap U$, and take a sequence V_n of open neighborhoods of A in U, converging to A. For every $\varphi \in \mathcal{D}_{\mathbf{R}}^{n-1,n-1}(U)$, we get:

$$\begin{aligned}\alpha_*[\alpha^* D](\varphi) &= [\alpha^* D](\alpha^*\varphi) = \frac{1}{\pi}\int_{\alpha^{-1}(A)} \log|f\circ\alpha| i\partial\overline{\partial}(\alpha^*\varphi) \\ &= \lim_{n\to\infty}\frac{1}{\pi}\int_{\alpha^{-1}(V_n)} \log|f\circ\alpha| i\partial\overline{\partial}(\alpha^*\psi) = \lim_{n\to\infty}\frac{1}{\pi}\int_{V_n}\log|f| i\partial\overline{\partial}\varphi \\ &= \frac{1}{\pi}\int_A \log|f| i\partial\overline{\partial}\varphi = [D](\varphi).\end{aligned}$$

Take a smooth representative of the cohomology class of $[D]$, that is, $\psi \in \mathcal{E}_{\mathbf{R}}^{1,1}(X)$ such that $\psi = [D] + (i/\pi)\partial\overline{\partial}u$ for a suitable current u of degree zero on X. Then $\log|f| + u \in C^\infty(U)$, and $\log|f\circ\alpha| \in L^1_{\mathrm{loc}}(\alpha^{-1}(U))$ since it is plurisubharmonic, which implies $u\circ\alpha \in L^1_{\mathrm{loc}}(X')$; thus $u\circ\alpha$ is a current on X'. Hence:

$$\alpha^*\psi = \frac{i}{\pi}\partial\overline{\partial}(\log|f\circ\alpha| + u\circ\alpha) = [\alpha^* D] + \frac{i}{\pi}\partial\overline{\partial}(u\circ\alpha)$$

that is, $[\alpha^* D] \in \alpha^* \langle [D] \rangle$. □

Remember the following property:

Lemma 3.5 (see e.g. Lemma 2.6 in [1]). *Let $X' \xrightarrow{\alpha} X$ be a proper modification. For every $x \in X$ there exist an open neighborhood V of x in X, a complex manifold Z and holomorphic maps $Z \xrightarrow{g} X'$, $Z \xrightarrow{h} V$ such that $h = \alpha\circ g$; moreover $Z \xrightarrow{g} \alpha^{-1}(V)$ is a blow-up and $Z \xrightarrow{h} V$ is obtained as a finite sequence of blow-ups with smooth centers.*

Lemma 3.6. *If $X' \xrightarrow{\alpha} X$ is a blow-up with smooth (connected) center, then the exceptional divisor E is not the component of a boundary, i.e. there is no current L on X' such that $[E] = \overline{\partial}L + \partial\overline{L}$.*

Proof. Let $n = \dim X$, and denote by $E \xrightarrow{i} X'$ the inclusion map; as in the proof of Theorem 2.3 in [1], we can build a closed compactly supported $(n-1, n-1)$-form Θ

on X', such that $i^*\Theta \geq 0$ and, for some $x \in E$, $i^*\Theta_x > 0$. This gives a contradiction:

$$0 < \int_E \Theta = L(\overline{\partial}\Theta) + \overline{L}(\partial\Theta) = 0. \qquad \square$$

More generally:

Theorem 3.7. *Let X' and X be complex manifolds and $X' \xrightarrow{\alpha} X$ be a proper modification with exceptional divisor $E = \cup E_k$. If $\sum_k c_k[E_k] = \overline{\partial}L + \partial\overline{L}$, with $c_k \in \mathbf{R}$ and L a suitable current on X', then $c_k = 0$ for every k.*

Proof. Fix k_0 and then $x' \in \operatorname{Reg}(E_{k_0})$; using Lemma 3.5, we shall consider $\alpha^{-1}(V) \xrightarrow{\alpha} V$. Hence, with no loss of generality, we suppose that there are a complex manifold Z and a blow-up $Z \xrightarrow{g} X'$ such that $\alpha \circ g = h\colon Z \to X$ is given by the composition of a finite number of blow-ups with smooth centers.

Let us denote by E'_k the strict transform of E_k in Z and by $\{G_j\}$ the irreducible components of the exceptional set G of the modification $Z \xrightarrow{g} X'$. Notice that, by (3.1), $g^*E_k = E'_k + \sum_j n_{kj}G_j$. Hence, by Remark 3.4, $\sum_k c_k[g^*E_k] = \sum_k c_k[E'_k] + \sum_{k,j} c_k n_{kj}[G_j]$ is a total transform of $\sum_k c_k[E_k]$. By hypothesis, $\left\langle \sum_k c_k[E_k] \right\rangle = 0$, hence also $\langle \sum_k c_k[E'_k] + \sum_{k,j} c_k n_{kj}[G_j] \rangle = 0$.

But $\{H_i\} = \{E'_k\} \cup \{G_j\}$ are the irreducible components of the exceptional divisor of $Z \xrightarrow{h} X$, so that we only need to prove that, if $\left\langle \sum_i r_i[H_i] \right\rangle = 0$ for some constants r_i, then the r_i vanish. This claim can be easily proved by induction on the number of the blow-ups with smooth center which give the map h, using Lemma 3.6. $\square$

REMARK 3.8. The following result, which is similar to Theorem 3.7, is well-known (see for example [16, p. 286]):

If $\sum_k c_k[E_k]$ represents the zero class in $H_{2n-2}(X')$, where $n = \dim X$, then $c_k = 0$ for every k.

But notice that in Theorem 3.7 the current L is not compactly supported, so that we cannot use homology.

And now we can prove the following:

Theorem 3.9. *Let T be a pluriharmonic $(1,1)$-current of order zero on X. If a total transform exists, it is unique and will be denoted by α^*T. In this case, also the strict transform T' exists, and*

$$\alpha^*T = T' + \chi_E\alpha^*T.$$

(Some examples will prove that, in general, the current $\chi_E\alpha^*T$ is not of the form $\sum_k c_k[E_k]$, so that the present result is not a direct consequence of Theorem 3.7).

Proof. Let R be a total transform of T; since $\operatorname{codim} Z > 1$, we get $\chi_Z T = 0$ by Proposition 2.3. Hence $\alpha_*(\chi_E R) = \chi_Z \alpha_* R = \chi_Z T = 0$, and so $(1 - \chi_E)R$ satisfies the conditions in Definition 3.1 and is the strict transform T' of T. We get $R = T' + \chi_E R$.

Now let R and $\tilde{R}$ be total transforms of T. Then $R - \tilde{R} = \chi_E R - \chi_E \tilde{R}$ is supported on E. Since both $R, \tilde{R} \in \alpha^* \langle T \rangle$, there is a $(1,0)$-current L on X' such that $R - \tilde{R} = \overline{\partial} L + \partial \overline{L}$. But $0 = \alpha_* R - \alpha_* \tilde{R} = \overline{\partial}(\alpha_* L) + \partial(\alpha_* \overline{L})$ so that $\partial \alpha_* L \in \Omega^2(X)$ is a holomorphic 2-form. For every $x' \in \operatorname{Reg}(E)$ and every pseudoconvex neighborhood U of $\alpha(x')$ in X, there is $\phi \in \Omega^1(U)$ such that $\partial(\alpha_* L) = \partial \phi$. Therefore $Q := L - \alpha^* \phi$ is a $(1,0)$-current on $\alpha^{-1}(U)$, and it satisfies $R - \tilde{R} = \overline{\partial} Q + \partial \overline{Q}$ and $\alpha_* \partial Q = 0$, so that $d(Q + \overline{Q})$ is supported in $\alpha^{-1}(U) \cap E$.

Let us consider $R - \tilde{R}$: it is a pluriharmonic current of order zero, hence it is **C**-flat (see Corollary 1.16(i) in [8]); moreover, it is real, so that by Corollary 1.16(ii) in [8], in a suitable neighborhood $V \subset \alpha^{-1}(U)$ of x' in X', $R - \tilde{R} = \overline{\partial} G + \partial \overline{G}$, where G is a $(1,0)$-current in V with coefficients in $L^1_{\text{loc}}(V)$. As before, $\partial(Q - G) \in \Omega^2(V)$ and also $\partial(Q - G) = \partial \psi$ for a suitable $\psi \in \Omega^1(V)$, when V is supposed to be pseudoconvex.

The following equality holds in V:

$$d(G + \psi + \overline{G} + \overline{\psi}) = \partial Q + \overline{\partial Q} + R - \tilde{R} = d(Q + \overline{Q})$$

so that $d(Q + \overline{Q})$ is a flat current, supported on $V \cap E$. With no loss of generality, we can suppose $V \cap \operatorname{Sing}(E) = \emptyset$, and apply the Federer flatness Theorem ([19, p. 194]) getting:

$$d(Q + \overline{Q}) = \sum_k f_k [V \cap E_k]$$

where $f_k \in L^1_{\text{loc}}(V \cap E_k)$. Since our current is closed, every f_k is a constant c_k, so that we get (for the part of bidegree $(1,1)$): $R - \tilde{R} = \sum_k c_k [V \cap E_k]$. Moreover, since x' is arbitrary, we get $R - \tilde{R} = \sum_k c_k [E_k]$ in $X' - \operatorname{Sing}(E)$, and also on the whole of X', because $\operatorname{codim}(\operatorname{Sing}(E)) > 1$, so that (by Proposition 2.3) $\chi_{\operatorname{Sing}(E)}(R - \tilde{R}) = 0$. By Theorem 3.7, it follows that $R - \tilde{R} = 0$. □

The following example shows that, in general, $\chi_E \alpha^* T$ is not a current on E (see Chapter 2).

EXAMPLE 3.10. Take

$$T = \delta(z_2) i (dz_1 \wedge d\overline{z}_2 + dz_2 \wedge d\overline{z}_1) = \overline{\partial} P + \partial \overline{P}$$

where $P := -(i/\pi z_2) dz_1$. T is a pluriharmonic current of order zero on $\mathbf{C}^3$; it is supported on $Y := \{z_2 = 0\}$ but is not a current on Y. Consider the blow-up $X' \xrightarrow{\alpha} \mathbf{C}^3$ with center $Z := \{z_2 = z_3 = 0\}$, and suppose that α is defined in terms of coordinates

$x_i,\ y_j$ in X' as

$$\begin{cases} z_1 = x_1 \\ z_2 = x_2x_3 \\ z_3 = x_3 \end{cases} \quad \text{and} \quad \begin{cases} z_1 = y_1 \\ z_2 = y_2 \\ z_3 = y_2y_3 \end{cases}$$

so that the equations defining E are, respectively, $x_3 = 0$ and $y_2 = 0$. Take the current

$$P' := \begin{cases} -\dfrac{i}{\pi x_2 x_3}\, dx_1 & \text{w. r. to the } x_i \\ -\dfrac{i}{\pi y_2}\, dy_1 & \text{w. r. to the } y_j \end{cases}$$

which is obtained by extending P_α across E, so that P' is the strict transform of P.

We would like to check that $\alpha^*T = \overline{\partial}P' + \partial\overline{P'}$: since $\alpha_*(\overline{\partial}P' + \partial\overline{P}') = \overline{\partial}P + \partial\overline{P} = T$ and also, obviously, $\langle\overline{\partial}P' + \partial\overline{P}'\rangle = 0$ and $\langle T\rangle = 0$, we have only to check that $\overline{\partial}P' + \partial\overline{P}'$ is of order zero:

$$\begin{aligned} &\overline{\partial}P' + \partial\overline{P}' \\ &= \begin{cases} \delta(x_2)i\left(\dfrac{1}{x_3}\, dx_1 \wedge d\overline{x_2} + \dfrac{1}{\overline{x_3}}\, dx_2 \wedge d\overline{x_1}\right) + \delta(x_3)i\left(\dfrac{1}{x_2}\, dx_1 \wedge d\overline{x_3} + \dfrac{1}{\overline{x_2}}\, dx_3 \wedge d\overline{x_1}\right) \\ \delta(y_2)i(dy_1 \wedge d\overline{y_2} + dy_2 \wedge d\overline{y_1}) \end{cases} \end{aligned}$$

Now it is clear that $\chi_E\alpha^*T$ is not of the form $\delta(x_3)f(x_1, x_2)(i/2)\, dx_3 \wedge d\overline{x_3} = f[E]$ (in terms of the coordinates x_i).

Let us go to an existence result for the total transform:

Theorem 3.11. *Let T be a positive pluriharmonic $(1,1)$-current on X: then there exists the total transform α^*T of T; moreover it is positive and*

$$\alpha^*T = T' + \sum_k f_k[E_k] \tag{3.2}$$

where T' is the strict transform of T and every f_k is a non-negative weakly plurisubharmonic function on E_k. In particular, T' is plurisuperharmonic (i.e. $i\partial\overline{\partial}T' \leq 0$). If moreover T is closed, or if E is compact, then every f_k is a constant, and T' is closed in the first case, pluriharmonic in the second case.

Proof. The first statement is contained in Theorem 3 in [2].

To check (3.2), by Theorem 3.9 we need only to prove that $\chi_E\alpha^*T = \sum_k f_k[E_k]$. Let $\chi_E = \sum_k \chi_{E_k} - \sum_j \chi_{Y_j}$ for suitable analytic subsets Y_j of codimension bigger than one: then $\chi_{Y_j}\alpha^*T = 0$ by Proposition 2.3 and $\chi_{E_k}\alpha^*T = f_k[E_k]$ for suitable weakly plurisubharmonic functions f_k by Theorem 4.10 in [8]. From (3.2) it follows that $i\partial\overline{\partial}T' = -\sum_k i\partial\overline{\partial} f_k \wedge [E_k] \leq 0$.

If T is closed, then locally $T = i\partial\overline{\partial} f$ for a suitable plurisubharmonic function f, and $\alpha^* T = i\partial\overline{\partial}(f \circ \alpha)$ (see the proof of Remark 3.4), i.e. $\alpha^* T$ is closed and positive. A classical result (see 12.3 in [24]) implies that $\chi_{E_k}\alpha^* T = c_k[E_k]$, $c_k \geq 0$; hence by (3.2), $dT' = 0$. If E_k is compact, then every f_k is constant, and by (3.2), $\partial\overline{\partial}T' = 0$. □

Let us show an example where (3.1) and (3.2) are really different, i.e. the functions f_k are not constant.

EXAMPLE 3.12. Take $X := \{z \in \mathbf{C}^3; |z_2| < 1\}$, $Z := \{z_2 = z_3 = 0\}$, $Y := \{z_2 = 0\}$. Let $X' \stackrel{\alpha}{\to} X$ be the blow-up with center Z and

$$T := -\frac{i}{2\pi} \log |z_2| \partial\overline{\partial}|z_1|^2 + \frac{1}{2}|z_1|^2[Y].$$

T is a positive pluriharmonic current. Using the same coordinates x_i and y_j as in Example 3.10, it is easy to compute T_α and its trivial extension

$$(T_\alpha)^0 = T' = \begin{cases} -\dfrac{i}{2\pi} \log |x_2| \partial\overline{\partial}|x_1|^2 - \dfrac{i}{2\pi} \log |x_3| \partial\overline{\partial}|x_1|^2 + \dfrac{1}{2}|x_1|^2[Y'] \\ -\dfrac{i}{2\pi} \log |y_2| \partial\overline{\partial}|y_1|^2 \end{cases}$$

where Y' is the strict transform of Y. Hence

$$i\partial\overline{\partial}T' = -i\partial\overline{\partial} f \wedge [E] = \begin{cases} -\dfrac{i}{2}\partial\overline{\partial}|x_1|^2 \wedge [E] \leq 0 \\ -\dfrac{i}{2}\partial\overline{\partial}|y_1|^2 \wedge [E] \leq 0 \end{cases}.$$

As regards the strict transform of $(1, 1)$-currents, we can relieve the hypotheses in Theorem 3.11 as follows:

Proposition 3.13. *Let T be a positive plurisubharmonic* $(1, 1)$*-current on X. Then there exists the strict transform T' of T, and it is positive.*

Proof. By Proposition 2.3, $\chi_Z T = 0$, and moreover $(\alpha|_{X'-E})_*^{-1}(T|_{X-Z})$ has locally finite mass across E (by Corollary 3.6 in [3], but see also Lemma 2.10 ibidem). We get the thesis by Proposition 3.2. □

In the following example we shall show that, when T is only positive, the strict transform T' may not exist; moreover, the example will show that the hypotheses in

Proposition 3.13 are well-chosen to the problem.

EXAMPLE 3.14. There exists a $(1,1)$-current T, which is positive and plurisuperharmonic, such that T_α has not finite mass across E, so that T' cannot be defined.

Let $X' \xrightarrow{\alpha} \mathbf{C}^2$ be the blow-up at the origin, and let $\omega := (i/2)\partial\overline{\partial}||z||^2$ be the Kähler form of the euclidean metric of $\mathbf{C}^2$; take $T := ||z||^{-2}\omega$. T is a positive well-defined $(1,1)$-current on $\mathbf{C}^2$: in fact, for every neighborhood U of the origin, the mass of T in $U-\{0\}$ is given by

$$\int_{U-\{0\}} T\wedge\omega = 2\int_{U-\{0\}} ||z||^{-2}\frac{\omega^2}{2} < \infty$$

hence T can be extended across the origin.

Since $X' \subset \mathbf{C}^2\times\mathbf{CP}_1$, the natural Kähler form on X' is $\omega+\theta$, where θ is the form of the Fubini-Study metric on $\mathbf{CP}_1$. Since $\alpha_*\theta = (i/\pi)\partial\overline{\partial}\log||z||$, we get $\alpha_*\theta\wedge\omega = (1/\pi)||z||^{-2}(\omega^2/2)$; hence the mass of T_α in $\alpha^{-1}(U-\{0\})$ is given by

$$\int_{\alpha^{-1}(U-\{0\})} T_\alpha\wedge(\omega+\theta) = \int_{U-\{0\}} T\wedge\alpha_*(\omega+\theta) = \int_{U-\{0\}} T\wedge\omega + \frac{1}{\pi}\int_{U-\{0\}} ||z||^{-4}\frac{\omega^2}{2} = +\infty.$$

Notice that T is plurisuperharmonic, because $i\partial\overline{\partial}T = 0$ in $\mathbf{C}^2-\{0\}$ and for every $u\in C_0^\infty(\mathbf{C}^2)$,

$$i\partial\overline{\partial}T(u) = \lim_{\varepsilon\to 0}\int_{||z||>\varepsilon} T\wedge i\partial\overline{\partial}u = -\lim_{\varepsilon\to 0}\frac{4}{\varepsilon^4}\int_{||z||>\varepsilon} u\frac{\omega^2}{2} = -2\pi^2 u(0).$$

The last example shows that, also when T is a pluriharmonic $(1,1)$-current of order zero, which has a total transform α^*T, and $\chi_E\alpha^*T$ is a current on E, we cannot deduce that $i\partial\overline{\partial}(\chi_E\alpha^*T)$ is of order zero.

EXAMPLE 3.15. Let $X' \xrightarrow{\alpha} \mathbf{C}^2$ be the blow-up at the origin, and let

$$T = \delta(z_2)i(dz_1\wedge d\overline{z}_2 + dz_2\wedge d\overline{z}_1) = \overline{\partial}P + \partial\overline{P}$$

where $P = -(i/\pi z_2)\,dz_1$ (see Example 3.10). As in Example 3.10 (notice that, there, T was a current on $\mathbf{C}^3$!), we get $\chi_E\alpha^*T = 4\mathrm{Re}(1/y_2)[E]$. But $i\partial\overline{\partial}(4\mathrm{Re}(1/z)) = 4\pi\{(\partial\delta/\partial z) - (\partial\delta/\partial\overline{z})\}(i/2)\,dz\wedge d\overline{z}$ is not of order zero.

4. Transforms of currents of bidimension (1, 1)

Let us use the same notation as in the previous chapter: in particular, $X' \xrightarrow{\alpha} X$ is a proper modification. Let T be a current of bidimension $(1,1)$ on X, i.e. $T\in\mathcal{D}'_{1,1}(X)$. First of all, we study the existence of the strict transform T' of T.

Theorem 4.1. *Assume that the exceptional divisor E of the modification $X' \stackrel{\alpha}{\to} X$ is compact and that there exists a Kähler neighborhood of E in X'. If $T \in \mathcal{D}'_{1,1}(X)$ is positive and plurisubharmonic, and $\chi_Z T = 0$, then there exists the strict transform T'.*

Proof. By means of Proposition 3.2, it is enough to show that the current $T_\alpha := (\alpha|_{X'-E})_*^{-1}(T|_{X-Z})$ on $X' - E$ has locally finite mass across E. We can choose a relatively compact neighborhood U of Z in X, such that $\alpha^{-1}(U)$ has a Kähler metric with Kähler form Ω; moreover, it holds $\alpha_*\Omega = \Phi + i\partial\overline{\partial} f$ for a suitable closed $(1,1)$-form $\Phi \in \mathcal{E}^{1,1}_{\mathbf{R}}(U)$ and a 0-current f on U. In order to apply the Regularization Theorem of Demailly ([13]) to the current $\alpha_*\Omega$ on U, remark that "…the method can be easily extended to non compact manifolds, but uniform estimates only hold on relatively compact open subsets…" (see [13, Introduction]); so (see also Lemma 4.1 in [18]), chosen a suitable hermitian metric on U with Kähler form u, it follows that, for every smooth $(1,1)$-form γ on U which satisfies $\alpha_*\Omega \geq \gamma$, there are a sequence $\{f_\mu\}_{\mu\geq 0}$ of smooth functions on U and a sequence $\{\lambda_\mu\}_{\mu\geq 0}$ of continuous functions on U such that:

(i) $\Phi + i\partial\overline{\partial} f_\mu \geq \gamma - \lambda_\mu u \quad \text{on} \quad U$

(ii) $\{f_\mu\}_{\mu\geq 0}$ is decreasing to f

(iii) $\{\lambda_\mu\}_{\mu\geq 0}$ is decreasing to the Lelong number $n(\alpha_*\Omega, x)$, for every $x \in U$.

Moreover, using Satz 1.8 and 1.9 in [23], it is not hard to see that the sequence $\{f_\mu\}$ can be chosen in such a way that

(iv) $\{f_\mu\}_{\mu\geq 0}$ converges in $C^\infty(U - Z)$ to f.

Now, let us choose a suitable family of forms on U: for every open neighborhood W of Z, $W \Subset U$, take a smooth $(1,1)$-form γ_W on U such that

$$\alpha_*\Omega \geq \gamma_W \geq 0 \quad \text{on} \quad U \quad \text{and} \quad \alpha_*\Omega = \gamma_W \quad \text{on} \quad U - W.$$

Let $V \Subset U$ be a fixed open neighborhood of Z; from above we get, for every $W \Subset V$:

$$\int_{\alpha^{-1}(V-W)} T_\alpha \wedge \Omega = \int_{V-W} T \wedge \alpha_*\Omega = \int_{V-W} T \wedge \gamma_W \leq \int_V T \wedge \Phi + \int_V T \wedge i\partial\overline{\partial} f_\mu + \int_V T \wedge \lambda_\mu u.$$

Choose $g \in C^\infty_0(V)$, $0 \leq g \leq 1$, $g = 1$ in a neighborhood of Z, and recall that $i\partial\overline{\partial} T \geq 0$. Thus, since $(1-g)f_\mu$ converges in $C^\infty(U)$ to $(1-g)f$, and gf_μ decreases to gf, we get:

$$\begin{aligned}\lim_{\mu\to\infty} \int_V T \wedge i\partial\overline{\partial} f_\mu &= \lim_{\mu\to\infty} \int_V T \wedge i\partial\overline{\partial}[(1-g)f_\mu] + \lim_{\mu\to\infty} \int_V i\partial\overline{\partial} T \wedge g f_\mu \\ &= \int_V T \wedge i\partial\overline{\partial}[(1-g)f] + \int_V i\partial\overline{\partial} T \wedge gf.\end{aligned}$$

Moreover, $n(\alpha_*\Omega, x)$ vanishes outside Z and $\chi_Z T = 0$, therefore $\lim_{\mu\to\infty}\int_V T\wedge\lambda_\mu u = 0$. This means that $||T_\alpha||(\alpha^{-1}(V) - E) = \sup_W ||T_\alpha||(\alpha^{-1}(V - W)) < \infty$. □

REMARK 4.2. In [21, p. 1144], there is an example which shows that the compactness hypothesis on E is necessary.

It would be interesting to look for a generalization of Theorem 4.1 to currents of every bidimension, and also to avoid the hypothesis on the Kähler neighborhood of E. A first answer is given in Proposition 4.5.

DEFINITION 4.3 (see Definition 2.3 in [18]). Let X be a complex manifold. A Kähler current Ω on X is a closed $(1, 1)$-current such that $\Omega - \omega$ is a positive current (in the sense of Lelong), where ω is the $(1, 1)$-form of a suitable hermitian metric on X.

REMARK 4.4. If ω is the $(1, 1)$-form of a Kähler metric on X, then ω is a Kähler current. More generally, if a compact manifold M belongs to the class $\mathcal{C}$ of Fujiki, then (see [29, Théorème 3]) there is a proper modification $M' \xrightarrow{\beta} M$ where M' is Kähler; for every $(1, 1)$-form ω' of a Kähler metric on M', $\beta_*\omega'$ is a Kähler current on M.

Proposition 4.5. Theorem 4.1 *still holds assuming, instead of "E is compact and has a Kähler neighborhood in X'", that T is compactly supported and there exists a Kähler current in a neighborhood of $\alpha^{-1}(\operatorname{Supp} T)$ in X'.*

Proof. Choose a relatively compact neighborhood U of $\operatorname{Supp}(T)$ in X, such that $\alpha^{-1}(U)$ has a Kähler current Ω, and write $\alpha_*\Omega = \Phi + i\partial\overline{\partial} f$ for a suitable closed $(1, 1)$-form Φ on U. Finally let ω be the $(1, 1)$-form of a hermitian metric on $\alpha^{-1}(U)$ such that $\Omega - \omega \geq 0$.

Apply the Regularization Theorem of Demailly to the current $\alpha_*\Omega$ on U as in the above proof (but remark that, since we do not know if f is smooth in $U - Z$, we cannot say that f_μ converges in $C^\infty(U - Z)$ to f).

Now let us choose a suitable family of forms; for every open neighborhood W of $U \cap Z$, $W \subset U$, take a smooth $(1, 1)$-form γ_W on U such that

$$\alpha_*\Omega \geq \gamma_W \geq 0$$

on U, while on $U - W$:

$$\alpha_*\omega = \gamma_W.$$

We get:

$$\int_{\alpha^{-1}(U-W)} T_\alpha \wedge \omega = \int_{U-W} T \wedge \alpha_* \omega = \int_{U-W} T \wedge \gamma_W$$
$$\leq \int_U T \wedge \Phi + \int_U T \wedge i\partial\overline{\partial} f_\mu + \int_U T \wedge \lambda_\mu u.$$

Since T is plurisubharmonic and compactly supported in U, and f_μ decreases to f,

$$\lim_{\mu\to\infty} \int_U T \wedge i\partial\overline{\partial} f_\mu = \lim_{\mu\to\infty} \int_U i\partial\overline{\partial} T \wedge f_\mu = \int_U i\partial\overline{\partial} T \wedge f.$$

Finally, $n(\alpha_*\Omega, x)$ is upper-semicontinuous, thus bounded from above in $\mathrm{Supp}(T)$, therefore

$$\lim_{\mu\to\infty} \int_U T \wedge \lambda_\mu u \leq C||T||(U).$$

This means that $\sup_W ||T_\alpha||(\alpha^{-1}(U - W)) < \infty$. □

Proposition 4.6. *Let $T \in \mathcal{M}_{1,1}(X)$ be a positive pluriharmonic* (*resp. positive closed*) *current on X. If the strict transform T' exists, then it is pluriharmonic* (*resp. positive closed*).

Proof. The strict transform of T is $(T_\alpha)^0$, where T_α is pluriharmonic. From Theorem 2 in [12], it follows

$$i\partial\overline{\partial}(T_\alpha)^0 = i\partial\overline{\partial}(T_\alpha)^0 - (i\partial\overline{\partial} T_\alpha)^0 \leq 0;$$

thus $i\partial\overline{\partial}(T_\alpha)^0$ is a measure $\mu \leq 0$ on X'; since

$$\alpha_*(i\partial\overline{\partial}(T_\alpha)^0) = i\partial\overline{\partial}\alpha_*(T_\alpha)^0 = i\partial\overline{\partial} T = 0$$

we get $\mu = 0$ and so $i\partial\overline{\partial}(T_\alpha)^0 = 0$. If T is closed, then T_α is closed too, thus also $(T_\alpha)^0$ is closed (see Théorème 1, p. 372 in [25]). □

Let us give a first application:

Theorem 4.7. *Let X be a complex manifold which is an open subset of a manifold in the class $\mathcal{C}$ and let S be a compact analytic subset of X. If $T \in \mathcal{M}_{1,1}(X)$ is positive, pluriharmonic and supported on S, then there exist currents R and P on X, supported on S, such that R is closed and of bidimension* $(1, 1)$ *and* $T = R + \overline{\partial} P + \partial \overline{P}$.

Proof. Recall that a compact manifold M in the class $\mathcal{C}$ is regular in the sense of Varouchas ([28]); in particular the natural morphism

$$\Lambda_{\mathbf{R}}^{p,p}(M) \to V_{\mathbf{R}}^{k,k}(M)$$

is an isomorphism; therefore, for every $\partial\overline{\partial}$-closed current T on M, there exist a closed current R and a current P such that $T = R + \overline{\partial}P + \partial\overline{P}$. By the Federer-type **C**-flatness Theorem 1.24 in [8], T is a current on S, which also belongs to $\mathcal{C}$; hence, if S is smooth, the proof is over. If S is singular, let us recall the following result:

Proposition 4.8 ([10, p. 43]**).** *Let X be a complex manifold and S an analytic subset of X. There exist a complex manifold X' and a holomorphic map $X' \overset{\alpha}{\to} X$ given by a finite sequence of blowing-ups*

$$X' = X_r \overset{\alpha_r}{\to} X_{r-1} \to \cdots \to X_1 \overset{\alpha_1}{\to} X_0 = X$$

with smooth centers Z_j, $j = 0, \ldots, r-1$, such that the images of the centers lie in S and the strict transform S' of S in X' is smooth.

By our hypothesis, X and also $X_1, \ldots, X_r$ (see 3.4(ii) in [29]) are open subsets of manifolds in $\mathcal{C}$; this implies, by Remark 4.4, that the modifications α_j are in the situation of Proposition 4.5.

Let $T_0 := T$, which is a positive pluriharmonic current of bidimension $(1,1)$; since Z_0 is smooth, also the current $(1 - \chi_{Z_0})T_0$ is pluriharmonic (see Corollary 2.3 in [4]). By Proposition 4.5 we get the strict transform T_1 of $(1 - \chi_{Z_0})T_0$ via the modification α_1, and by Proposition 4.6, T_1 is pluriharmonic. In this manner, when we got T_j on X_j, it is defined the current T_{j+1} as the strict transform of $(1 - \chi_{Z_j})T_j$ via α_{j+1}, $j = 0, \ldots, r-1$.

For every j, the current $\chi_{Z_j}T_j$ is a pluriharmonic current of bidimension $(1,1)$ on Z_j which belongs to $\mathcal{C}$; hence there are currents R_j and P_j on Z_j, such that:

$$\chi_{Z_j}T_j = R_j + \overline{\partial}P_j + \partial\overline{P}_j$$

(notice that R_j is supposed to be closed and of bidimension $(1,1)$, and on X' we get: $T_r = R_r + \overline{\partial}P_r + \partial\overline{P}_r$). Since

$$T = \chi_{Z_0}T_0 + \alpha_{1*}(\chi_{Z_1}T_1) + \alpha_{1*}\alpha_{2*}(\chi_{Z_2}T_2) + \cdots + \alpha_{1*}\cdots\alpha_{r*}T_r,$$

we get the thesis if $R := R_0 + \alpha_{1*}R_1 + \cdots + \alpha_{1*}\cdots\alpha_{r*}R_r$ and $P := P_0 + \alpha_{1*}P_1 + \cdots + \alpha_{1*}\cdots\alpha_{r*}P_r$. □

5. Quasi-projective 1-convex manifolds

DEFINITION 5.1. Let Y be a complex space; a couple (N, Σ) is said a *compactification* of Y if N is a connected compact complex space, $\Sigma \neq \emptyset$ is a closed nowhere dense analytic subset of N and $N - \Sigma$ is biholomorphic to Y. If Y has a projective compactification N, then Y is said a quasi-projective space.

DEFINITION 5.2. A complex manifold X is said 1-convex (or strongly pseudoconvex) if there exist a proper surjective holomorphic map (called the Remmert reduction) $X \xrightarrow{f} Y$ onto a Stein space Y, and a finite set $B \subset Y$ such that, if $S := f^{-1}(B)$, the induced map $X - S \xrightarrow{f} Y - B$ is biholomorphic and $\mathcal{O}_Y \simeq f_* \mathcal{O}_X$. Actually, since X is a manifold, Y has only isolated singularities which are contained in B.

Let X be a 1-convex manifold and $X \xrightarrow{f} Y$ the Remmert reduction. There is a natural correspondence between the set of the compactifications of Y and that of the compactifications of X: for instance, if (N, Σ) is a compactification of Y, then gluing together $N - B$ and X we get a compactification (M, Σ) of X $(M := (N - B) \cup X)$ and a holomorphic map $M \xrightarrow{F} N$ which extends F and is the identity on Σ.

In particular, we are interested in the case where Y is quasi-projective, i.e. when Y has a projective compactification (N, Σ). If necessary, we can blow-up the singularities in $N - B$, so that $\mathrm{Sing}(N) = \mathrm{Sing}(Y)$; as said before, we get a smooth compactification (M, Σ) of X.

Precisely, the situation we shall study is the following:

$(*)$ X is a 1-convex manifold of dimension $n \geq 3$, $X \xrightarrow{f} Y$ is its Remmert reduction, where Y is a Stein quasi-projective space. Let (N, Σ) be a compactification of Y such that N is projective and $\mathrm{Sing}(N) = \mathrm{Sing}(Y)$ (i.e. the corresponding compactification (M, Σ) of X is smooth).

REMARK 5.3. In the situation $(*)$, Σ is connected and of pure codimension 1.

Proof. Notice that $H^i(M, \Sigma; \mathbf{R}) = H_{2n-i}(X, \mathbf{R})$; indeed Σ is an Euclidean Neighborhood Retract (see f.i. [14, Propositions IV.8.12, VIII.6.12 and VIII.7.2]). Thus the exact sequence of cohomology groups of the couple (M, Σ) is:

$$0 \to H^0_c(X, \mathbf{R}) \to H^0(M, \mathbf{R}) \to H^0(\Sigma, \mathbf{R}) \to H^1_c(X, \mathbf{R}) \to \cdots$$

The following exact diagram is related to the Remmert reduction $X \xrightarrow{f} Y$ (see [16, Satz 4.1]):

$$\begin{array}{ccccccccc} \cdots & \to & H_k(X) & \to & H_k(X, S) & \to & H_{k-1}(S) & \to & \cdots \\ & & \downarrow & & \downarrow\simeq & & \downarrow & & \\ \cdots & \to & H_k(Y) & \to & H_k(Y, B) & \to & H_{k-1}(B) & \to & \cdots \end{array}$$

Since $H_k(B) = 0$ for $k \geq 1$, we get $H_k(Y) \simeq H_k(Y, B) \simeq H_k(X, S)$ for $k > 1$, and the diagram gives the following exact sequence:

$$0 \to H_{2n}(S) \to H_{2n}(X) \to H_{2n}(Y) \to H_{2n-1}(S) \to H_{2n-1}(X) \to H_{2n-1}(Y) \to \cdots$$

But $H_{2n-1}(S) = H_{2n}(S) = 0$, since $\dim S < n$, and $H_{2n-1}(Y) = H_{2n}(Y) = 0$ because Y is a Stein space (see Theorem 3 in [20]). Hence, by Poincaré duality,

$$H_c^1(X, \mathbf{R}) \simeq H_{2n-1}(X) = 0$$

and

$$H_c^0(X, \mathbf{R}) \simeq H_{2n}(X) = 0.$$

Thus $H_0(\Sigma, \mathbf{R}) \simeq H_0(M, \mathbf{R}) \simeq \mathbf{R}$, since M is connected.

Finally, if A is an irreducible component of Σ with $\operatorname{codim} A \geq 2$, we can extend the holomorphic functions on Y across A, but this is impossible since Y is Stein. □

Now we can establish the

Theorem 5.4 (**Main Theorem**). *Assume the situation* $(*)$.

If the map $H_2(X, \mathbf{R}) \xrightarrow{i_*} H_2(M, \mathbf{R})$, *induced by the inclusion* $X \xrightarrow{i} M$, *is injective, then the following properties are equivalent*:

(i) X *is Kähler*

(ii) X *is embeddable*

(iii) M *is projective* (*in particular* X *is quasi-projective*).

Proof. If X has a smooth projective compactification, then X carries a positive line bundle, so that, by Theorem III in [15], X becomes embeddable and hence Kähler. So we need only to prove that, if X is Kähler, then M is Kähler too, because this implies that M is projective (notice that, by $(*)$, M is Moishezon). We shall use the characterization of compact Kähler manifolds by means of positive currents (see [17, Theorem 14]); let T be a positive current on M of bidimension $(1, 1)$ which is the $(1, 1)$-component of a boundary: it is enough to show that $T = 0$.

Since N is projective, there is an embedding $N \xrightarrow{h} \mathbf{CP}_m$, for a suitable m. Let $M \xrightarrow{F} N$ be the extension of the Remmert reduction and let θ be the Fubini-Study form on $\mathbf{CP}_m$. The form $\Omega := F^* h^* \theta$ is a closed positive form on M, which is strictly positive outside of S. Since T is the $(1, 1)$-component of a boundary:

$$0 = T(\Omega) = \int_M \Omega_x(\vec{T}_x) d||T||$$

hence $\mathrm{Supp}(T) \subset S$. From Theorem 4.7 it follows that on X:

$$T = R + \overline{\partial} P + \partial \overline{P}$$

where R and P are supported on S and R is closed. Since T is the component of a boundary in M, there is a current L on M such that $T = \overline{\partial} L + \partial \overline{L}$. Thus

$$i_* R = \overline{\partial}(L - i_* P) + \partial(\overline{L} - i_* \overline{P}).$$

Therefore $i_* R$ is closed and is the component of a boundary; but M is Moishezon, thus regular (see [28]), therefore $i_* R$ is $\partial\overline{\partial}$-exact. In particular, $i_* R$ represents the zero class of $H_2(M, \mathbf{R})$. Since $H_2(X, \mathbf{R}) \xrightarrow{i_*} H_2(M, \mathbf{R})$ is injective, $R = dQ$ for a suitable current Q compactly supported on X. Thus, on X, $T = dQ + \overline{\partial} P + \partial \overline{P}$. But X has a Kähler form, say α, and P, Q have compact support, so that $T = 0$, because $T(\alpha) = (dQ + \overline{\partial} P + \partial \overline{P})(\alpha) = 0$. □

REMARK 5.5. We do not know if the hypothesis about $H_2(X, \mathbf{R}) \xrightarrow{i_*} H_2(M, \mathbf{R})$ is really necessary.

REMARK 5.6. Assume $(*)$. If $\dim S = 1$, then X is Kähler if and only if it is embeddable.

Proof. From the exact homology sequence of the couple (M, X) we get:

$$H_3(M, X; \mathbf{Z}) \to H_2(X, \mathbf{Z}) \xrightarrow{i_*} H_2(M, \mathbf{Z})$$

Thus, since $H^{2n-3}(\Sigma, \mathbf{Z}) \simeq H_3(M, X; \mathbf{Z})$ and $H_2(M, \mathbf{Z})$ are finitely generated, it follows that $H_2(X, \mathbf{Z})$ is finitely generated too. This is enough thanks to Theorem II in [5]. □

In the last part of the paper, we shall suppose that Σ is smooth, and investigate some simple conditions which imply that:

$$H_2(X, \mathbf{R}) \xrightarrow{i_*} H_2(M, \mathbf{R}) \quad \text{is} \quad \text{injective.} \tag{5.1}$$

(this hypothesis is used in the Main Theorem 5.4)

Proposition 5.7. *Assume* $(*)$. *If* Σ *is smooth, then*

$$H_1(\Sigma, \mathbf{R}) = 0 \tag{5.2}$$

implies (5.1).

Proof. Since $\dim_{\mathbf{R}} \Sigma = 2n - 2$ (see Remark 5.3), then, by means of Poincaré duality:

$$H_1(\Sigma, \mathbf{R}) \simeq H^{2n-3}(\Sigma, \mathbf{R}) \simeq H_3(M, X; \mathbf{R}).$$

The thesis follows from the exact homology sequence of the couple (M, X):

$$H_3(M, X; \mathbf{R}) \to H_2(X, \mathbf{R}) \xrightarrow{i_*} H_2(M, \mathbf{R}). \qquad \square$$

The exact sequence of the couple (M, X) also gives:

$$H_1(X, \mathbf{R}) \to H_1(M, \mathbf{R}) \to H_1(M, X; \mathbf{R}).$$

As before, $H_i(M, X; \mathbf{R}) \simeq H^{2n-i}(\Sigma, \mathbf{R})$. For dimensional reasons,

$$H^{2n}(\Sigma, \mathbf{R}) = H^{2n-1}(\Sigma, \mathbf{R}) = 0,$$

therefore, if $H_1(X, \mathbf{R}) = 0$, then $H_1(M, \mathbf{R}) = 0$.

Proposition 5.8. *Assume* $(*)$. *If* Σ *is smooth, then*

$$\text{(5.3)} \qquad \operatorname{codim} S > 1 \quad \textit{and} \quad H_1(M, \mathbf{R}) = 0 \quad \textit{(or} \quad H_1(X, \mathbf{R}) = 0)$$

implies (5.2) *and thus* (5.1).

Proof. Arguing as in the proof of Remark 5.3 we get

$$H_{2n-1}(S) \to H_{2n-1}(X) \to H_{2n-1}(Y) \to H_{2n-2}(S) \to H_{2n-2}(X) \to H_{2n-2}(Y)$$

and, since $n \geq 3$, $H_{2n-1}(Y) = H_{2n-2}(Y) = H_{2n-1}(S) = H_{2n-2}(S) = 0$. Therefore

$$0 = H_{2n-1}(X) = H_c^1(X, \mathbf{R})$$

and

$$0 = H_{2n-2}(X) = H_c^2(X, \mathbf{R})$$

Using these facts and the exact sequence of cohomology groups of the couple (M, Σ):

$$0 = H_c^1(X, \mathbf{R}) \to H^1(M, \mathbf{R}) \to H^1(\Sigma, \mathbf{R}) \to H_c^2(X, \mathbf{R}) = 0$$

we get $H^1(M, \mathbf{R}) \simeq H^1(\Sigma, \mathbf{R})$, so that

$$0 = H_1(M, \mathbf{R}) \simeq H^1(M, \mathbf{R}) \simeq H^1(\Sigma, \mathbf{R}) \simeq H_1(\Sigma, \mathbf{R}). \qquad \square$$

Proposition 5.9. *Assume* $(*)$ *and let* Σ *be smooth. If*

(5.4) Σ *is a complete intersection in some* $\mathbf{CP}_m$

or

(5.5) N *is a complete intersection in some* $\mathbf{CP}_m$

or

(5.6) Σ *is embeddable in* $\mathbf{CP}_m$, *with* $m \leq 2n - 3$

then (5.1) *holds.*

Proof. If (5.4) holds, then Proposition 8 in [6] says that $\mathbf{CP}_m - \Sigma$ is q-complete, where q is the number of equations which define Σ; thus $q = m - (n - 1)$. And when (5.5) holds, then $\mathbf{CP}_m - N$ is q-complete, for $q = m - n$. But $Y = N - \Sigma$ is a Stein space, hence by a classical result of Siu it has a Stein open neighborhood U in $\mathbf{CP}_m - \Sigma$. So we can consider $\mathbf{CP}_m - \Sigma$ as given by the union of two open sets, $\mathbf{CP}_m - N$, which is $(m-n)$-complete, and U, which is 1-complete. Therefore $\mathbf{CP}_m - \Sigma$ is $(m - n + 1)$-complete. If (5.6) holds, then $\mathbf{CP}_m - \Sigma$ is q-complete (see [22]), with

$$q = 2(\operatorname{codim}_{\mathbf{CP}_m} \Sigma) - 1 = 2m - 2n + 1.$$

In all cases, since "q-complete" implies "cohomologically q-complete", we can use a result of Sorani (see [26, Teorema 4.4]) which asserts that, for such a manifold Z, $H_k(Z, \mathbf{C}) = 0$ for $k \geq q + \dim_{\mathbf{C}} Z$. Thus $H_{2m-2}(\mathbf{CP}_m - \Sigma, \mathbf{R}) = 0$ if $2m - 2 \geq q + m$, and by the exact sequence

$$0 = H^1(\mathbf{CP}_m, \mathbf{R}) \to H^1(\Sigma, \mathbf{R}) \to H^2(\mathbf{CP}_m, \Sigma; \mathbf{R}) \simeq H_{2m-2}(\mathbf{CP}_m - \Sigma, \mathbf{R})$$

condition (5.1) follows.

But in the first case, $2m - 2 \geq q + m$ precisely when $n \geq 3$, and in the last case when $2n - 3 \geq m$. □

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